The Stability and Transition of the Boundary
Layer on Rotating Bodies

by

Stephen John Garrett

Corpus Christi College

This dissertation is submitted to the University of Cambridge
for the degree of Doctor of Philosophy

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This thesis is dedicated to the memory of my grandparents  

Joan Luton & Frank Harry Garrett.
Declaration

This thesis describes research carried out in the Engineering Department of Cambridge University from September 1999 to April 2002. It contains 39,701 words and 69 figures, which does not exceed the word limit set by the Engineering Degree Committee. Except where explicitly stated in the text, this thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration. No part of the work contained herein has been submitted to any other university or place of learning for any degree, diploma or other qualification.

Stephen John Garrett

Corpus Christi College
Cambridge
May 2002
Summary

The majority of this work is concerned with the \textit{local}-linear stability of the incompressible boundary-layer flows over rotating spheres and rotating cones; convective and absolute instabilities are investigated and the effects of viscosity and streamline-curvature are included in each analysis. Preliminary investigations into the linear \textit{global}-mode behaviour of the rotating-disk, rotating-cone and rotating-sphere boundary layers are also presented.

The local rotating-sphere analyses are conducted at various latitudes from the axis of rotation ($\theta$), and the local rotating-cone analyses are conducted at points along cones of various half-angles ($\psi$), in each case convective and absolute instabilities are found within specific parameter spaces. The predictions of the Reynolds number, vortex angle and vortex speed at the onset of convective instability are consistent with existing experimental measurements for both boundary-layer types.

It is suggested that absolute instability may cause the onset of transition of the rotating-sphere boundary layer when $\theta < 70^\circ$, and for the rotating-cone boundary layer when $\psi \geq 60^\circ$. Axial flow is found to stabilize each boundary layer with respect to convective and absolute instabilities.

The global behaviour of the boundary-layer flows over rotating disks, cones and spheres is considered by taking into account the slowly varying basic state along each body surface. The absolute frequency of the inviscid problem is determined
as a function of a slow-spatial variable ($\bar{X}$), and the problem is immersed in the complex $\bar{X}$-plane by means of a rational-function approximation. The locations of saddle points in the absolute frequency are determined which give the leading-order estimate of the global frequency. For the rotating disk and rotating cones, the global frequency indicates that disturbances in the boundary-layer flow are globally damped; and for rotating spheres, the global frequency indicates that the boundary layer may support neutrally stable global modes when a region of absolute instability exists.

**Key Words**

Convective and absolute instabilities, transition to turbulence, three-dimensional boundary layers, rotating spheres and cones, linear global-mode analysis.
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Chapter 1

Introduction

The majority of this thesis is concerned with the stability and transition of the incompressible boundary-layer flows over rotating spheres and cones. The work was prompted by the discovery by Lingwood (1995a, 1996) of radial absolute instability in the boundary-layer flow over a disk rotating in still fluid. It is known that a region of absolute instability can give rise to a self-excited global mode which may promote transition to turbulence (Huerre & Monkewitz, 1990), and the close agreement found between the predicted onset of absolute instability and the experimentally measured onset of turbulence led Lingwood to suggest that this is precisely the mechanism for transition on the rotating disk. Given the similarity between the boundary layers on the rotating disk, the rotating sphere and rotating cones, it is reasonable to expect that rotating spheres and cones might well exhibit absolute instability as well, and this point will be investigated in this thesis. In each case linear stability theory is used and a parallel-flow approximation is made, thus restricting each analysis to a local analysis. A parallel-flow approximation was also made by Lingwood (1995a), and in this thesis we extend her work by presenting preliminary linear global-mode analyses of the boundary layers on rotating disks, rotating cones and
Chapter 1: Introduction

rotating spheres.

Visualisation experiments on the rotating disk, such as those conducted by Gregory, Stuart & Walker (1955), show a region of laminar flow at the centre of the disk followed by spiral vortices at larger radii. The spiral vortices are co-rotating and are characteristic of crossflow instabilities; they are observed to be stationary with respect to the disk surface. At larger radii still, the flow undergoes transition and becomes fully turbulent. Many experimental investigations have been performed on the rotating disk and each report transitional Reynolds numbers within a 3% scatter of the local Reynolds number $R_x \approx 2.6 \times 10^5$. See, for example, Kobayashi, Kohama & Takamadate (1988) and Malik, Wilkinson & Orszag (1981). Transition on a rotating disk is therefore repeatedly consistent despite the instability waves first appearing across a wide parameter range with varying rotation rate. This is in contrast to the boundary layer on, say, a flat plate where onset is sudden but the location is highly dependent on the disturbance environment. The repeatable transition location on the rotating disk is consistent with a well-defined location of absolute instability, and it is this that prompted Lingwood in her investigations of the rotating-disk boundary layer.

(a) The rotating sphere

When a sphere rotates in still fluid a flow is induced in which the fluid moves over the outer surface from the poles and is ejected radially from the equator. The experimental papers of Sawatzki (1970) and Kohama & Kobayashi (1983) report that the flow around the pole remains laminar, with co-rotating spiral vortices characteristic of crossflow instabilities appearing at a higher latitude; this is illustrated in figure 1.1. At a higher latitude still, the flow undergoes transition and becomes
fully turbulent. The measurements taken by Kohama & Kobayashi show that the onset of turbulence occurs at a repeatedly consistent local Reynolds number that is roughly equal to $R_X = 2.5 \times 10^5$ at all latitudes below $\theta = 70^\circ$, this is despite the instability waves first appearing across a wide parameter range with varying rotation rate and sphere radius. This observation is highly suggestive that the boundary layer becomes absolutely unstable at, or just before, the transition location which would then lead to the temporal growth of the disturbances and trigger the nonlinear behaviour characteristic of the onset of transition. This is precisely the mechanism that was claimed by Lingwood (1995a) to pertain to transition on the rotating disk as discussed above, and will be investigated for the rotating-sphere boundary layer in §2.5. The effect of axial flow on the absolute instability of the boundary layer will be investigated in §3.4.

Although the local Reynolds number at transition is independent of the transition location, Sawatzki and Kohama & Kobayashi observe that the location of the
transitional and turbulent regions move nearer to the pole with increased rotation rate. They also observe that the spiral vortices make an angle of 14° to a circle parallel to the equator at the onset of instability, but this reduces to around 4°–8° as the rotation rate is increased. The number of spiral vortices at the onset of instability increases with increased rotation rate and appears to asymptotically approach a value of 31 or 32, which is the same number as observed experimentally on the rotating disk.

The transition of the boundary layer on a rotating sphere is also studied in the experimental paper of Kobayashi & Arai (1990). For rotating disks and cones, experiments have shown that the spiral vortices are fixed on the body’s surface at all times, see Kohama (1984b) and Kobayashi, Kohama, Arai & Ukaku (1987). However, Kobayashi & Arai observe that the spiral vortices are fixed on the sphere surface when the rotation rate is large, while they move relative to the sphere surface when the rotation rate is small. The relative speed of these slow vortices is always 0.76 times the surface speed of the sphere.

The problem of a sphere rotating in a uniform axial flow has received little experimental attention. Noordzij & Rotte (1967, 1968); Tanaka & Tago (1975) and Furuta et al. (1975) have looked at the mass transfer to a rotating sphere in a uniform axial flow and Luthander & Rydberg (1935) have measured the drag induced by the sphere as the axial flow rate is increased. El-Shaarawi, Kemry & El-Bedeawi (1987) have made measurements of the velocity components in the boundary layer and also the separation point as the axial flow rate is increased. More relevant to the work presented in this dissertation though, are the experiments of Kobayashi, Arai & Nakajima (1988) and Kobayashi & Arai (1990). These papers detail the spiral vortex behaviour in the transition region of the boundary layer under various rotation and
axial flow rates. Their results show that as the axial flow rate is increased, a faster rotation rate is required to observe laminar-turbulent transition at each latitude, i.e. axial flow has the effect of stabilising the boundary layer. Unfortunately, unlike the investigation by Kohama & Kobayashi (1983) on the sphere rotating in still fluid, these papers provide no details of the number of vortices or their orientation. However, Kobayashi & Arai do report that the changeover between the stationary and slow vortices is delayed with axial flow, i.e. a larger rotation rate is required for the changeover as the axial flow increases. No measurements are made for the onset of turbulence for non-zero axial flow rates.

It appears that only a small number of theoretical papers exist that are concerned with the rotating-sphere boundary layer. For the sphere rotating in still fluid, the steady laminar boundary-layer flow was first investigated theoretically by Howarth (1951). Howarth made boundary-layer approximations to the steady Navier–Stokes equations and used a series solution to calculate the mean flow. Banks (1965) uses Howarth’s series solution and Manohar (1967) and Banks (1976) use more accurate finite difference techniques. In a uniform axial flow, El-Shaarawi, El-Refaie & El-Bedewi (1985) develop a finite-difference scheme to solve the equations governing the steady laminar boundary-layer flow.

The only theoretical paper that appears to exist on the stability of the rotating-sphere boundary layer is by Taniguchi, Kobayashi & Fukunishi (1998). The paper uses local-linear stability theory to predict the onset of convective instability and hence the appearance of the spiral vortices on a sphere rotating in still fluid. The perturbation equations derived in their paper are solved at each latitude using the approximate mean-flow profiles of Banks (1965). Taniguchi et al. show that both crossflow and streamline-curvature instabilities appear in the boundary layer. The
crossflow instability dominates near to the poles and the streamline-curvature instability dominates near to the equator. The number of vortices at the onset of instability are also calculated at each latitude and are seen to decrease with increasing latitude. At the onset of instability the spiral vortices are predicted to be stationary with respect to the sphere surface, with the region of instability moving closer to the pole with increased rotation rate. The discrepancy between the experimental critical Reynolds numbers for the onset of spiral vortices and those calculated increases slightly with latitude when below 70° but still shows reasonably good agreement. However, above a latitude of 70° the predicted critical Reynolds numbers diverge sharply from the experimental values.

It is expected that the stability characteristics of rotating-sphere boundary layer tend towards those of the rotating disk as the analysis moves towards the pole. The results of Taniguchi et al. do not do this, and so, although they have had success at other locations on the sphere, their theory is not satisfactory at all latitudes. Also, it appears that no stability analyses exist for the boundary layer on a sphere rotating in a uniform axial flow. For these reasons we conduct local convective instability analyses on the rotating-sphere boundary layer using a different formulation to Taniguchi et al. in §2.4. The effect of axial flow on the convective instabilities is investigated in §3.3. In each case an attempt to clarify the appearance of the slow and stationary vortices (as observed by Kobayashi & Arai, 1990) is also made. As mentioned earlier, local absolute instability analyses are conducted in §§2.5 & 3.4 for spheres rotating in still fluid and uniform axial flows respectively.
(b) The rotating cone

The transition of the boundary layer on the surface of rotating cones has been the subject of a number of experimental investigations. Early experimental studies of rotating cones in still fluids and uniform axial flows were limited to measurements of the transitional Reynolds numbers, for example Kreith, Ellis & Giesing (1962); Tein & Campbell (1963) and Kappesser, Greif & Cornet (1973) for rotating cones in still fluids and Salzberg & Kezios (1965) for rotating cones in uniform axial flows. Each of these experiments measured the transitional Reynolds numbers with reasonable accuracy, but the experimental techniques used were unable to clarify the structure in the transition region. Not until detailed flow visualisation and hot-wire measurements were used was it discovered that spiral vortices characterise the transition region of the boundary layer (Kobayashi & Izumi, 1983; Kobayashi, Kohama & Kurosawa, 1983; Kobayashi & Kohama, 1984; Kohama, 1984a); this is illustrated in figure 1.2. The vortices are known to be fixed on the cone for all rotation and axial flow rates, which is in contrast to the vortices found on the rotating sphere.

Using cones with half-angles of 15°–75° in 15° increments rotating in an otherwise still fluid, Kobayashi & Izumi (1983) have found that the critical and transition Reynolds numbers become larger as the cone angle is increased, the vortex angle and number of spiral vortices also increases and approaches the value observed on the rotating disk. When expressed in terms of a local Reynolds number, Kobayashi & Izumi’s data shows that the onset of turbulence occurs at the same local Reynolds numbers for all cones with half-angles between 60° and 90°. This may suggest that absolute instability is initiating the transition to turbulence on these cones. Given the similarity between the boundary layers on the rotating disk and cones with large
half-angles, it is reasonable to expect that these cones might also exhibit absolute instability, and this is investigated in §4.6 of this thesis. The effect of axial flow on the absolute instability will be investigated in §5.4.

Using a cone with a half-angle of 15° rotating in a uniform axial flow, Kobayashi et al. (1983); Kobayashi & Kohama (1984) and Kohama (1984a) have found that, for a fixed rotation rate, the critical and transition Reynolds numbers are increased with increased axial flow. The vortex angle and number are also increased with increased axial flow.

As well as presenting the experimental results discussed above, Kobayashi & Izumi (1983) use local-linear stability theory to predict the onset of convective instability and hence the appearance of the spiral vortices. The study considers cones with half-angles of 15°–90° in 15° increments; critical Reynolds numbers for the onset of the vortices, the spiral-vortex angle and the number of vortices are predicted as a
function of the half-angle. The predicted critical Reynolds numbers agree reasonably well with the experimental values given in the paper. However, the experimental measurements for the onset of instability conducted by Kreith et al. (1962) and Kappesser et al. (1973) agree well with each other but are substantially different from those of Kobayashi & Izumi. Also, Kobayashi & Izumi provide no information on the instability modes that govern the transition of the boundary layer, and so the relative importance of the crossflow and streamline-curvature modes at each half-angle remains unknown. We attempt to clarify these points in §4.5 of this thesis with our own convective instability analyses of these boundary layers.

Kobayashi (1981) and Kobayashi, Kohama & Kurosawa (1983) use local-linear stability theory to predict the onset of spiral vortices in the boundary-layer flows over cones rotating in a uniform axial flow. Both papers consider a cone with a half-angle of 15° and predict the critical Reynolds number for the onset of the vortices, the vortex angle and the number of vortices as a function of an axial flow parameter. The predictions are found to be consistent with the experimental measurements of Salzberg & Kezios (1965) and also the experimental data presented by Kobayashi, Kohama & Kurosawa (1983). However, neither paper gives information on the relative importance of the instability modes that govern the transition in an axial flow, and this will be addressed in §5.3 of this thesis.

We have therefore seen evidence to suggest that the transition of the rotating-sphere and rotating-cone boundary layers may be prompted by an absolute instability, much in the same way as was found by Lingwood on the rotating disk. We have also seen that very little is known about the convective-instability modes that lead to the onset of the spiral vortices in both types of boundary layer. These points
will be addressed in this thesis, where we consider the boundary layers on spheres and cones rotating in still fluid and also in axial flows. In each case, spatial and temporal analyses are used separately to study the convective instability when the boundary-layer flow is not absolutely unstable, and then a spatio-temporal analysis is used to study the absolute instability.

Briggs (1964) and Bers (1975) have shown that an absolute instability can be identified by singularities corresponding to zeros of the dispersion relation. The singularities occur when waves that propagate energy in different directions coalesce, with the point of coalescence known as a pinch point. Pinch points can appear as the Reynolds number is varied, and at these points the flow changes from a convectively unstable regime to an absolutely unstable regime. Here the Briggs–Bers method is used to distinguish between absolutely and convectively unstable time-asymptotic responses to impulsive excitation of each of the boundary layers. Other examples of absolute instability in fluid mechanics include Huerre & Monkewitz (1985); Niew (1993) and Lingwood (1995a, 1997). Similar ideas have applied in the context of the boundary layer on a swept wing, see for example Taylor & Peake (1998).

As mentioned above, a parallel-flow approximation will be used in the analyses of the rotating-sphere and rotating-cone boundary layers presented in chapters 2–5, and so we are restricted to the local stability characteristics of each flow. However, a region of local absolute instability can give rise to a self-excited global mode and it is the nonlinear results of the global instability that lead to the onset of turbulence. The aim of a global analysis is to obtain a spatially extended structure made up of waves governed by the local properties of the medium and tuned at an overall global frequency. Using a linear global analysis, Chomaz, Huerre & Redekopp (1991) have demonstrated that, at leading order, the complex global frequency is
determined by a saddle-point condition applied to the local linear dispersion relation for the linear complex Ginzburg–Landau equation with spatially varying coefficients. Monkewitz, Huerre & Chomaz (1993) have shown that the same criterion also holds for the Navier–Stokes equations linearized about an arbitrary slowly varying basic flow. Similar ideas have been applied to a variety of physical systems, for example the Taylor–Couette flow between concentric spheres (Soward & Jones, 1983) and low-speed axisymmetric jets (Cooper & Crighton, 2000). These flows display a spatially varying basic state and so have a spatial dependence of the local instability characteristics, much like the rotating disk, cone and sphere boundary layers. The methods employed in those investigations will be discussed in more detail in chapter 6, where similar ideas are used in preliminary investigations into the linear global-mode behaviour of the rotating-disk, rotating-cone and rotating-sphere boundary layers.

The work presented in chapter 6 is related to that of Davies & Carpenter (2001), who study the linear global behaviour corresponding to the absolute instability of the rotating-disk boundary layer. This is done by conducting numerical simulations of the complete linearized Navier–Stokes equations with and without a parallel-flow approximation, (they term these two cases as homogenous and inhomogeneous respectively). Impulse-like excitations are used which creates disturbances in the form of wave-packets, initially containing a wide range of frequencies. In the homogenous case their findings are consistent with those of Lingwood (1995a), i.e. convective and absolute behaviour is observed in the parameter regions predicted by Lingwood. In the inhomogeneous case their findings are consistent with the experimental behaviour observed in the experiments of Lingwood (1996), and in particular there is close agreement between the simulation and experiment for the
ray paths traced out by the leading and trailing edges of the wave-packets. However, more relevant to the work presented in this thesis are their findings which indicate that, in the inhomogeneous case, convective behaviour eventually dominates at all Reynolds numbers after sufficient time. This means that the absolute instability of the rotating-disk boundary layer does not produce a linear amplified global mode. We shall return to this point in chapter 6 where we compare our findings with those of Davies & Carpenter.
Chapter 2

The rotating-sphere boundary layer in still fluid

This chapter is concerned with the instability mechanisms within the boundary-layer flow over the outer surface of a sphere rotating in an otherwise still fluid, and has appeared in the literature as Garrett & Peake (2002). Viscous and streamline-curvature effects are included, and local stability analyses are conducted between latitudes of 10°–80° from the axis of rotation in 10° increments.

In §2.1 the solution of the steady boundary-layer equations that give the laminar-flow profiles is described, and the unsteady perturbation equations for the stability problem are derived in §2.2. In §2.4 the perturbation equations are solved and we study the local convective instability of the boundary layer at each latitude. It is here that the methods used in the solution of each set of perturbation equations in this thesis are described. The local absolute instability analyses are conducted in §2.5. In each section our theoretical results are compared to existing experimental measurements where possible.
2.1 The steady mean flow

Experiments have shown that the onset of transition occurs at a well-defined Reynolds number based on the local velocity and distance along the surface of the sphere from the pole. For the onset of transition, it is therefore relatively unimportant if disturbances convect in the azimuthal direction in either a fixed or rotating frame of reference; it is the development of disturbances in the direction of increasing latitude that is of primary importance. An absolute instability at a fixed distance from the pole, in either frame of reference, will lead to an unbounded linear response at that distance which is likely to promote non-linearity and transition. All previous work on the rotating-sphere boundary layer has been conducted in a fixed frame of reference, see for example the laminar-profile solutions of Howarth (1951) & Banks (1965) and the convective instability analysis of Taniguchi et al. (1998). In this chapter and the next we consider the sphere to be rotating in a fixed frame of reference. This is in contrast to Lingwood’s work on the rotating disk, where a frame of reference that rotates with the disk is used. As a result, the Coriolis terms that appear in Lingwood’s perturbation equations do not appear in this work, and the centrifugal effects due to rotation are included via the boundary conditions in the solution of the steady-laminar profiles.

Consider spherical polar coordinates fixed in space with origin located at the centre of the sphere. The radius of the sphere is $a^*$, and it rotates at a constant angular frequency $\Omega^*$. The distance $r^*$ is measured radially from the centre of the sphere, $\theta$ is the angle of latitude measured from the axis of rotation and $\phi$ is the angle of azimuth. The coefficient of kinematic viscosity is denoted by $\nu^*$ and density by $\rho^*$ (asterisks indicate dimensional quantities).

The equations that govern the mean flow are stated in, amongst others, Banks
(1965) as

\[
\begin{align*}
W^* \frac{\partial U^*}{\partial r^*} + U^* \frac{\partial U^*}{a^* \partial \theta} - \frac{V^{*2}}{a^*} \cot \theta &= \frac{\partial^2 U^*}{\partial r^{*2}}, \\
W^* \frac{\partial V^*}{\partial r^*} + U^* \frac{\partial V^*}{a^* \partial \theta} + U^* V^* \frac{\partial \theta}{a^*} &= \frac{\partial^2 V^*}{\partial r^{*2}}, \\
\frac{\partial W^*}{\partial r^*} + \frac{1}{a^*} \frac{\partial U^*}{\partial \theta} + \frac{U^*}{a^*} \cot \theta &= 0,
\end{align*}
\]

(2.1) \hspace{1cm} (2.2) \hspace{1cm} (2.3)

where \( U^* \), \( V^* \) & \( W^* \) are the dimensional velocities in the \( \theta \), \( \phi \) and \( r^* \) directions respectively. These equations are derived from the steady Navier–Stokes and continuity equations by making the boundary-layer assumptions that \( W^* \sim O(\delta^*), U^* \sim O(1), V^* \sim O(1) \) and \( (\partial / \partial \theta) \sim O(1) \), where \( \delta^* = (\nu^* / \Omega^*)^{1/2} \) is the boundary-layer thickness. Using these in the continuity equation we find that \( \partial / \partial r^* \sim O(\delta^{-1}) \), and in the radial Navier–Stokes equation we find \( P^* = P^*(\theta) \). Since, in this chapter, the sphere is rotating in otherwise still fluid \( P^* = \text{constant} \). By assuming \( \delta^*/a^* \ll 1 \) we can replace the \( r^* \) multiplying terms in the Navier–Stokes and continuity equations by \( a^* \) – this represents a parallel-flow assumption.

In the fixed frame of reference (2.1)–(2.3) are subject to the boundary conditions

\[
\begin{align*}
U^* = W^* = V^* - a^* \Omega^* \sin \theta &= 0 \quad \text{on } r^* = a^*, \\
U^* = V^* &= 0 \quad \text{as } r^* \to \infty.
\end{align*}
\]

(2.4)

The first of these equations represents the no-slip condition on the sphere surface, and the second represents the quiescent fluid condition away from the sphere.

The non-dimensional mean-flow variables are defined as

\[
U(\eta, \theta) = \frac{U^*}{\Omega^* a^*}, \quad V(\eta, \theta) = \frac{V^*}{\Omega^* a^*}, \quad W(\eta, \theta) = \frac{W^*}{[\nu^* \Omega^*]^{1/2}},
\]

(2.5)

where \( \eta = (\Omega^* / \nu^*)^{1/2}(r^* - a^*) \) is the non-dimensional distance from the sphere surface in the radial direction. Note that \( \eta \) is scaled on the boundary-layer thickness \( \delta^* \), and the latitudinal and azimuthal velocities are scaled on the equatorial speed of
the sphere surface. Using these non-dimensionalizing scales we non-dimensionalize (2.1)–(2.3) as

\[ W \frac{\partial U}{\partial \eta} + U \frac{\partial U}{\partial \theta} - V^2 \cot \theta = \frac{\partial^2 U}{\partial \eta^2}, \tag{2.6} \]

\[ W \frac{\partial V}{\partial \eta} + U \frac{\partial V}{\partial \theta} + UV \cot \theta = \frac{\partial^2 V}{\partial \eta^2}, \tag{2.7} \]

\[ \frac{\partial W}{\partial \eta} + \frac{\partial U}{\partial \theta} + U \cot \theta = 0. \tag{2.8} \]

The boundary conditions (2.4) are non-dimensionalized to produce

\[ U = W = V = \sin \theta = 0 \quad \text{on } \eta = 0, \]

\[ U = V = 0 \quad \text{as } \eta \to \infty. \tag{2.9} \]

Manohar (1967) and Banks (1976) solve (2.6)–(2.9) using finite difference techniques to produce accurate basic flow profiles at each latitude. Both of these calculations require finite difference codes written especially for the problem, but modern commercial routines are now available that are efficient and easy to use. Here the system of equations (2.6)–(2.9) is solved using the NAG routine D03PEF to find the mean flow at each latitude. This NAG routine is a general PDE solver that reduces the system of PDEs to a system of ODEs in \( \eta \). The resulting system of ODEs is solved at each latitude by marching from a given complete solution at \( \theta = 5^\circ \) towards the equator \( \theta = 90^\circ \) in one degree increments. At each latitude a backward differentiation formula method is used over a regularly spaced grid of 2000 data points between \( \eta = 0 \) and \( \eta = 20 \). The initial solution at \( \theta = 5^\circ \) is found using the series solution method of Banks (1965) and is described in appendix B. The resulting profiles have been compared to the finite difference results of Banks (1976) and complete agreement is found up to the equator. Figure 2.1 shows the three velocity components at latitudes of \( \theta = 10^\circ \text{–} 80^\circ \) in ten degree increments. Note that the latitudinal velocity, \( U \), is inflectional at all latitudes, and also that fluid is entrained.
Figure 2.1: Mean velocity profiles $U(\eta), V(\eta), W(\eta)$ at latitudes of $\theta = 10^\circ$–$80^\circ$ (left to right) in ten degree increments.

radially into the boundary layer at all the latitudes shown.

2.2 Derivation of the perturbation equations

In this section we formulate the stability problem. The perturbation equations are derived and the Reynolds numbers that will be used in this investigation are discussed.

The stability analysis conducted at a particular latitude on the rotating-sphere boundary layer involves imposing infinitesimally small perturbations on the steady basic flow at that latitude. We denote the dimensional velocity and pressure of the perturbed flow by bared upper-case quantities. These quantities are formed from a dimensional basic flow component (denoted by an upper-case quantity) and a
perturbing quantity (denoted by a lower-case hatted quantity).

\[
(U^*, V^*, W^*, P^*) = (U^* + \hat{u}^*, V^* + \hat{v}^*, W^* + \hat{w}^*, P^* + \hat{p}^*).
\] (2.10)

Physically, secondary instabilities can occur if the perturbations are large enough to significantly distort the mean velocity profiles. Throughout this thesis we assume that the perturbing quantities are small enough for the transition process to be controlled by the stability of the mean flow rather than any secondary instabilities. The perturbations are also considered to be sufficiently small so that products can be ignored and a linear analysis conducted. Bypass transitions, where the initial perturbations are sufficiently large that non-linear effects dominate from the start, are not considered.

Figure 2.2 shows the coordinate system used in the analysis. The distance measured over the surface of the sphere from the pole to the latitude under consideration is \(a^*\theta\), and the dimensional wavenumber of disturbances in this direction is \(\alpha^*\). The distance measured along a circular cross section of the sphere by a plane perpen-
Chapter 2: Rotating sphere in still fluid

dilute to the axis of rotation is \( a^* \phi \sin \theta \), and the dimensional wavenumber in this
direction is \( \beta^* \).

The dimensional perturbation variables (denoted by lower case hatted quantities)
are assumed to have the normal-mode form

\[
(\hat{u}^*, \hat{v}^*, \hat{w}^*, \hat{p}^*) = \left( u^*(r^*), v^*(r^*), w^*(r^*), p^*(r^*) \right) e^{i(\alpha'^* a^* \theta + \beta'^* a^* \phi \sin \theta - \gamma'^* t^*)}. \tag{2.11}
\]

The wavenumber in the \( \theta \) direction, \( \alpha \), and frequency, \( \gamma \), are in general complex,
as required by the spatio-temporal analysis of §2.5; we write these quantities as
\( \alpha = \alpha_r + i\alpha_i \) and \( \gamma = \gamma_r + i\gamma_i \). In contrast, the azimuthal wave number, \( \beta \), is real.
The angle that the phase fronts make with a circle parallel to the equator is denoted
\( \epsilon \), and is found from

\[
\epsilon = \tan^{-1} \left( \beta / \alpha_r \right). \tag{2.12}
\]

The integer number of complete cycles of the disturbance round the azimuth is

\[
n = \beta R \sin \theta. \tag{2.13}
\]

Later in this chapter, we will identify \( \epsilon \) and \( n \) as being the angle and number of
spiral vortices on the sphere surface.

To derive the dimensional perturbation equations we substitute the dimensional
perturbed variables (2.10) into the dimensional Navier–Stokes and continuity equa-
tions written in the spherical polar coordinate system. We write the flow variables
using (2.11), and after linearization we find the first-order perturbation equations to be
\[
W^* \frac{d u^*}{dr^*} + \left[ i \left\{ \left( \alpha^* U^* + \beta^* V^* \right) \frac{a^*}{r^*} - \gamma^* \right\} + \frac{1}{r^*} \left( \frac{\partial U^*}{\partial \theta^*} + W^* \right) \right] u^*
- \frac{2 V^* \cot \theta}{r^*} v^* + \left( \frac{\partial U^*}{\partial r^*} + \frac{U^*}{r^*} \right) w^* = - \frac{i \alpha^* a^* p^*}{r^* p^*}
+ \nu^* \left[ \frac{d^2 u^*}{dr^*} + \frac{2}{r^*} \frac{du^*}{dr^*} + \left\{ - (\alpha^* \beta^2) \frac{a^*}{r^*} + \frac{2 i \beta^2 a^*}{r^*} \right\} u^* - \frac{2 i \beta^* a^*}{r^*} w^* \right],
\]

\[
W^* \frac{d v^*}{dr^*} + \left[ i \left\{ \left( \alpha^* U^* + \beta^* V^* \right) \frac{a^*}{r^*} - \gamma^* \right\} + \frac{1}{r^*} (U^* \cot \theta + W^*) \right] v^*
+ \left( \frac{\partial V^*}{\partial \theta^*} + V^* \cot \theta \right) u^* + \left( \frac{\partial V^*}{\partial r^*} + \frac{V^*}{r^*} \right) w^* = - \frac{i \beta a^* p^*}{r^* p^*}
+ \nu^* \left[ \frac{d^2 v^*}{dr^*} + \frac{2}{r^*} \frac{dv^*}{dr^*} + \left\{ - (\alpha^* \beta^2) \frac{a^*}{r^*} + \frac{i \alpha^* a^* \cot \theta}{r^*} \right\} v^* + \frac{2 i \beta^* a^*}{r^*} u^* \right],
\]

\[
W^* \frac{d w^*}{dr^*} + \left[ i \left\{ \left( \alpha^* U^* + \beta^* V^* \right) \frac{a^*}{r^*} - \gamma^* \right\} + \frac{\partial W^*}{\partial r^*} \right] w^* - \frac{2 V^*}{r^*} v^*
+ \left( \frac{\partial W^*}{\partial \theta^*} - 2 U^* \right) u^* = - \frac{1}{r^*} \frac{\partial p^*}{\partial r^*} + \nu^* \left[ \frac{d^2 w^*}{dr^*} + \frac{2}{r^*} \frac{dw^*}{dr^*} \right]
+ \left\{ - (\alpha^* \beta^2) \frac{a^*}{r^*} + \frac{i \alpha^* a^* \cot \theta}{r^*} - \frac{2}{r^*} \right\} w^* - \frac{2 (i \alpha^* a^* + \cot \theta)}{r^*} u^* - \frac{2 i \beta^* a^*}{r^*} v^*,
\]

\[
\frac{i \alpha^* a^* + \cot \theta}{r^*} u^* + \frac{i \beta^* a^*}{r^*} v^* + \frac{2}{r^*} w^* + \frac{dw^*}{dr^*} = 0.
\]

We non-dimensionalize (2.14)–(2.17) using the boundary-layer thickness \( \delta^* \) as the length scale and the maximum rotation speed of the sphere surface \( U^*_m = a^* \Omega^* \) as the velocity scale. This is consistent with the non-dimensionalization of the mean-flow variables in §2.1, and means that the non-dimensional perturbing quantities are
written as
\[ u = u^*/U^*_m, \quad v = v^*/U^*_m, \quad w = w^*/U^*_m, \]
\[ \alpha = \alpha^* \delta^*, \quad \beta = \beta^* \delta^*, \quad \gamma = \gamma^* \delta^*/U^*_m, \]
\[ R = U^*_m \delta^*/\nu^*, \quad \delta_1 = \delta^*/\alpha^* = 1/R, \quad l = 1/(1 + \delta_1 \eta). \]

After neglecting \( O(R^{-2}) \) terms the non-dimensional perturbation equations are found to be
\[
\delta_1 W \frac{du}{d\eta} + \left[ i \left\{ (\alpha U + \beta V) l - \gamma \right\} + \delta_1 l \frac{U}{\partial \theta} \right] u - 2V \delta_1 l \cot \theta v \\
+ \left( \frac{\partial U}{\partial \eta} + \delta_1 l U \right) w = -i \alpha lp + \frac{1}{R} \left[ \frac{d^2 u}{d\eta^2} - \ell^2 (\alpha^2 + \beta^2) u \right], \tag{2.19}
\]
\[
\delta_1 W \frac{dv}{d\eta} + \left[ i \left\{ (\alpha U + \beta V) l - \gamma \right\} + \delta_1 l U \cot \theta \right] v \\
+ \left( \frac{\partial V}{\partial \theta} + V \cot \theta \right) \delta_1 l u + \left( \frac{\partial V}{\partial \eta} + \delta_1 l V \right) w = -i \beta lp + \frac{1}{R} \left[ \frac{d^2 v}{d\eta^2} - \ell^2 (\alpha^2 + \beta^2) v \right], \tag{2.20}
\]
\[
\delta_1 W \frac{dw}{d\eta} + \left[ i \left\{ (\alpha U + \beta V) l - \gamma \right\} + \delta_1 W \frac{\partial W}{\partial \eta} \right] w - 2U \delta_1 lu \\
- 2V \delta_1 lv = - \frac{\partial p}{\partial \eta} + \frac{1}{R} \left[ \frac{d^2 w}{d\eta^2} - \ell^2 (\alpha^2 + \beta^2) w \right], \tag{2.21}
\]
\[
\frac{dw}{d\eta} + 2 \delta_1 lw = - \left\{ (i \alpha + \delta_1 \cot \theta) u + i \beta v \right\}. \tag{2.22}
\]

Factors \( l = 1/(1 + \delta_1 \eta) \) appear multiplying terms in the perturbation equations, and in the analysis these factors are set to unity in an approximation that is similar to the parallel-flow approximations made in many other boundary-layer investigations. The parallel-flow approximation limits the analysis to a local analysis at each value of \( \theta \), and its validity at low and high latitudes is discussed in §2.6.1. Note also that the equations have terms multiplied by \( 1/R \) and \( \delta_1 \), these factors indicate terms arising from viscosity and streamline curvature respectively.

The perturbation equations (2.19)–(2.22) can be written as a set of six first-order
ordinary differential equations using the transformed variables

\[
\phi_1(\eta; \alpha, \beta, \gamma; R, \theta) = (\alpha - i \cot \theta / R)u + \beta v, \tag{2.23}
\]

\[
\phi_2(\eta; \alpha, \beta, \gamma; R, \theta) = (\alpha - i \cot \theta / R)u' + \beta v', \tag{2.24}
\]

\[
\phi_3(\eta; \alpha, \beta, \gamma; R, \theta) = w, \tag{2.25}
\]

\[
\phi_4(\eta; \alpha, \beta, \gamma; R, \theta) = p, \tag{2.26}
\]

\[
\phi_5(\eta; \alpha, \beta, \gamma; R, \theta) = (\alpha - i \cot \theta / R)v - \beta u, \tag{2.27}
\]

\[
\phi_6(\eta; \alpha, \beta, \gamma; R, \theta) = (\alpha - i \cot \theta / R)v' - \beta u', \tag{2.28}
\]

where a prime denotes differentiation with respect to \(\eta\). Writing \(\alpha_1 = \alpha - [i \cot \theta / R]_s\), these equations are

\[
\phi'_1 = \phi_2, \tag{2.29}
\]

\[
\begin{bmatrix} \phi'_2 \\ \frac{\phi'_2}{R} \end{bmatrix} = \frac{1}{R} \left( [\alpha^2 + \beta^2]_v + iR (\alpha U + \beta V - \gamma) \right) \phi_1 + \left[ \frac{W \phi_2}{R} \right]_s

+ \left( \alpha_1 U' + \beta V' + \left[ \frac{1}{R} (\alpha_1 U + \beta V) \right]_s \right) \phi_3

+ i \left( \alpha^2 + \beta^2 - \left[ \frac{i \alpha \cot \theta}{R} \right]_s \right) \phi_4 - \left[ \frac{V \cot \theta \phi_5}{R} \right]_s

+ \left[ \frac{1}{R} \left( \left( \alpha_1 \frac{\partial U}{\partial \theta} + \beta \frac{\partial V}{\partial \theta} \right) u - (\alpha_1 V - \beta U) \cot \theta \right) \right]_s,
\]

\[
\phi'_3 = -i \phi_1 - \left[ \frac{2 \phi_3}{R} \right]_s, \tag{2.31}
\]

\[
\phi'_4 = \left[ \frac{i W \phi_1}{R} \right]_s - \left[ \frac{i \phi_2}{R} \right]_v + \left[ \frac{2}{R} (U u + V v) \right]_s

- \frac{1}{R} \left( [\alpha^2 + \beta^2]_v + iR (\alpha U + \beta V - \gamma) + DW_s \right) \phi_3, \tag{2.32}
\]

\[
\phi'_5 = \phi_6, \tag{2.33}
\]
\[
\begin{align*}
\left[ \frac{\phi_6'}{R} \right]_v &= \left[ \frac{V \cot \theta \phi_1}{R} \right]_s + \left[ \frac{W \phi_6}{R} \right]_s + \left[ \frac{\beta \cot \theta \phi_4}{R} \right]_s \\
&\quad + \left( \alpha_1 V' - \beta U' + \left[ \frac{1}{R} (\alpha_1 V - \beta U) \right]_s \right) \phi_3 \\
&\quad + \frac{1}{R} \left( \left( \alpha_1 \frac{\partial V}{\partial \theta} - \beta \frac{\partial U}{\partial \theta} \right) u + (\alpha_1 U + \beta V) v \cot \theta \right) ]_s \\
&\quad + \frac{1}{R} \left( [a^2 + \beta^2]_v + i R (a U + \beta V - \gamma) \right) \phi_5.
\end{align*}
\] (2.34)

Equations (2.29)–(2.34) are the perturbation equations upon which the stability analyses of §§2.4 & 2.5 are performed. The subscripts \( v \) and \( s \) indicate which of the \( O(R^{-1}) \) terms arise from the viscous and streamline-curvature effects respectively. Since a stationary frame of reference is used Coriolis terms do not appear in the equations. Note also that the perturbation velocities \( u \) and \( v \) still appear explicitly in (2.29)–(2.34), but can be expressed in terms of \( \phi_1 \) and \( \phi_2 \) via

\[
\begin{align*}
u &= \frac{1}{\alpha_1 + \beta^2} (\alpha_1 \phi_1 - \beta \phi_5), \\
v &= \frac{1}{\alpha_1 + \beta^2} (\alpha_1 \phi_5 + \beta \phi_1).
\end{align*}
\]

By neglecting those terms in (2.29)–(2.34) that arise from streamline curvature, we find the Orr–Sommerfeld equation for the rotating sphere in the form

\[
(i/R)(\phi_3''' - 2(\alpha^2 + \beta^2)\phi_3'' + (\alpha^2 + \beta^2)^2 \phi_3) + \\
(\alpha U + \beta V - \gamma) (\phi_3'' - (\alpha^2 + \beta^2) \phi_3) - (\alpha U'' + \beta V'') \phi_3 = 0. \tag{2.35}
\]

Ignoring both the streamline-curvature and viscous terms in the perturbation equations leads to Rayleigh’s equation (2.36). By doing this we assume that viscosity acts in establishing the steady basic flow but has a negligible effect on the instability waves;

\[
(\alpha U + \beta V - \gamma) (\phi_3'' - (\alpha^2 + \beta^2) \phi_3) - (\alpha U'' + \beta V'') \phi_3 = 0. \tag{2.36}
\]

The Reynolds numbers that characterise the boundary-layer flow over the rotating sphere can be formed from a number of different length and velocity scales.
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The formulation of this problem described above has been such that the Reynolds number is defined as \( R = \frac{U_m^* \delta^*}{\nu^*} \). However, the experimental results of Kohama & Kobayashi (1983) are presented in terms of the spin Reynolds number. The spin Reynolds number is based on the sphere radius, \( a^* \), and the maximum surface rotation speed, \( U_m^* = a^* \Omega^* \). It is therefore defined as \( R_S = \frac{a^* \Omega^*}{\nu^*} \). By using the definition of the boundary-layer thickness we see that \( R_S = R^2 \), and this relationship will enable a comparison between our predictions and the experimental results of Kohama & Kobayashi.

A Reynolds number can also be formed that is based on the local velocity at a point on the sphere surface, \( a^* \Omega^* \sin \theta \), and the distance from the pole over the sphere surface, \( a^* \theta \). This local Reynolds number is defined as \( R_X = \frac{(a^* \Omega^*/\nu^*) \theta \sin \theta}{\nu^*} \), and will be important in our discussion of the absolute instability of the rotating-sphere boundary layer in §2.5.

Lingwood (1995a) and Malik (1985) use a Reynolds number based on the local velocity at a point on the rotating disk and the boundary-layer thickness there. For the rotating sphere we can write the equivalent Reynolds number as \( R_L = R \sin \theta \). This relationship will enable a comparison between Lingwood’s results and our own later in this chapter.

\[ \begin{align*}
\phi_t &= 0 & \text{on} & \quad \eta = 0, \\
\phi_t &\to 0 & \text{as} & \quad \eta \to \infty, 
\end{align*} \]
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where \( i = 1, 2 \ldots 6 \). Although (2.29)–(2.34) are specific to the boundary layer on the surface of a rotating sphere, the methods discussed here are common to all the stability analyses conducted in this thesis. Only slight modifications are needed when dealing with other boundary layers, and these will be discussed in the relevant chapters of this thesis. The techniques used are similar to those used by Lingwood (1995b) in her analysis of the rotating-disk stability problem.

The system (2.29)–(2.34) represents a sixth-order differential equation system at each latitude and permits six independent solutions for each transformed variable solution. These are denoted by \( \phi^i_j(\eta; \alpha, \beta, \gamma; R, \theta) \) with the superscript \( j \) indicating one of the six independent solutions of the transformed variable solutions denoted by \( i \). As we shall see, three of the independent solutions for each \( i \) exponentially decay as \( \eta \to \infty \) and three exponentially grow. The second boundary condition of (2.37) shows that only the decaying solutions are relevant, and each transformed variable solution is formed from a linear combination of these with coefficients \( C_{1,3,5} \). These coefficients can be calculated from the boundary conditions of the transformed variables at \( \eta = 0 \). In this unforced problem the boundary values of the perturbing quantities are all zero and so we can write

\[
\begin{pmatrix}
\phi_1(0) \\
\phi_3(0) \\
\phi_5(0)
\end{pmatrix} =
\begin{pmatrix}
\phi_1^1(0) & \phi_3^1(0) & \phi_5^1(0) \\
\phi_1^3(0) & \phi_3^3(0) & \phi_5^3(0) \\
\phi_1^5(0) & \phi_3^5(0) & \phi_5^5(0)
\end{pmatrix}
\begin{pmatrix}
C_1 \\
C_3 \\
C_5
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

(2.38)

This matrix equation has a non-trivial solution only when the determinant of the coefficient matrix is zero. We call the determinant \( D_S \) and require

\[
D_S(\alpha, \beta, \gamma; R, \theta) = 0.
\]

(2.39)

This is the dispersion relation for the rotating sphere at a latitude \( \theta \) and allows an unknown parameter (\( \alpha, \beta \) or \( \gamma \)) to be calculated given the others at each \( R \). The
dispersion relation is satisfied when the coefficient matrix is singular, and allows $C_{1,3,5}$ to be determined from singular value decomposition.

At the outer edge of the boundary layer the basic flow quantities are such that $U \to 0$, $V \to 0$ and $W \to W_\infty$ at each latitude. As $\eta \to \infty$ the perturbation equations therefore take the form

\begin{align}
\phi'_1 &= \phi_2, \\
\frac{\phi'_2}{R} &= \frac{1}{R} \left( \alpha^2 + \beta^2 - i R \gamma \right) \phi_1 + \frac{W \phi_2}{R} + i \left( \alpha^2 + \beta^2 - \frac{i \alpha \cot \theta}{R} \right) \phi_4, \\
\phi'_3 &= - i \phi_1 - \frac{2 \phi_3}{R}, \\
\phi'_4 &= \frac{i W_\infty}{R} - \frac{i}{R} \phi_2 - \frac{1}{R} \left( \alpha^2 + \beta^2 - i R \gamma \right) \phi_3, \\
\phi'_5 &= \phi_6, \\
\frac{\phi'_6}{R} &= \frac{V \cot \theta \phi_1}{R} + \frac{\beta \cot \theta \phi_4}{R} + \frac{1}{R} \left( \alpha^2 + \beta^2 - i R \gamma \right) \phi_5 + \frac{W_\infty \phi_6}{R}.
\end{align}

Equations (2.40)-(2.45) permit solutions in the form

$$\phi^j_i(\eta \to \infty; \alpha, \beta, \gamma; R, \theta) = c_i^j e^{\kappa_j \eta},$$

where the coefficient $c_i^j$ and exponent $\kappa_j$ are constant with respect to $\eta$, and can be found by substituting (2.46) into (2.40)-(2.45). After ignoring $O(R^{-2})$ terms we find the exponents to be

\begin{align}
\kappa_{1,2} &= \frac{W_\infty}{2} \pm \left[ \left( \frac{W_\infty}{2} \right)^2 + \alpha^2 + \beta^2 - i R \gamma \right]^{1/2}, \\
\kappa_{3,4} &= \frac{W_\infty}{2} \pm \left[ \left( \frac{W_\infty}{2} \right)^2 + \alpha^2 + \beta^2 - i R \gamma \right]^{1/2}, \\
\kappa_{5,6} &= \mp \left[ \alpha^2 + \beta^2 - \frac{i \alpha \cot \theta}{R} \right]^{1/2},
\end{align}

where the real parts of the complex square roots are taken to be positive.

As $\eta \to \infty$ all perturbations must decay, therefore only the $j = 1, 3$ and 5 solutions are relevant. The required co-efficients in (2.46) for the transformed variables
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with $i = 1, 3$ and 5 are found to be

\[
\begin{align*}
    c_1^1 &= 1 & c_3^3 &= 0 & c_5^5 &= i\kappa_5, \\
    c_2^1 &= \kappa_1 & c_3^3 &= 0 & c_5^5 &= i\kappa_5^2, \\
    c_3^3 &= -i/\kappa_1 & c_3^3 &= 0 & c_5^5 &= 1, \\
    c_4^1 &= 0 & c_4^3 &= 0 & c_4^5 &= (1/R\kappa_5) \left[ iR\gamma - \frac{i\alpha\cot \theta}{R} - W_\infty \kappa_5 \right], \\
    c_5^1 &= 0 & c_5^3 &= 1 & c_5^5 &= 0, \\
    c_6^1 &= 0 & c_6^3 &= \kappa_3 & c_6^5 &= 0.
\end{align*}
\]

Given values of $\alpha, \beta, \gamma, R$ and $\theta$, the perturbation equations (2.29)-(2.34) are numerically integrated down towards the sphere surface from each of the initial solutions defined by (2.46)-(2.50) at the edge of the boundary layer. As with the basic flow, the outer edge of the boundary layer is approximated by $\eta = 20$ and a double-precision fourth-order Runge-Kutta integrator is used. The integrations result in nine independent solutions over the domain of integration from which the coefficient matrix of equation (2.38) can be found. Difficulties arise in this process because the perturbation equations form a stiff set with two different length scales over which the solutions develop; the independent solutions denoted by $j = 1$ and 3 grow much more rapidly than the solution denoted by $j = 5$ as the integration proceeds towards the sphere surface. The round-off error of the numerical integration follows the most rapidly growing solution and so it becomes impossible to preserve the linear independence of the independent solutions when simply integrating the equations. To maintain linear independence Gram-Schmidt orthogonalization is applied each time the solutions loose their independence, and details of this technique can be found in Nicholson (1995).

To calculate the spatial branches at each $\theta$ and $R$ for a given frequency $\gamma$, the
wavenumber $\beta$ is varied and a second-order Newton–Raphson search method is used find the complex value of $\alpha$ that produces a singular coefficient matrix, i.e. satisfies the dispersion relation. To calculate the temporal branches at each $\theta$ and $R$ for a given wavenumber $\beta$ and imaginary part of the frequency $\gamma_i$, the real part of the frequency is varied and $\alpha$ is calculated in the same way. The numerical code used was provided by R. J. Lingwood (personal communication, 1999), and the subroutines of the code originate from Press, Teukolshy, Vetterling & Flannery (1992).

2.4 The convective instability analysis

In this section we solve the perturbation equations (2.29)–(2.34) with the aim of studying the occurrence of convective instabilities. Since we are supposing in the first instance that the flow is not absolutely unstable, it follows that in the Briggs–Bers procedure we can reduce the imaginary part of the frequency down to zero, so that $\gamma_i = 0$. To produce the neutral curves for convective instability a number of approaches can be taken. One approach involves finding the region of convective instability at each latitude for different fixed vortex angles, so that $\beta$ is known in terms of $\alpha$ from (2.12). For a particular $\theta$ the real part of the complex frequency, $\gamma_r$, and $\alpha$ are calculated with the perturbation equations solved for neutral stability, $\gamma_i = 0$, at each Reynolds number. This was the approach taken by Taniguchi et al. (1998) in their temporal analysis. Another approach is to insist that the vortices rotate at some fixed multiple of the sphere surface velocity, thereby fixing the ratio $\gamma_r/\beta$, and then $\alpha$ and $\beta$ are calculated using a spatial analysis. This is the approach taken in §§2.4.1 & 2.4.2. The non-dimensional speed of the surface of the sphere is $\sin \theta$, and equating the relevant multiple of this with the disturbance phase velocity
in the same direction, \( \gamma_r / \beta \), leads to \( \gamma_r = c \beta \sin \theta \). This relationship must be satisfied with \( c = 1.0 \) if the vortices are to rotate with the sphere and \( c = 0.76 \) if the vortices are those reported by Kobayashi & Arai (1990) at high latitudes.

In §2.4.3 a further approach is taken in which the neutral curves are plotted at each latitude for various integer \( n \). The ‘global’ neutral curve at a particular latitude is then the envelope of the neutral curves pertaining to each single \( n \). This method enables a prediction of the speed of the spiral vortices with respect to the sphere surface rather than making a-priori assumptions about the longitudinal wave speed.

### 2.4.1 Stationary vortices

We begin by considering vortices that rotate with the surface of the sphere, i.e. we fix \( c = 1.0 \). At each latitude we have found that two spatial branches determine the convective instability characteristics of the system. Figure 2.3 shows these spatial branches in the complex \( \alpha \)-plane at \( \theta = 10^\circ \) and \( R = 2400 \). Branch 1 is due to the crossflow-instability mode and branch 2 arises here from the effects of streamline curvature; this was shown by the analysis of the Orr–Sommerfeld equation (2.35) where branch 2 is not found, which is consistent with existing work on the rotating disk, see Lingwood (1995a). A branch with a region lying below the \( \alpha_r \)-axis indicates convective instability. Figure 2.4 shows the two branches at \( R = 2500 \), where we see that an exchange of modes has occurred between them. The modified branch 1 now determines the regions of convective instability. Increasing the value of \( R \) causes the peak inbetween the two minima on branch 1 to move downwards and the points where the branch crosses the \( \alpha_r \)-axis move apart, thereby widening the regions of instability and mapping out two lobes on the neutral curve defined by \( \alpha_i = 0 \). Above a certain value of \( R \) the peak moves below the real \( \alpha \)-axis and further increases in
Figure 2.3: The two spatial branches at $\theta = 10^\circ$ and $R = 2400$ showing convective instability from branch 1 only.

$R$ change the region of instability, producing the upper and lower branches of the neutral curve.

This branch behaviour is typical for all latitudes below $\theta = 66^\circ$. Above $\theta = 66^\circ$ the two branches only ever appear like those described when the peak in the modified branch 1 has moved below the $\alpha_r$-axis after the branch exchange. The neutral curves for latitudes above $\theta = 66^\circ$ therefore do not have the two lobed structure. The branches at $\theta = 70^\circ$ and $R = 100$ are shown in figure 2.5.

Figure 2.7 shows the neutral curves for convective instability in the $(R, \alpha_r)$- and $(R, \beta)$-planes at latitudes of $\theta = 10^\circ-70^\circ$. Each curve encloses a region that is convectively unstable. The neutral curves shown in figure 2.7 were calculated using the full perturbation equations, and a comparison with the neutral curves calculated using the Orr–Sommerfeld equation (2.35) at $\theta = 20^\circ$ is shown in figure 2.6. At each latitude the Orr–Sommerfeld neutral curves were found to be single-lobed, with
Figure 2.4: The two spatial branches after the exchange showing regions of both streamline-curvature and crossflow instability at $\theta = 10^\circ$ and $R = 2500$.

Figure 2.5: The two spatial branches at $\theta = 70^\circ$ and $R = 100$ showing a kink in branch 2 and a region of instability caused by branch 1.
Figure 2.6: A comparison between both neutral curves calculated from the full perturbation equations and Orr–Sommerfeld equations (dotted line) at $\theta = 20^\circ$.  

critical Reynolds numbers lower than the most dangerous modes in figure 2.7. The neutral curves calculated from the full system and the Orr–Sommerfeld equation were found to match for large $R$, with the discrepancy at low $R$ being reduced with increased latitude.

The neutral curves in figure 2.7 show that the rotating-sphere boundary layer is increasingly stable as we decrease the latitude from the equator towards the pole, which is consistent with the experimental results of Sawatzki (1970) and Kohama & Kobayashi (1983). At all latitudes in the figure except $\theta = 70^\circ$, a two-lobed structure is seen. The larger lobe, characterised by higher wavenumbers, is due to crossflow instabilities and the smaller lobe, characterised by smaller wave numbers, is due to streamline-curvature instabilities. The crossflow mode possesses the lowest critical Reynolds number at moderate $\theta$, but the streamline-curvature mode becomes increasingly important as we approach $\theta = 66^\circ$ from below. At $\theta = 66^\circ$
Figure 2.7: The neutral curves of convective instability for stationary vortices at latitudes of \( \theta = 10^\circ - 70^\circ \) (right to left).

the streamline-curvature mode becomes dominant, and so in the \( \theta = 70^\circ \) curve we see a single lobed neutral curve formed from the streamline-curvature mode; the crossflow mode is seen to distort the shape of the lobe but does not form a lobe itself. It is interesting to note that the dominance of the streamline-curvature mode coincides with the appearance of a region of reverse flow in the radial component of the steady-laminar flow close to the sphere surface.

As described in §2.2, Malik (1986) and Lingwood (1995a), in their investigations on the rotating disk, use a Reynolds number based on the local disk velocity at the radius under investigation and the local boundary-layer thickness. The equivalent Reynolds number in our investigation is written as \( R_L = R \sin \theta \). Using this Reynolds number a comparison between our results and those of Malik (1986) for the rotating disk is made. Although not shown here, plots of the neutral curves at \( \theta = 10^\circ \) in the \((R_L, \alpha_r)\)- and \((R_L, \beta)\)-planes are very similar to Malik’s neutral curves. Figure
Figure 2.8: A comparison of the critical $R_L$ values for convective instability at each latitude with those of the Malik (1986) for the rotating disk (horizontal lines).

2.8 shows that the convective instability critical Reynolds numbers of the rotating-sphere boundary layer approach those of the rotating disk as we approach the pole, i.e. as $\theta \to 0$.

For a comparison with the theoretical neutral curves of Taniguchi et al. (1998) we consider our results in the $(R_S, n)$-plane. Here $R_S = \Omega^2 a^2 / \nu^* = R^2$ is the spin Reynolds number, as described in §2.2, and $n$ is the number of vortices from (2.13). Figure 2.9 shows the neutral curves in this plane, where we see that the number of spiral vortices at the onset of instability decreases with increased latitude. This property was also found by Taniguchi et al, with the values of $n$ and the critical Reynolds numbers similar to ours. Kohama & Kobayashi (1983) observe the number of spiral vortices at the onset of instability increasing with increased rotation rate, i.e. increased $R_S$, and tending to the value observed on the rotating disk ($n \approx 32$). Figure 2.9 is consistent with this behaviour although we find $n$
Figure 2.9: The neutral curves of convective instability in the \((R_S, n)\)-plane for stationary vortices at \(\theta = 10^\circ - 70^\circ\) (right to left).

tending to approximately 22 at the onset of the crossflow mode as we move towards the pole, i.e. the value of \(n\) associated with the smallest value of \(R_S\) on the upper right-most neutral curve in figure 2.9. This discrepancy occurs because the spiral vortices are not neutral, but instead grow spatially in the latitudinal direction. The calculated value of \(n\) is equal to that calculated by Malik (1986) for the rotating disk. Using (2.12), the vortex angles show a roughly constant value of \(\epsilon \approx 11.4^\circ\) and \(19.4^\circ\) at the onset of the crossflow and streamline-curvature instability modes respectively at all latitudes. This is consistent with the results of Malik (1986) on the rotating disk, where the vortex angles are calculated to be \(\epsilon = 11.4^\circ\) and \(19.5^\circ\) respectively, and are reasonably close to the experimental observation of \(\epsilon \approx 14^\circ\) reported by Kohama & Kobayashi (1983) on the rotating sphere. Taniguchi et al. fail to observe the streamline-curvature instability lobe in their neutral curves for latitudes lower than \(\theta = 40^\circ\). The explanation for this must lie in the approach
Figure 2.10: The critical Reynolds numbers predicted for stationary vortices, vortices rotating at 0.76 times the sphere surface speed and those experimentally measured by Kohama & Kobayashi (1983). The vertical dashed line indicates the observed onset of vortices rotating at 0.76 times the sphere surface by Kobayashi & Arai (1990).

taken in their calculations. Recall that they fix the vortex angle at 14°, which is in contrast to the approach taken in this paper where the speed of the spiral vortices is fixed with respect to the surface rotation speed of the sphere.

In figure 2.10, a comparison is made between the predicted critical Reynolds numbers calculated using the stationary-vortex assumption and the experimental results of Kohama & Kobayashi (1983). For latitudes lower than $\theta = 60^\circ$ we see very good agreement between the experimental critical Reynolds numbers of Kohama & Kobayashi’s larger sphere and the predicted onset of streamline-curvature instabilities. This is unexpected as the experiments should measure the lowest critical Reynolds numbers and so pick up the onset of crossflow instabilities at latitudes below $\theta = 66^\circ$. However, note that Kohama & Kobayashi’s critical Reynolds num-
bers for their smaller sphere do match onto the predicted onset of crossflow instabilities, although unfortunately the critical Reynolds numbers for $\theta < 60^\circ$ appear not to have been measured on the smaller sphere. These findings may suggest that different instability modes were being measured on the different sized spheres. Further experimental investigations are required to clarify this. At a latitude of $\theta = 60^\circ$ we see a discrepancy between the experimental critical values and those predicted for stationary vortices, and this discrepancy quickly increases beyond this latitude. Note that this behaviour is similar to that found by Taniguchi et al., and starts at the Reynolds number measured by Kobayashi & Arai for the onset of the slow vortices. This strongly suggests that the discrepancy is due to the stationary vortex assumption being invalid at higher latitudes, and this point will be addressed now.

2.4.2 Slow vortices

The convective instability of the boundary layer has been studied for a variety of vortex speeds at a number of latitudes. The neutral curves at each of these latitudes show a similar behaviour as the vortex speed is reduced, i.e. the streamline-curvature lobe is smoothed out as the speed is decreased, producing a single lobed neutral curve. The critical Reynolds numbers are found to increase with reductions in the vortex speed, with the neutral curve eventually closing up and disappearing below a critical value. This means that the boundary layer is increasingly stable to slowly rotating vortices and cannot support them below a certain value of vortex speed. The critical vortex speed has been found to decrease with latitude. Figure 2.11 shows the set of neutral curves at a latitude of $\theta = 20^\circ$, but the behaviour is typical for all latitudes. Figure 2.10 shows the results of the analysis with the slow vortex assumption (i.e. speed 0.76 times the sphere surface speed). By considering slowly
Figure 2.11: Neutral curves for convective instability in the \((R, \alpha_r)\)-plane at \(\theta = 20^\circ\) calculated for vortices rotating at speeds of 1.0, 0.76, 0.69 & 0.674 times the surface speed of the sphere.

rotating vortices the discrepancy between the experimental and predicted results is removed at \(\theta = 70^\circ\), and greatly reduced at \(\theta = 80^\circ\).

2.4.3 A prediction of the vortex speed

To investigate the vortex speed another approach can be taken, in which we plot the neutral curves defined by \(\gamma_i = \alpha_i = 0\) that correspond to each integer \(n\) defined by (2.13). The ‘global’ neutral curve at a particular latitude would then be the envelope of the neutral curves pertaining to each single \(n\). This approach does not require us to fix the longitudinal wave speed \(c\), but allows its prediction from the critical values of \(\gamma_r\) and \(R\) using \(c = \gamma_r R/n\). This approach has been taken at latitudes between \(\theta = 20^\circ-70^\circ\) in 10\(^\circ\) increments and the critical values of \(R\) for various integer \(n\) are shown in figure 2.12.
Figure 2.12: Critical Reynolds numbers at latitudes of $\theta = 20^\circ - 70^\circ$ (top to bottom) for neutral curves defined by fixing $n$ at various integer values. The circles indicate the results of §2.4.1.

For $\theta \leq 60^\circ$ we see that the curves have a single minimum. This minimum is the critical Reynolds number of the crossflow lobe of the enveloping neutral curve, and is identical to that calculated in §2.4.1 when $c$ was fixed at unity. The points where $c = 1.0$ for each $\theta$ are also indicated on figure 2.12. The critical values of $\gamma_r$ and $R$ at the minimum in the curve at each latitude lead to the prediction that $c \approx 1.0$ at each latitude as can be seen in table 2.1. This agreement shows that fixing the longitudinal wave speed to $c = 1.0$ at the low latitudes, as we did earlier in this section, is correct. This is in full agreement with the experimental results of Kohama & Kobayashi (1983).

Figure 2.12 also shows that as the latitude is increased the curves flatten out, until at $\theta = 70^\circ$ an inflection point is seen rather than the minimum. Further investigation of latitudes between $\theta = 60^\circ$ and $70^\circ$ have shown that the inflection
<table>
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<td>0.0547</td>
<td>10</td>
<td>1.034</td>
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</tbody>
</table>

Table 2.1: Critical parameters at latitudes of $\theta = 20^\circ$–$60^\circ$ for neutral curves defined by fixing $n$ at various integer values.

point first appears at $\theta = 66^\circ$, which coincides with the latitude at which the streamline-curvature mode becomes dominant. Hence, for $\theta \geq 66^\circ$, our approach predicts that stationary vortices no longer occur. However, since for $\theta \geq 66^\circ$ figure 2.12 does not predict a minimum critical $R$, we are not able to fix the value of the longitudinal wave speed at these high latitudes from our theory; the value $c = 0.76$, which we applied earlier in this section, has to be taken from experiment.

## 2.5 The absolute instability analysis

If the response to a transient disturbance is unbounded for large time at all points in space, the flow is absolutely unstable. In principle it is possible to explicitly determine the response of the system to an impulsive forcing by integrating the perturbation equations (2.29)–(2.34). However, the complexity of the system means that it is easier to consider the time-asymptotic solutions at each latitude, and the method due to Briggs (1964) and Bers (1975) enables us to do precisely this. Briggs
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and Bers have shown that an absolute instability can be identified by singularities corresponding to zeros of the dispersion relation. The singularities occur when waves that propagate energy in different directions coalesce, and can appear as the Reynolds number is varied. At these points the flow changes from a convectively unstable regime to an absolutely unstable regime.

Mathematically, the Briggs–Bers criterion is applied with fixed $\beta$ to distinguish between convectively and absolutely unstable time-asymptotic responses to an initial boundary-value perturbation. The perturbation is provided by an impulsive longitudinal line forcing, $\delta(\theta - \theta_s)\delta(t) e^{i m \phi}$, where $\delta(\theta - \theta_s)$ and $\delta(t)$ are the Dirac delta functions at the latitude $\theta_s$ and at time $t = 0$ respectively. The criterion for absolute instability requires branch-point singularities between at least two spatial branches of the dispersion relation. Two of these branches must lie in distinct half $\alpha$-planes when $\gamma_i$ is sufficiently large and positive. Such a singularity is known as a pinch-point. If $\gamma_i > 0$ at the pinch-point the flow is absolutely unstable, otherwise the flow is only convectively unstable or stable. The latitudinal group velocity $\partial \gamma / \partial \alpha$ is identically zero at a pinch-point. A branch-point singularity between two spatial branches that lie in the same half $\alpha$-plane for large positive values of $\gamma_i$ does not cause absolute instability. The value of $\gamma$ at a pinch-point is denoted by $\gamma^o$ and $\alpha(\gamma^o) = \alpha^o$.

By considering the Rayleigh equation, Lingwood (1995a) has shown that the absolute instability of the rotating-disk boundary layer is inviscid in origin. By considering the Orr–Sommerfeld equation with very high Reynolds number, we have found that this is also the case for the rotating sphere at each latitude. For this reason we consider only the full perturbation equations in this section, and solve (2.29)–(2.34) subject to the boundary conditions (2.37) with the aim of studying
Figure 2.13: Branches of type 1 and 2 at $\theta = 70^\circ$ when $R = 150$, $n = 9$ and $\gamma_i = a)$ $3 \times 10^{-3}$, b) $1 \times 10^{-4}$, c) $-4.146 \times 10^{-4}$ and d) $-4.5 \times 10^{-4}$. The branch point is indicated by a cross ($\times$).

the occurrence of absolute instabilities. To enable a spatio-temporal analysis to be completed, $\alpha$ and $\gamma$ are both complex quantities, while $\beta$ remains real in order to enforce periodicity round the azimuth. The term ‘spatial branch’ will be used again, but this time to refer to the solutions of the dispersion relation in the $\alpha$-plane for given complex $\gamma$.

Branches have been calculated using the methods described in §2.3, and figure 2.13 shows the progress of branches 1 and 2 with $R = 150$ and $n = 9$ ($\beta = 0.06$) at a latitude of $\theta = 70^\circ$. In §2.4 it was shown that branches of type 1 and 2 arise from crossflow and streamline-curvature effects respectively, and this branch behaviour is typical for all latitudes. The figure shows the branches when $\gamma_i = 3 \times 10^{-3}, 1 \times 10^{-4}, -4.146 \times 10^{-4}$ and $-4.5 \times 10^{-4}$. A branch point has been found when $\gamma = 4.445 \times 10^{-2} - i4.146 \times 10^{-4}$ and $\alpha = 0.140 + i3.377 \times 10^{-2}$ and is indicated
Figure 2.14: Branches of type 1 and 3 at $\theta = 70^\circ$ when $R = 300$, $n = 30$ and $\gamma_t =$ (a) $4 \times 10^{-3}$, (b) $1.157 \times 10^{-3}$ and (c) $1 \times 10^{-3}$.

in figure 2.13(c), but this is not a pinch point as branches 1 and 2 originate in the same half $\alpha$-planes for large $\gamma_t$, see figure 2.13(a). Branches of type 1 and 2 originate in the upper half $\alpha$-plane for all values of the parameters at each latitude, and so represent disturbances convecting away from the source towards the equator. To find an absolute instability we therefore need to consider a branch originating in the lower half $\alpha$-plane. Such a branch has been found that is equivalent to the branch 3 of Lingwood (1995a).

A consequence of the fact that pinch points occur only between branches that originate in distinct halves of the $\alpha$-plane is that the total number of crossings of the real axis must be odd (Kupfer, Bers & Ram, 1987). It follows that an absolutely unstable region must be surrounded by a region in which one mode is convectively unstable on one side of the source and the other mode stable in the opposite direction.
Figure 2.15: Absolute instability neutral curves at $\theta = 70^\circ$ in the $(R, \alpha_\gamma^0)$-, $(R, \alpha_i^0)$-, $(R, \gamma_\gamma^0)$- and $(R, \beta)$-planes.

Figure 2.14 shows the progress of branches 1 and 3 with $R = 300$ and $n = 30$ ($\beta = 0.1$) at $\theta = 70^\circ$. The figure shows the branches when $\gamma_i = 4 \times 10^{-3}, 1.157 \times 10^{-3}$ and $1 \times 10^{-3}$. As branches 1 and 3 originate in distinct half $\alpha$-planes there is a pinch-point at $\gamma^0 = 6.652 \times 10^{-2} + i1.157 \times 10^{-3}$ and $\alpha^0 = 0.163 - i7.628 \times 10^{-2}$ in figure 2.14(b). Note the branch exchange after the pinch point in figure 2.14(c).

Pinch-points with $\gamma_i > 0$ have been found at all latitudes, and so the boundary-layer on the rotating sphere is shown to be absolutely unstable for certain values of $R$ and $\beta$ at each latitude. Neutral curves for absolute instability can be found by plotting the points at which $\gamma_i^0 = 0$, and figure 2.15 shows the neutral stability curves for absolute instability in the $(R, \gamma_\gamma^0)$-, $(R, \alpha_\gamma^0)$-, $(R, \alpha_i^0)$- and $(R, \beta)$-planes at $\theta = 70^\circ$, the flow being absolutely unstable inside each curve. The shape of these curves is typical for all latitudes as can be seen in §A.1 where, for completeness, we present the neutral curves at each latitude. The critical Reynolds numbers for
Table 2.2: Values of the Reynolds number $R$ and local Reynolds number $R_X$ at the onset of absolute instability at each latitude $\theta$.

The onset of absolute instability at each latitude are shown in Table 2.2 expressed in terms of $R$, $R_L$ and $R_X$. When expressed in terms of $R$, the results show that the boundary layer is increasingly stable to absolute instability as we move towards the pole, i.e. for a fixed sphere radius, a faster rotation rate is required for the onset of absolute instability at lower latitudes. When expressed in terms of $R_L$, we see that the critical values for absolute instability at each latitude approach those of the rotating disk, $R_L = 510.63$, as we move towards the pole. The values of $\alpha_r^c$, $\alpha_i^c$ and $\beta$ in the neutral curve calculated at $\theta = 10^\circ$ show very good agreement with the neutral curves of figure 8 in Lingwood (1995a) for the rotating disk, although the values of $\gamma_r^c$ are different due to the differing frames of reference used.

Figure 2.16 shows a comparison between the predicted onset of absolute instability and the experimentally measured transition points from Kohama & Kobayashi (1983), using the local Reynolds number $R_X$ at each latitude. For latitudes up
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Figure 2.16: A comparison of the predicted critical $R_X$ values for absolute instability with the transitional $R_X$ values measured by Kohama & Kobayashi (1983).

To and including $\theta = 70^\circ$, the experimental results show that transition occurs at roughly the same local Reynolds number at each latitude, this is despite the instability waves first appearing across a wide parameter range with varying rotating rate and sphere radius. This observation leads us to expect that the underlying transition mechanism here is an absolute instability. At $\theta = 80^\circ$ we see the transition point at a rather lower local Reynolds number, perhaps indicating that the underlying mechanism is different at high latitudes. The predicted onset of absolute instability is seen to match onto the experimental value well at a latitude of $\theta = 30^\circ$. At this latitude the experimental value is $R_X = 2.2 \times 10^5$, but it should be noted that all experimental data used has simply been read off the graphical results of Kohama & Kobayashi (1983) and so must be considered to be approximate. Beyond this latitude the discrepancy increases, but still remains relatively small below $\theta = 70^\circ$, again perhaps indicating that the transition mechanism may not be an absolute
instability at the highest latitudes.

2.6 Conclusion

2.6.1 The validity of the parallel-flow approximation

In the derivation of the governing equations in §2.2, factors

$$1/(1 + \eta/R), \quad (2.51)$$

that multiplied terms in the perturbation equations have been replaced by unity. This approximation is similar to the parallel-flow approximation found in many other boundary-layer investigations and, after conducting the stability analyses, we are now in a position to comment on the validity of the approximation at low and high latitudes. As shown in figure 2.1, at low latitudes the boundary layer is seen to be fully developed at around $\eta = 5$ but at higher latitudes it is seen to be fully developed only above $\eta = 10$. This thickening of the boundary layer, together with the fact that the critical Reynolds numbers for each type of instability decrease with increased latitude, means that our approximation is less valid near to the equator. The inaccuracy is smallest near to the pole where, at $\theta = 10^\circ$, the onset of convective instability occurs above $R = 1500$. This means that close to the pole the factor (2.51) is approximated by unity with an inaccuracy of about 0.3%. At $\theta = 70^\circ$ the convective instability in the analysis of slow vortices has a critical Reynolds number of around $R = 150$. Here we see that the factor (2.51) is approximated by unity with an error of around 6.7%. For the absolute instability calculations at this latitude the inaccuracy is around 4%. Although the inaccuracy caused by this approximation grows substantially as the analysis moves towards the equator, it is
in the author’s opinion that it is still not sufficiently large to affect the conclusions of this investigation.

2.6.2 Summary of the results

It is known that the steady laminar flow profiles of the rotating-sphere boundary layer reduce to those of the rotating disk near to the pole, see appendix B, and the investigation conducted in this chapter shows that the neutral curves for both convective and absolute instability of the boundary layer on the rotating sphere approach those of the rotating disk as we approach the pole. The convective instability neutral curves of Taniguchi et al. (1998) do not do this and it is expected that this is due to the different approach taken in their calculations.

The convective instability analysis shows that crossflow instabilities dominate below $\theta = 66^\circ$, whilst streamline-curvature instabilities dominate above this latitude due to a region of reverse flow in the radial component of the mean flow. For latitudes lower than $\theta = 60^\circ$ we see very good agreement between the experimental critical Reynolds numbers of Kohama & Kobayashi (1983) and the predicted onset of streamline-curvature instabilities for stationary vortices. However, the critical Reynolds numbers for experiments conducted on a smaller sphere match onto the predicted onset of crossflow instabilities. This suggests that different instability modes were being measured by Kohama & Kobayashi on the different sized spheres. Above a latitude of $\theta = 60^\circ$ we see discrepancies between the experimental critical values and those predicted by assuming stationary vortices. These discrepancies start at the Reynolds number measured by Kobayashi & Arai (1990) for the onset of vortices rotating at 0.76 times the sphere surface speed, and are greatly reduced when we consider vortices rotating at this slower speed. Using a different approach
in the analysis we have shown that the occurrence of the slowly rotating vortices may be associated with the first point at which the streamline-curvature mode becomes dominant, i.e. $\theta = 66^\circ$. However, we are not able to fix the slower longitudinal wave speed at the high latitudes from this theory.

The number of spiral vortices at the onset of instability are predicted to decrease with increased latitude, which is consistent with the observations of Kohama & Kobayashi. As the analysis moves towards the pole we predict that the number of spiral vortices approaches $n \approx 22$, the theoretical prediction for the rotating disk (Malik, 1986). This value differs from the experimental observations for both the sphere and the disk of $n \approx 31$ because the spiral vortices are not neutral but growing spatially in the latitudinal direction. At the onset of instability the stationary vortices at each latitude were predicted to have roughly the same vortex angles for each mode, the values found being $\epsilon \approx 11.4^\circ$ and $19.4^\circ$ at the onset of crossflow instabilities and streamline-curvature instabilities respectively. These values agree well with those of the rotating disk, and are reasonably close to the experimentally observed value of $\epsilon \approx 14^\circ$ on the rotating sphere.

Expressing the observed transition points of Kohama & Kobayashi (1983) in terms of a local Reynolds number, we find that transition occurs at roughly the same value at all latitudes up to and including $\theta = 70^\circ$. At $\theta = 80^\circ$ the transition point is slightly lower. We have tried to associate transition with the onset of absolute instability, and the predicted onset of the latitudinal absolute instability is seen to match onto the experimental value well at a latitude of $\theta = 30^\circ$. Beyond this the discrepancy increases but still remains close when below $\theta = 70^\circ$. The theoretical results show that the transition mechanism for the boundary layer on a rotating sphere may be an absolute instability in the latitudinal direction for latitudes up to
\( \theta = 70^\circ \). Beyond this latitude the absolute instability may be suppressed and other transition mechanisms apply.

Both the convective and absolute instability results show a large discrepancy at \( \theta = 80^\circ \). The boundary layer is known to erupt at the equator causing the boundary-layer assumptions to be invalid there, so that \( \theta = 80^\circ \) may be close enough to be affected by this, giving an explanation for these discrepancies.
Chapter 3

The rotating-sphere boundary layer in uniform axial flows

This chapter is concerned with the effect of axial flow on the instability mechanisms of the boundary-layer flow over the surface of a rotating sphere. Viscous and streamline-curvature effects are included, and local stability analyses are conducted between latitudes of 10°–70° from the axis of rotation under various axial flow rates. The formulation is similar to that described in the still fluid analysis presented in chapter 2, and is such that the unsteady perturbation equations are identical to (2.29)–(2.34). This formulation means that the effect of non-zero axial flow is simply to change the basic flow profiles upon which the stability analyses are performed.

The solutions of the steady boundary-layer equations and perturbation equations are discussed in §§3.1 & 3.2 respectively. In §3.3 the perturbation equations are solved, and we study the local convective instability of the boundary layer at latitudes between \( \theta = 10°–80° \) in 10° increments for different axial flow rates. The effect of axial flow on the local absolute instability at each latitude is studied in
§3.4. In each section our theoretical results are compared to existing experimental measurements where possible.

3.1 The steady mean flow

The formulation of the problem is similar to that used in §2.1, but now the sphere rotates in a uniform axial flow with free-stream velocity $U_\infty^*$. At the edge of the boundary layer the dimensional latitudinal surface velocity distribution (i.e. the slip velocity) is $U_0^*(\theta)$ and is related to the pressure, $P^*(\theta)$, by the $\theta$-momentum equation

$$
\frac{U_0^* \partial U_0^*}{a^* \partial \theta} = \frac{1}{a^* \varrho^*} \frac{\partial P^*}{\partial \theta}.
$$

(3.1)

The form of the surface velocity distribution $U_0^*$ will be discussed later in this section.

The slip velocity $U_0^*$ is non-dimensionalized on the free-stream velocity $U_\infty^*$, while other steady velocities are non-dimensionalized using $\Omega^* a^*$. The non-dimensional mean flow variables are then defined as

$$
U(\eta, \theta) = \frac{U^*}{\Omega^* a^*}, \quad V(\eta, \theta) = \frac{V^*}{\Omega^* a^*}, \quad W(\eta, \theta) = \frac{W^*}{(\varrho^* \Omega^*)^{1/2}}, \quad U_0(\theta) = \frac{U_0^*}{U_\infty^*},
$$

(3.2)

where $U$, $V$ & $W$ are the non-dimensional velocities in the $\theta$, $\phi$ and $r^*$ directions respectively.

The equations that govern the mean flow in the boundary layer are stated by Mangler (1945) and are non-dimensionalized using (3.2) as

$$
U \frac{\partial U}{\partial \theta} + W \frac{\partial U}{\partial \eta} - V^2 \cot \theta = T^2 U_\theta U_\theta + \frac{\partial^2 U}{\partial \eta^2},
$$

(3.3)

$$
U \frac{\partial V}{\partial \theta} + W \frac{\partial V}{\partial \eta} + U V \cot \theta = \frac{\partial^2 V}{\partial \eta^2},
$$

(3.4)

$$
\frac{\partial W}{\partial \eta} + U \cot \theta + \frac{\partial U}{\partial \theta} = 0,
$$

(3.5)
where $T = U_\infty^*/a^*\Omega^*$ is the non-dimensional axial flow parameter being the ratio of the free-stream flow speed to the speed of the points on the sphere equator. These equations are based on the usual boundary-layer assumption of large Reynolds number.

The formulation of this problem is such that the axial flow control parameter $T$ appears only in the pressure term of the steady-flow equation (3.3). The pressure term is determined completely by the slip velocity, and for this reason the relevant unsteady perturbation equations are identical to (2.29)–(2.34). Details of the derivation of these perturbation equations can be found in §2.2. This formulation means that the effect of non-zero axial flow is simply to change the basic flow profiles upon which the stability analyses are performed. We can then look at the relatively small axial flow rates that are realistically attainable in experiments, i.e. $T < 0.25$ \footnote{For example, a maximum flow speed of $U_\infty^* \approx 30\text{ms}^{-1}$ incident on a sphere with radius $a^* = 0.25\text{m}$ rotating with angular velocity $\Omega^* = 500\text{s}^{-1}$, is equivalent to $T_{\text{max}} \sim 0.24$.}, which approach the limit of zero axial flow studied in chapter 2. The alternative choice of scaling the variables on $U_\infty^*$ would lead to the parameter $a^*\Omega^*/U_\infty^*$ appearing in both the basic flow and the perturbation equations. The limit $a^*\Omega^*/U_\infty^* \to \infty$ would then correspond to zero axial flow, making it hard to work with small axial flow rates, and our choice of $\Omega^*a^*$ for the mean flow scaling is therefore very much to be preferred.

Using (3.2) the non-slip boundary condition on the surface of the sphere and the condition at the edge of the boundary layer non-dimensionalize to

\begin{equation}
\begin{aligned}
U & = W = V - \sin \theta = 0 \quad \text{on } \eta = 0, \\
V & = U - TU_\infty = 0 \quad \text{as } \eta \to \infty.
\end{aligned}
\end{equation}
conditions (3.6) reduce to their zero-axial-flow equivalents given by (2.6)–(2.8) and (2.9) respectively.

We now return to the choice of the surface velocity distribution $U_\circ$. In potential flow the slip velocity distribution is given by

$$U_\circ(\theta) = \frac{3}{2} \sin \theta.$$ 

However, this inviscid solution is not a good approximation, due to the boundary-layer separation from the sphere surface. A more realistic scaled velocity distribution, as measured by Fage (1936), fits the curve

$$U_\circ(\theta) \approx 1.5\theta_{rad} - 0.4371\theta_{rad}^3 + 0.1481\theta_{rad}^5 - 0.0423\theta_{rad}^7, \quad (3.7)$$

for $0 \leq \theta \leq 85^\circ$, and $\theta_{rad}$ denotes $\theta$ to be measured in radians. This empirical velocity distribution reaches a maximum of $U_\circ = 1.274$ at $\theta = 74^\circ$, whereas the potential-flow distribution reaches a maximum of $U_\circ = 1.5$ at $\theta = 90^\circ$. We use the empirical velocity distribution (3.7) throughout this investigation.

Non-dimensionalizing the relation (3.1) at the edge of the boundary layer leads to

$$\frac{\partial P}{\partial \theta} = -T^2U_\circ \frac{dU_\circ}{d\theta}, \quad (3.8)$$

where we have scaled the dimensional pressure as $P^* = \rho^* (a^* \Omega^*)^2 P$. Using the empirical velocity distribution, $dU_\circ/d\theta > 0$ for $\theta < 74^\circ$, and substituting this in (3.8) shows that

$$\frac{\partial P}{\partial \theta} < 0 \text{ for } \theta < 74^\circ.$$ 

Therefore at latitudes below $\theta = 74^\circ$, increasing the axial flow rate gives rise to a favourable pressure gradient in the boundary-layer, and so is expected to have a stabilizing effect compared to the zero axial flow case.
Figure 3.1: The latitudinal velocity \( U \) at \( \theta = 10^\circ \) and \( 70^\circ \) with \( T = 0.00-0.25 \) in 0.05 increments (left to right).

The solution of (3.3)–(3.5) subject to conditions (3.6) follows the method outlined by §2.1 for zero axial flow: For each value of \( T \) the NAG routine D03PEF is used to find the mean flow at each latitude by marching from a given complete solution at \( \theta = 5^\circ \) towards the equator \( \theta = 90^\circ \) in one degree increments. The initial solution at \( \theta = 5^\circ \) for each \( T \) is found using the series solution method as described by Banks (1965). The details are broadly as given in appendix B, but now we have extra terms at each order in the expansion of (3.3) which arise from the non-zero pressure term in that equation. Figures 3.1–3.3 show the basic velocities \( U, V \) and \( W \) respectively for \( T = 0.00 \) to 0.25 in increments of 0.05 at \( \theta = 10^\circ \) and \( 70^\circ \). Although not shown here, the velocity profiles at latitudes between these show a smooth continuation from the \( \theta = 10^\circ \) profile to the \( \theta = 70^\circ \) profile as the latitude is increased.

It is interesting to note the behaviour of the radial velocity profile with increasing axial flow as shown in figure 3.3. At each latitude we see \( W \) tending to a constant
Figure 3.2: The azimuthal velocity $V$ at $\theta = 10^\circ$ and $70^\circ$ with $T = 0.00$–$0.25$ in $0.05$ increments (top to bottom).

Figure 3.3: The radial velocity $W$ at $\theta = 10^\circ$ and $70^\circ$ with $T = 0.00$–$0.25$ in $0.05$ increments (right to left).
gradient as $\eta \to \infty$. The gradient changes with $T$ and can be calculated from (3.5) using the second condition of (3.6), i.e.

$$\frac{\partial W}{\partial \eta} \sim -T \left( U_0 \cot \theta + \frac{\partial U_0}{\partial \theta} \right).$$

Physically, this behaviour cannot be maintained arbitrarily far from the surface, as it predicts that fluid is entrained into the boundary layer with unbounded speed as the radial distance is increased, and is of course a consequence of the boundary-layer approximations. The paper by El-Shaarawi et al. (1987) shows experimental measurements of these velocity components in the boundary layer, and measured $W$-profiles do indeed behave as in figure 3.3 in the boundary layer. Very close to the sphere surface, figures 3.1–3.3 show that the flow is independent of the axial flow rate.

In the analysis presented here the boundary layer is approximated by the region $0 \leq \eta \leq 20$ and so the $W$-profiles used are those shown in figure 3.3. It is important to note that the region is sufficiently large for the velocity components $U$ and $V$ to be fully developed. In the governing equations, $W$ is scaled by the Reynolds number $R$, which is necessarily large in the boundary-layer approximation, so that the cramping of the boundary layer should not cause major inaccuracies. In any event, different computational boundary-layer thicknesses were tested in order to check that an outer boundary of $\eta = 20$ did indeed give converged unsteady flow results.

### 3.2 Solution of the perturbation equations

We now discuss the amendments to the methods described in 2.3 that will be used to solve the eigenvalue problem defined by (2.29)–(2.34) with the boundary
conditions (2.37) in the case of non-zero axial flow. The eigenvalue problem will be solved for certain combinations of values of $\alpha$, $\beta$ and $\gamma$ at each Reynolds number $R$, and for particular values of $\theta$ and $T$. From these we form the dispersion relation, 

$$D_S(\alpha, \beta, \gamma; R, \theta, T) = 0,$$

at each $\theta$ and $T$, with the aim of calculating the instability branches.

As in §2.3, we assume that the disturbance quantities tend to zero exponentially as $\eta \to \infty$. An approximate solution at the outer edge of the boundary layer can be found by solving the perturbation equations at the edge of the boundary layer

$$\phi'_1 = \phi_2,$$  

$$\phi'_2 = \left[ \alpha^2 + \beta^2 + iR(\alpha U_{\infty} - \gamma) \right] \phi_1 + W_{\infty} \phi_2 + \alpha_1 U_{\infty} \phi_3$$

$$+ iR \left[ \alpha^2 + \beta^2 - \frac{i\alpha}{R} \cot \theta \right] \phi_1 + \frac{\partial U_{\infty}}{\partial \theta} u + \beta U_{\infty} v \cot \theta,$$  

$$\phi'_3 = -i\phi_1 + \frac{2}{R} \phi_3,$$  

$$R\phi'_4 = iW_{\infty} \phi_1 - i\phi_2 + 2U_{\infty} u - \left[ \alpha^2 + \beta^2 + iR(\alpha U_{\infty} - \gamma) + W_{\infty}' \right] \phi_3,$$  

$$\phi'_5 = \phi_6,$$  

$$\phi'_6 = -\beta U_{\infty} \phi_3 \beta \cot \theta \phi_4 + \left[ \alpha^2 + \beta^2 + iR(\alpha U_{\infty} - \gamma) \right] \phi_5$$

$$+ W_{\infty} \phi_6 - \frac{\partial U_{\infty}}{\partial \theta} + \alpha_1 U_{\infty} v \cot \theta.$$  

Equations (3.9)-(3.14) arise from the full perturbation equations (2.29)-(2.34) using the numerical values of the mean flow variables at $\eta = 20$, denoted by the subscript "∞". Note that for non-zero axial flow, both $U_{\infty}$ and $W_{\infty}$ are non zero. Equations (3.9)-(3.14) permit a solution of the form $c_j^j e^{\kappa_j \eta}$ and substitution of this assumed form enables the numerical solution for $\kappa_j$ for each independent solution defined by $j$. From these we form an approximate solution of the perturbation equations at $\eta = 20$ for each of the transformed variables. Note that since we require an exponentially decaying solution we use only the solutions with $\kappa_j < 0$. Integrating
the full perturbation equations from these initial solutions down towards $\eta = 0$ takes into account the behaviour of $W(\eta)$ over the solution domain and enables the correct eigenvalues to be calculated. Extensive experimentation with the maximum value of $\eta$ chosen has shown that the eigenvalues are independent of the domain as long as it allows the fully developed $U$ and $V$ velocity components to be used.

### 3.3 The convective instability analysis

In this section we solve the perturbation equations (2.29)–(2.34) with the aim of studying the occurrence of convective instabilities in case of non-zero axial flow. Since we are supposing in the first instance that the flow is not absolutely unstable, it follows that in the Briggs–Bers procedure we can reduce the imaginary part of the frequency down to zero, so that $\gamma_i = 0$. To investigate convective instability we use the approaches described in §2.4. In §3.3.1 we insist that the vortices rotate with the surface of the sphere, i.e. $c = 1.0$ in the notation of §2.4, and in §3.3.2 a further approach is taken in which we attempt to predict the onset of the slow vortices under increased axial flow rates.

#### 3.3.1 Stationary vortices

We begin by considering vortices that rotate with the surface of the sphere, i.e. we fix $c = 1.0$ in the methods described in §2.4. Two spatial branches were found that determine the convective instability characteristics at each latitude for each axial flow rate. These branches arise from crossflow and streamline-curvature instability modes and are identical to branches 1 & 2 discussed in §2.4.1; for this reason we do not discuss them here.
Figure 3.4: Neutral curves for the convective instability of stationary vortices at $\theta = 30^\circ$ with $T = 0.00-0.25$ (left to right) in 0.05 increments.

Figure 3.4 shows the neutral curves in the $(R, \alpha_r)$- and $(R, \beta)$-planes at $\theta = 30^\circ$ for each $T$ (for completeness, the neutral curves calculated at latitudes between $\theta = 10^\circ-70^\circ$ are shown in §A.2). Each curve encloses a region that is convectively unstable. The two lobed structure is seen in the neutral curves for each $T$ at all latitudes below $\theta = 60^\circ$. The larger lobe, characterised by higher wavenumbers, is due to crossflow instabilities and the smaller lobe, characterised by smaller wavenumbers, is due to streamline-curvature instabilities. At $\theta = 60^\circ$ when $T > 0.05$ and at $\theta = 70^\circ$ for all $T$, the cusp separating the two lobes is not seen and so these curves are considered to be single lobed. The single lobe is due to the streamline-curvature instability with the crossflow instability defining the shape of the upper branch. However, these calculations assume stationary vortices ($c = 1.0$), which is not a valid assumption for large latitudes. For this reason the calculations conducted at $\theta = 70^\circ$ will not be compared with the experimental results of Kobayashi & Arai.
Table 3.1: The critical Reynolds number R for the onset of the crossflow mode of convective instability at latitudes \( \theta = 10^\circ - 60^\circ \) for axial flow rates \( T = 0.05-0.25 \), - indicates that a crossflow lobe is not seen.


Figure 3.4 shows that increasing the axial flow rate has the effect of increasing the critical Reynolds number of each instability mode. This is seen at all latitudes, as shown in tables 3.1 & 3.2. The two-lobed structure of the neutral curves is found to be exaggerated by the axial flow when \( \theta < 60^\circ \) with the streamline-curvature lobe becoming more important with respect to the crossflow lobe with increased \( T \). At all latitudes below \( \theta = 60^\circ \) the streamline-curvature mode becomes the most dangerous mode above a certain value of \( T \) that decreases with increased latitude. The observation that an increased axial flow rate increases the strength of the streamline-curvature mode is sensible, as figure 3.3 shows that axial flow increases the amount of fluid entrained into the boundary layer, leading to more streamline curvature. In §2.4.1 we found that the streamline-curvature mode becomes more important as the latitude is increased, as a consequence of the geometry of the boundary layer on the sphere. This is consistent with the observation that the value of \( T \) at which the streamline curvature mode is seen to dominate decreases with latitude.
Table 3.2: The critical Reynolds number $R$ for the onset of the streamline-curvature mode of convective instability at latitudes $\theta = 10^\circ - 60^\circ$ for axial flow rates $T = 0.05 - 0.25$.

The lower branch of the neutral curves in figure 3.4 is seen to be independent of $T$ for sufficiently large $R$ at all latitudes. In contrast, the upper branch of each neutral curve depends on the value of $T$. Increasing $T$ moves the upper branch higher up the wavenumber axis in both parameter planes. This suggests that even though axial flow stabilizes the boundary layer in the sense that it increases the critical Reynolds numbers for each instability mode, axial flow increases the range of wavenumbers over which instability can occur.

Kobayashi & Arai (1990) have measured the locations of the onset of spiral vortices on a sphere rotating in a non-zero axial flow, and we now try to associate the appearance of the spiral vortices with the onset of convective instability. The experimental results are expressed in terms of a local Reynolds number $Re_{ax,l}$ that is based on the local latitudinal flow velocity at the outer edge of the boundary layer and the boundary-layer thickness there. $Re_{ax,l}$ is related to our Reynolds number by $Re_{ax,l} = TU_o R$, and their axial flow parameter is defined by $S = 3/(2T)$. The critical values of $R$ that were predicted in our analysis have been converted to $Re_{ax,l}$ at each $S$ and are plotted in figure 3.5. Experimental measurements for the onset of the
Figure 3.5: Plot of the critical local Reynolds numbers ($\text{Re}_{ax,t}$) for the onset of crossflow (top) and streamline-curvature (bottom) modes at latitudes $\theta = 10^\circ$–$60^\circ$ (top to bottom) in $10^\circ$ increments. The highlighted points are from experiments conducted by Kobayashi & Arai (1990) on three different spheres each under three different axial flow rates.

Spiral vortices are also indicated on the figure at three different axial flow rates, and we see good agreement between our predicted onset of the crossflow mode and the experimental data. The experimental measurements were conducted by Kobayashi & Arai (1990) on three different sized spheres for each axial flow rate.

Figure 3.6 gives the predicted values of the vortex angle, $\epsilon$, against $T$ at the onset of each instability mode at each latitude. The value of $\epsilon$ at the onset of the crossflow mode is seen to be roughly independent of latitude and increases linearly with $T$. At the onset of the streamline-curvature mode the predicted value of $\epsilon$ is dependent on both the latitude and axial flow rate. The dependence on latitude sees $\epsilon$ decreasing with increased $\theta$ at each $T$. Again, $\epsilon$ is seen to increase with increased $T$. Unfortunately, no experimental data appears to exist concerning the orientation of the vortices on a sphere rotating in an axial flow, and the theoretical predictions cannot be compared with experimental data at present.
Figure 3.6: The vortex angle $\epsilon$, in degrees, against axial flow rate at the onset of each instability mode type at latitudes of $10^\circ$–$70^\circ$ (top to bottom in the second figure).

Figure 3.7: The number of vortices against axial flow rate at the onset of each instability mode type at latitudes of $10^\circ$–$70^\circ$ (top to bottom).
Kohama & Kobayashi (1983) report observations of the number of spiral vortices, \( n \), at the onset of instability on a sphere rotating in an otherwise still fluid. They find that the number increases towards 31 or 32 as the instability region moves towards the pole. The previous analysis in §2.4.1 on stationary vortices had reasonable success in predicting this number when \( T = 0 \) by looking at \( n \) at the critical point on each neutral curve. However, the spiral vortices are experimentally observed to be growing spatially in the latitudinal direction and are not neutrally stable. This method of predicting \( n \) is therefore incorrect, but gives an indication of the qualitative behaviour with varied flow parameters. Figure 3.7 gives the predicted values of \( n \) against \( T \) at each latitude for the onset of both crossflow and streamline-curvature instability modes. At each latitude, and for both instability mode types, \( n \) is predicted to increase with axial flow speed. For each \( T \), \( n \) is predicted to increase as we approach the pole, which is consistent with the zero-axial flow observations. Unfortunately no experimental data can be found concerning the number of vortices on a sphere rotating in a non-zero axial flow, and again these theoretical predictions cannot be compared with experimental data.

### 3.3.2 A prediction of the vortex speed

To investigate the vortex speed the approach first described in §2.4.2 is taken, in which we plot the neutral curves defined by \( \gamma_i = \alpha_i = 0 \) that correspond to each integer \( n \). The ‘global’ neutral curve at a particular latitude would then be the envelope of the neutral curves pertaining to each single \( n \). This approach does not require us to fix the longitudinal wave speed \( c \), but allows its prediction from the critical values of \( \gamma_r \) and \( R \) using \( c = \gamma_r R / n \). This approach has been taken at latitudes between \( \theta = 10^\circ - 70^\circ \) in increments of \( 10^\circ \) for \( T = 0.05, 0.15 \& 0.25 \).
Figure 3.8: Critical Reynolds numbers, $R_c$, at latitudes $\theta = 60^\circ$ (a) and $\theta = 70^\circ$ (b) with axial flow rates $T = 0.05$, 0.15 & 0.25 for neutral curves defined by fixing $n$ at various integer values.

Figure 3.8(a) shows that at $\theta = 60^\circ$ the curves have a single minimum and, although not shown here, this is the case at all latitudes below $\theta = 60^\circ$ as well. The minimum occurs at the critical Reynolds number of the crossflow lobe of the enveloping neutral curve and is identical to that calculated in §3.3.1 for each $T$ when $c$ was fixed at unity. The critical values of $\gamma_r$ and $R$ at the minimum of each curve lead to the prediction that $c \approx 1.0$. This agreement shows that fixing the longitudinal wave speed to $c = 1.0$ at these latitudes for each $T$, as was done in §3.3.1, is correct. As the latitude is increased from $\theta = 60^\circ$ the curves flatten out and an inflection point is seen at $\theta = 70^\circ$ when $T < 0.25$, as shown in figure 3.8(b). Further investigation of latitudes between $\theta = 60^\circ$ and $70^\circ$ have shown that when $T = 0.05$ the inflection point first appears at $\theta = 67^\circ$, and when $T = 0.15$ the inflection point first appears at $\theta = 69^\circ$. For $T = 0.25$, figure 3.8(b) shows that the
curve is just at the changeover point, and at $\theta = 71^\circ$ an inflection point would be found. Since this method does not predict a minimum critical $R$ for the onset of slow vortices, we are not able to fix the value of the longitudinal wave speed at these high latitudes from our theory. The value $c = 0.76$ has to be taken from experiment.

The experimental observations of Kobayashi & Arai (1990) suggest that the effect of axial flow is to delay the changeover between slow and stationary vortices, i.e. axial flow increases the Reynolds number at which the changeover occurs. Our results are consistent with this observation, and we are also able to predict that the latitude for the changeover increases slightly, from $\theta = 66^\circ$ for $T = 0$ to $\theta = 71^\circ$ for $T = 0.25$. Unfortunately, details of the latitude at which changeover occurs are not given by Kobayashi & Arai (1990).

In §2.4.1 we have shown that in zero axial flow, the onset of the slow vortices occurs at $\theta = 66^\circ$. At this latitude a small region of reverse flow occurs in the radial component of the mean flow close to the wall (see figure 3.3), and this coincides with the dominance of the streamline-curvature mode. However, for non-zero $T$ we find that although the onset of slow vortices still occurs close to this latitude and reverse flow is still seen here, it does not coincide with the dominance of the streamline-curvature instability mode.

### 3.4 The absolute instability analysis

In this section we solve the perturbation equations (2.29)–(2.34) subject to the boundary conditions (2.37) with the aim of studying the effect of axial flow on the absolute instability of the rotating-sphere boundary layer. The Briggs–Bers criterion is applied with fixed $\beta$ to distinguish between convectively and absolutely unstable
time-asymptotic responses to an initial boundary-value perturbation, exactly as was
done in §2.5. To enable the spatio-temporal analysis to be completed, \( \alpha \) and \( \gamma \) are
both complex quantities, while \( \beta \) remains real in order to enforce periodicity round
the azimuth. The term ‘spatial branch’ will be used again, but this time to refer
to the solutions of the dispersion relation in the \( \alpha \)-plane for given complex \( \gamma \). The
branch structure found in this analysis for each \( T \) is identical to that in §2.5 and is
not discussed here.

Pinch-points with \( \gamma_i > 0 \) have been found at all latitudes for each axial flow
rate, and so the boundary-layer on the sphere rotating in a uniform axial flow is
absolutely unstable for certain values of \( R \) and \( \beta \) at each latitude. The neutral curves
for absolute instability at each latitude are of a similar shape to those calculated in
§2.5, and for this reason they are not shown here. The critical Reynolds numbers
for the onset of absolute instability at each latitude are shown in table 3.3 for axial
flow rates between \( T = 0.05-0.25 \) in increments of 0.05. These results show that
axial flow has the effect of stabilizing the boundary layer to absolute instabilities
at each latitude, i.e. the critical Reynolds numbers are increased with increased \( T \)
at each \( \theta \). The critical Reynolds numbers increase a great deal with increased axial
flow, and much more than the critical Reynolds numbers for the onset of convective
instability. For example, at \( \theta = 30^\circ \) the critical Reynolds number for the onset
of absolute instability increases by a factor greater than 20 between \( T = 0.05 \) and
\( T = 0.25 \), while in contrast table 3.1 shows that the critical Reynolds number for
the onset of convective instability increases only by a factor of 2. This suggests that
absolute instability is by far the more sensitive to axial flow.

The effect of axial flow on the absolute instability of the boundary layer can
also be judged by looking at the behaviour of the absolute growth rates at each
Table 3.3: The critical Reynolds number \( R \) for the onset of absolute instability at each latitude \( \theta \) and axial flow rate \( T \).

<table>
<thead>
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<th>( T )</th>
<th>( \theta = 10^\circ )</th>
<th>( \theta = 20^\circ )</th>
<th>( \theta = 30^\circ )</th>
<th>( \theta = 40^\circ )</th>
<th>( \theta = 50^\circ )</th>
<th>( \theta = 60^\circ )</th>
<th>( \theta = 70^\circ )</th>
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<td>1321</td>
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<td>709</td>
<td>519</td>
<td>344</td>
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<td>3602</td>
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<tr>
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<td>19971</td>
<td>19138</td>
<td>15809</td>
<td>14194</td>
</tr>
</tbody>
</table>

Figure 3.9: The behaviour of the maximum growth rates as we move into the region of instability at \( \theta = 10^\circ \) for \( T = 0.00-0.25 \) (top to bottom) in increments of 0.05.
Figure 3.10: Predicted critical $R_X$ values for absolute instability for $T = 0.00-0.25$ in 0.05 increments (bottom to top).

latitude as the axial flow rate is increased. More precisely, we can look at how the maximum growth rates behave as we move further into the region of absolute instability of each neutral curve. Figure 3.9 shows the maximum absolute growth rates at various points in the region of absolute instability for each $T$ at $\theta = 10^\circ$. Although the maximum growth rates decrease with increasing latitude, the results are qualitatively similar at each latitude and so it is sufficient to show only this figure. The most important observation to make from this figure is that the growth rates reduce with increased axial flow, and tend to zero as $T$ increases. This suggests that axial flow suppresses the absolute instability within the boundary layer. Further evidence for this can be found in the behaviour of the critical values of $\alpha_r^\circ$, $\alpha_i^\circ$, $\beta$ and $\gamma_r^\circ$ at the onset of absolute instability with variation in $\theta$ and $T$. At any latitude these critical parameters tend towards zero as the axial flow rate is increased.

For a sphere rotating in an otherwise still fluid, we have shown in §2.5 that the
local Reynolds numbers at the onset of absolute instability are roughly the same at all latitudes below $\theta = 70^\circ$, but have a slight deviation that increases with latitude; above $\theta = 70^\circ$ the critical local Reynolds number is substantially lower. Figure 3.10 shows that this is also true for non-zero axial flow (note that we also include $T = 0$ results for comparison). From this figure we see that the local Reynolds number, $R_X$, at the onset of absolute instability is roughly independent of the latitude for each value of $T$ when close to the pole ($R_X$ is defined in §2). The value of this critical local Reynolds number increases with increased $T$, hence demonstrating the stabilizing effect of the axial flow. We also see that the deviation from the constant value near to the pole that occurs as we move up the sphere is accelerated with increased axial flow.

The comparisons made in §2.5 between the predicted onset of absolute instability and the experimental data of Kohama & Kobayashi (1983) suggest that if the boundary-layer transition is governed by absolute instability, then it would only be for latitudes where the critical local Reynolds number is approximately independent of latitude. Figure 3.10 therefore indicates that if absolute instability does promote transition in the boundary layer then it can only do so in an increasingly limited region close to the pole as the axial flow rate is increased. However, for this to be correct the critical local Reynolds numbers would need to be sufficiently close to experimental measurements for the onset of turbulence and, unfortunately, it appears that these measurements do not yet exist.

3.5 Conclusion

In this chapter we have followed the formulation of chapter 2 by scaling the boundary-layer variables on the boundary-layer thickness and the surface rotation
rate at the equator. This formulation allows the introduction of axial flow into the problem by continuously increasing the axial flow parameter $T = U^*_\infty/(a^\star \Omega^\star)$ from zero. By using this formulation the perturbation equations relevant to the axial flow problem are identical to (2.29)-(2.34). The effect of non-zero axial flow on the analysis is therefore to change the basic flow profiles upon which the analysis is performed.

For stationary vortices, we have seen that axial flow stabilizes the boundary layer with respect to convective instabilities. This is shown by the increased critical Reynolds numbers for the onset of each instability mode with increased axial flow that occurred at all latitudes. When below $\theta = 70^\circ$ the two lobed structure of the neutral curves is exaggerated, with the streamline-curvature mode becoming more important with increased $T$. At each latitude there exists a certain value of $T$ at which the streamline-curvature mode becomes the most dangerous, and this value is seen to decrease with increased latitude. At all latitudes below $\theta = 60^\circ$ we see good agreement with the predicted onset of the crossflow mode and the measurements of Kobayashi & Arai (1990) for the onset of instability. Predictions of the vortex angle and an estimate of the number of vortices were also made. The vortex angle was found to increase at each latitude with increased axial flow, as does the number of vortices.

To investigate the possible variation of the vortex speed, another approach was taken in which the neutral curves defined by $\gamma_i = \alpha_i = 0$ that correspond to each integer $n$ (the azimuthal mode order) were plotted. The global neutral curve at a particular latitude is then the envelope of the neutral curves pertaining to each single $n$, and the corresponding vortex speed is predicted from the critical values of $\gamma_r$ and $R$ using $c = \gamma_r R/n$. Kobayashi & Arai’s experimental observations suggest that the
effect of axial flow is to delay the changeover between slow and stationary vortices, i.e. axial flow increases the Reynolds number at which the changeover occurs. Our results are consistent with this observation, and we were also able to predict that the latitude for the changeover increases slightly from $\theta = 66^\circ$ for $T = 0$ to $\theta = 71^\circ$ for $T = 0.25$.

We have also seen that axial flow stabilizes the boundary layer to absolute instability. This was shown by the critical Reynolds numbers increasing with axial flow at each latitude, and also by the maximum absolute growth rates being reduced at each latitude with increased axial flow. The critical Reynolds numbers increase a great deal with increased axial flow, and much more than the critical Reynolds numbers for the onset of convective instability. At each axial flow rate the onset of absolute instability is seen to occur at a roughly constant value as we move away from the pole. This behaviour was also found in §2.5 for a sphere rotating in an otherwise still fluid. However, the deviation from this constant value as one moves away from the pole is accelerated with increased axial flow. The comparisons made in §2.5 between the theoretical results and the experimentally measured transition locations of Kohama & Kobayashi (1983) do suggest that if the boundary-layer transition is governed by absolute instability, then it would only be for latitudes where the critical local Reynolds number is approximately independent of latitude. Our results for axial flow therefore indicate that if absolute instability does promote transition in the boundary layer then it will be increasingly limited to the region close to the pole. However, for this suggestion to be correct our predicted critical local Reynolds numbers need to be sufficiently close to experimental measurements for the onset of turbulence and, unfortunately, it appears that such data is not currently available.
Chapter 4

The rotating-cone boundary layer in still fluid

This chapter is concerned with the instability mechanisms within the boundary-layer flow over the outer surface of cones rotating in an otherwise still fluid. Viscous and streamline-curvature effects are included, and local stability analyses are conducted on cones with half-angles of $20^\circ$–$90^\circ$ in $10^\circ$ increments. Note that the limiting case of a $90^\circ$ half-angle is equivalent to the rotating disk studied by Malik (1986) and Lingwood (1995a).

In §4.1 we begin with the derivation of the continuity and Navier–Stokes equations in the relevant geometry. The boundary-layer equations that govern the steady basic flow are derived and solved in §4.2, and the unsteady perturbation equations for the stability problem are derived in §4.3. In §§4.5 & 4.6 we conduct local convective and absolute instability analyses respectively on the boundary layer for each half-angle. In each section our theoretical results are compared to existing experimental measurements where possible.
4.1 The cone geometry

Consider the orthogonal curvilinear coordinate system \((x^*, \theta, z^*)\) shown in figure 4.1 that is fixed in space with origin located at the tip of the cone. The half-angle of the cone is \(\psi\), and it has a local surface radius of \(r^*_\circ = x^* \sin \psi\) as measured from the axis of rotation. As with the rotating-sphere analyses presented in chapters 2 & 3, the cone rotates within this fixed frame of reference with a constant angular frequency \(\Omega^*\), and so Coriolis effects will not appear in the governing equations. The coordinate system \((x^*, \theta, z^*)\) is related to the cartesian coordinate system \((X^*, Y^*, Z^*)\) via

\[
\begin{align*}
X^* &= x^* \cos \psi - z^* \sin \psi, \\
Y^* &= (x^* \sin \psi + z^* \cos \psi) \sin \theta, \\
Z^* &= (x^* \sin \psi + z^* \cos \psi) \cos \theta.
\end{align*}
\]

(4.1)

Using standard expressions for the vector calculus identities in a general orthogonal coordinate system, (4.1) enables the derivation of the dimensional continuity and Navier–Stokes equations in this cone geometry, and these are given as (4.2) and
(4.3)–(4.5) respectively:

\[
\frac{\partial \tilde{U}^*}{\partial x^*} + \frac{1}{h^*} \frac{\partial \tilde{V}^*}{\partial \theta} + \frac{\partial W^*}{\partial z^*} + \frac{1}{h^*} \left( \tilde{U}^* \sin \psi + \tilde{W}^* \cos \psi \right) = 0,
\]

(4.2)

\[
\frac{\partial \tilde{U}^*}{\partial t^*} + \tilde{U}^* \frac{\partial \tilde{U}^*}{\partial x^*} + \frac{\tilde{V}^*}{h^*} \frac{\partial \tilde{U}^*}{\partial \theta} + \frac{\partial \tilde{U}^*}{\partial z^*} - \frac{\tilde{V}^*}{h^*} \sin \psi = \frac{1}{\rho^*} \frac{\partial \tilde{P}^*}{\partial x^*}
\]

\[
+ \nu^* \left( \nabla^2 \tilde{U}^* - \frac{2 \sin \psi}{h^*} \frac{\partial \tilde{V}^*}{\partial \theta} - \frac{\sin \psi}{h^*} \left( \tilde{U}^* \sin \psi + \tilde{W}^* \cos \psi \right) \right),
\]

(4.3)

\[
\frac{\partial \tilde{V}^*}{\partial t^*} + \tilde{U}^* \frac{\partial \tilde{V}^*}{\partial x^*} + \frac{\tilde{V}^*}{h^*} \frac{\partial \tilde{V}^*}{\partial \theta} + \frac{\partial \tilde{V}^*}{\partial z^*} + \frac{\tilde{V}^*}{h^*} \left( \tilde{U}^* \sin \psi + \tilde{W}^* \cos \psi \right) =
\]

\[
- \frac{1}{h^* \rho^*} \frac{\partial \tilde{P}^*}{\partial \theta} + \nu^* \left( \nabla^2 \tilde{V}^* + \frac{2}{h^*} \left( \sin \psi \frac{\partial \tilde{U}^*}{\partial \theta} + \cos \psi \frac{\partial \tilde{W}^*}{\partial \theta} \right) - \frac{\tilde{V}^*}{h^*} \right),
\]

(4.4)

\[
\frac{\partial \tilde{W}^*}{\partial t^*} + \tilde{U}^* \frac{\partial \tilde{W}^*}{\partial x^*} + \frac{\tilde{V}^*}{h^*} \frac{\partial \tilde{W}^*}{\partial \theta} + \frac{\partial \tilde{W}^*}{\partial z^*} - \frac{\tilde{V}^*}{h^*} \cos \psi = \frac{1}{\rho^*} \frac{\partial \tilde{P}^*}{\partial z^*}
\]

\[
+ \nu^* \left( \nabla^2 \tilde{W}^* - \frac{2 \cos \psi}{h^*} \frac{\partial \tilde{V}^*}{\partial \theta} - \frac{\cos \psi}{h^*} \left( \tilde{U}^* \sin \psi + \tilde{W}^* \cos \psi \right) \right),
\]

(4.5)

where

\[
\nabla^2 = \frac{\partial^2}{\partial x^*^2} + \frac{1}{h^*} \frac{\partial^2}{\partial \theta^*} + \frac{\partial}{\partial z^*^2}.
\]

Here \(\tilde{U}^*, \tilde{V}^*, \text{ and } \tilde{W}^*\) are the dimensional velocity components in the \(x^*, \theta\) and \(z^*\) directions respectively, \(\tilde{P}^*\) is the dimensional pressure and \(h^* = r^*_e + z^* \cos \psi\). Note that when \(\psi = 90^\circ\), (4.2) and (4.3)–(4.5) reduce to the continuity and Navier–Stokes equations in a stationary cylindrical-polar coordinate system.

### 4.2 The steady mean flow

In this section we derive the boundary-layer equations relevant to the steady mean boundary-layer flow on each rotating cone defined by \(\psi\). We assume that \(\delta^*/r^*_e \ll 1\), with \(\delta^* = (\nu^*/\Omega^*)^{1/2}\) the boundary-layer thickness, and \(\tilde{U}^* \sim O(1)\),
\( \tilde{V}^* \sim O(1), \tilde{W}^* \sim O(\delta^*) \) and \( \partial/\partial r^* \sim O(\delta^*-1) \). Similar boundary-layer assumptions were used in the analyses of the rotating-sphere boundary layer in chapters 2 & 3. By assuming a steady axisymmetric mean flow with components \( U^*, V^* \) and \( W^* \), the boundary-layer assumptions enable us to write (4.2)-(4.4) as

\[
\frac{U^*}{x^*} + \frac{\partial U^*}{\partial x^*} + \frac{\partial W^*}{\partial z^*} = 0, \quad (4.6)
\]

\[
U^* \frac{\partial U^*}{\partial x^*} + W^* \frac{\partial U^*}{\partial z^*} - \frac{V^*}{x^*} = \nu^* \frac{\partial^2 U^*}{\partial x^*}, \quad (4.7)
\]

\[
U^* \frac{\partial V^*}{\partial x^*} + W^* \frac{\partial V^*}{\partial z^*} + \frac{U^* V^*}{x^*} = \nu^* \frac{\partial^2 V^*}{\partial z^*}, \quad (4.8)
\]

at \( O(1) \). Equation (4.5) non-dimensionalizes to be \( O(\delta^*) \) and so is not considered here.

In a fixed frame of reference (4.6)-(4.8) are subject to the boundary conditions

\[
U^* = W^* = V^* - x^* \Omega^* \sin \theta = 0 \quad \text{on} \quad z^* = 0, \quad (4.9)
\]

\[
U^* = V^* = 0 \quad \text{as} \quad z^* \to \infty.
\]

The first of these equations represents the no-slip condition on the cone surface, and the second represents the quiescent fluid condition away from the cone. Note that (4.6)-(4.8) subject to (4.9) are identical to the Mangler equations for a rotating cone (Mangler, 1945), and so verify our formulation.

By using a transformation pointed out by Wu (1959) and Tein (1960), the steady laminar boundary-layer flow around the rotating cone can be obtained directly from von Kármán’s differential equations for the rotating disk (von Kármán, 1921). However, the scalings involved in this transformation would lead to a Reynolds number appearing in the perturbation equations that is based on the distance over the cone surface, and existing experimental results are given in terms of a Reynolds numbers based on the local surface radius. Care would therefore need to be taken with factors of \( \sin \psi \) in any comparisons between the predicted and measured values. In the work
presented here, we use a modified version of Wu and Tein’s transformation that is based on the local surface radius, and this leads to boundary-layer equations that are parameterised by the half-angle. As we shall see in §4.3, the scalings are such that the resulting Reynolds number is simply the square root of the experimental Reynolds number, and the perturbation equations turn out to be similar to those derived by Lingwood (1995a) for the rotating disk. Note that the results of the analysis should be independent of the non-dimensionalizing scales, provided that they are used consistently in both the steady mean flow and unsteady perturbation equations.

The dimensionless similarity variables at each half-angle are functions of \( \eta \), and are defined as

\[
U(\eta; \psi) = \frac{U^*}{x^* \Omega^* \sin \psi}, \quad V(\eta; \psi) = \frac{V^*}{x^* \Omega^* \sin \psi}, \quad W(\eta; \psi) = \frac{W^*}{(\nu^* \Omega^*)^{1/2}}, \quad (4.10)
\]

where \( \eta = z^*/(\nu^* / \Omega^*)^{1/2} \). Note that \( \eta \) is scaled on the boundary-layer thickness \( \delta^* \), and \( U \) and \( V \) are scaled on the local surface speed of the cone. Using (4.10), (4.6)–(4.8) non-dimensionalize as the following set of coupled nonlinear ordinary differential equations at each \( \psi \):

\[
WU' + (U^2 - V^2) \sin \psi = U'', \quad (4.11)
\]
\[
WV' + 2UV \sin \psi = V'', \quad (4.12)
\]
\[
W' + 2U \sin \psi = 0, \quad (4.13)
\]

where a prime denotes differentiation with respect to \( \eta \). The boundary conditions (4.9) non-dimensionalize as

\[
U = W = V - 1 = 0 \quad \text{on} \ \eta = 0, \quad (4.14)
\]

\[
U = V = 0 \quad \text{as} \ \eta \to \infty.
\]
Equations (4.11)–(4.13) subject to (4.14) are solved numerically for each $\psi$ using a fourth-order Runge–Kutta integrator with a Newton–Raphson searching routine. Figure 4.2 shows the three components of the similarity solution for the steady mean flow on rotating cones with half-angles of $\psi = 20^\circ$–$90^\circ$ in $10^\circ$ increments. Note that the flow along the cone surface, $U$, is inflectional for each cone angle, and the formulation is such that $U$ and $V$ do not significantly change with $\psi$.

4.3 Derivation of the perturbation equations

In this section we formulate the stability problem. The perturbation equations are derived and the Reynolds numbers used in this investigation are discussed.

To conduct a local stability analysis at a location $x_0^*$ along the cone surface, we
impose infinitesimally small disturbances on the mean flow at that point. The non-dimensionalizing length, velocity, pressure and time scales are \( \delta^*, r_{\alpha,\delta}^* \Omega^*, \rho^* r_{\alpha,\delta}^{*2} \Omega^{*2} \) and \( \delta^*/r_{\alpha,\delta}^* \) respectively, and these scalings lead to the Reynolds number \( R \) at \( x_s^* \), where

\[
R = \frac{x_s^* \Omega^* \sin \psi \delta^*}{\nu^*} = \frac{x_s^* \sin \psi}{\delta^*} = x_s \sin \psi = r_{\alpha,\delta}^*.
\]

Note that these scalings are consistent with the scalings (4.10) that were used in the derivation of the steady mean-flow equations.

We denote the non-dimensional velocity and pressure of the perturbed flow by bared upper-case quantities. These quantities are formed from a non-dimensional basic flow component (denoted by an upper-case quantity) and a perturbing unsteady quantity (denoted by a lower-case hatted quantity).

\[
\bar{U}(\eta, x, \theta, t; R, \psi) = \frac{r_{\alpha}^*}{R} U(\eta; \psi) + \hat{u}(\eta, x, \theta, t; R, \psi),
\]

\[
\bar{V}(\eta, x, \theta, t; R, \psi) = \frac{r_{\alpha}^*}{R} V(\eta; \psi) + \hat{v}(\eta, x, \theta, t; R, \psi),
\]

\[
\bar{W}(\eta, x, \theta, t; R, \psi) = \frac{1}{R} W(\eta; \psi) + \hat{w}(\eta, x, \theta, t; R, \psi),
\]

\[
\bar{P}(\eta, x, \theta, t; R, \psi) = \frac{1}{R^2} P(\eta; \psi) + \hat{p}(\eta, x, \theta, t; R, \psi).
\]

The factors multiplying the basic-flow quantities are a consequence of the non-dimensionalizing scales and the similarity normalization.

The dimensionless continuity and Navier–Stokes equations are linearized with respect to the perturbation quantities to give

\[
\frac{\partial \hat{u}}{\partial t} + \hat{u} \sin \psi \frac{\partial}{\partial x} + \hat{\psi} \frac{\cos \psi}{h} + \frac{1}{\theta} \frac{\partial \hat{v}}{\partial \theta} + \frac{\partial \hat{w}}{\partial \eta} = 0,
\]

\[
\frac{\partial \hat{u}}{\partial t} + \frac{r_{\alpha}^* \hat{u}}{R} \frac{\partial}{\partial x} + \frac{r_{\alpha}^* \hat{V}}{h R} \frac{\partial}{\partial \theta} + \frac{W \hat{\psi}}{R} \frac{\partial}{\partial \eta} + \frac{r_{\alpha}^* \hat{w}}{R} \frac{\partial U}{\partial \eta} + \frac{U \hat{\psi}}{R} = 0,
\]

\[
- \frac{2r_{\alpha}^* \hat{v} \sin \psi}{h R} = \frac{\partial \hat{\psi}}{\partial x} + \frac{1}{R} \left( \frac{\partial^2 \hat{u}}{\partial x^2} + \frac{1}{R^2} \frac{\partial^2 \hat{u}}{\partial \theta^2} + \frac{\partial^2 \hat{u}}{\partial \eta^2} \right) + \frac{2 \sin \psi \hat{\psi}}{h^2} \frac{\partial}{\partial \theta} - \sin \psi \frac{\partial \psi}{h^2} \left( \hat{u} \sin \psi + \hat{w} \cos \psi \right),
\]
\begin{align*}
\frac{\partial\hat{v}}{\partial t} + \frac{r_s U}{R} \frac{\partial\hat{v}}{\partial x} + \frac{U \hat{v} \sin \psi}{R} + \frac{r_s V}{hR} \frac{\partial\hat{v}}{\partial \theta} + \frac{W \hat{v}}{R} \frac{\partial\hat{v}}{\partial \eta} + \frac{r_s \hat{w}}{R} \frac{\partial\hat{V}}{\partial \eta} \\
+ \frac{\hat{v}}{h} \left( \frac{r_s U}{R} \sin \psi + \frac{W}{R} \cos \psi \right) + \frac{r_s V}{hR} \left( \hat{u} \sin \psi + \hat{w} \sin \psi \right) = -\frac{1}{hR} \frac{\partial \hat{p}}{\partial \eta} \tag{4.21}
\end{align*}

\begin{align*}
\frac{\partial\hat{w}}{\partial t} + \frac{r_s U}{R} \frac{\partial\hat{w}}{\partial x} + \frac{r_s V}{R} \frac{\partial\hat{w}}{\partial \theta} + \frac{W \hat{w}}{R} \frac{\partial\hat{w}}{\partial \eta} + \frac{\hat{w}}{R} \frac{\partial\hat{W}}{\partial \eta} \\
- \frac{2r_s \hat{v} \hat{w} \cos \psi}{hR} = -\frac{\partial \hat{p}}{\partial \eta} + \frac{1}{R} \left( \frac{\partial^2 \hat{v}}{\partial x^2} + \frac{1}{R^2} \frac{\partial^2 \hat{v}}{\partial \theta^2} + \frac{\partial^2 \hat{w}}{\partial \eta^2} \right) \tag{4.22}
\end{align*}

To make these perturbation equations separable in \(x\), \(\theta\) and \(t\), it is necessary to make additional parallel-flow and boundary-layer approximations, i.e. to ignore variation in the Reynolds number with local surface radius and also to assume that \(\eta/r_s \ll 1\). This involves replacing the variable \(h = r_s + \eta \cos \psi\), which appears in the coefficients of (4.19)–(4.22), by \(R\); the validity of this approximation at each half-angle is discussed in §4.7.1. Terms \(O(R^{-2})\) are neglected and the perturbing quantities are expressed in normal mode form

\begin{align*}
\hat{u} & = u(\eta; \alpha, \gamma, \beta; R, \psi) e^{i(\alpha x \sin \psi + \beta \theta - \gamma t)}, \\
\hat{v} & = v(\eta; \alpha, \gamma, \beta; R, \psi) e^{i(\alpha x \sin \psi + \beta \theta - \gamma t)}, \\
\hat{w} & = w(\eta; \alpha, \gamma, \beta; R, \psi) e^{i(\alpha x \sin \psi + \beta \theta - \gamma t)}, \\
\hat{p} & = p(\eta; \alpha, \gamma, \beta; R, \psi) e^{i(\alpha x \sin \psi + \beta \theta - \gamma t)}. \tag{4.26}
\end{align*}

Here \(\alpha\) and \(\beta\) are wavenumbers in the \(x\) and \(\theta\) directions respectively, and \(\gamma\) is the frequency of the disturbance. The quantities \(\alpha\) and \(\gamma\) are in general complex, as required by the spatial-temporal analysis of §4.6; we write these quantities as \(\alpha = \alpha_r + i\alpha_i\) and \(\gamma = \gamma_r + i\gamma_i\). In contrast, the circumferential wave number, \(\beta\), is
real. The angle that the phase fronts make with a circle parallel to the base of the cone is denoted $\epsilon$, and is found from

$$
\epsilon = \tan^{-1} (\beta / R \alpha_r).
$$  \hspace{1cm} (4.27)

The integer number of complete cycles of the disturbance round the circumference is

$$
n = \beta.
$$  \hspace{1cm} (4.28)

Later in this chapter we will identify $\epsilon$ and $n$ as being the angle and number of spiral vortices on the cone surface.

The perturbation equations may be written as a set of six first-order ordinary-differential equations using the transformed variables

$$
\phi_1(\eta; \alpha, \gamma, \beta; R, \psi) = (\bar{\alpha} - i \sin \psi / R) u + \bar{\beta} v,
$$  \hspace{1cm} (4.29)

$$
\phi_2(\eta; \alpha, \gamma, \beta; R, \psi) = (\bar{\alpha} - i \sin \psi / R) u' + \bar{\beta} v',
$$  \hspace{1cm} (4.30)

$$
\phi_3(\eta; \alpha, \gamma, \beta; R, \psi) = w,
$$  \hspace{1cm} (4.31)

$$
\phi_4(\eta; \alpha, \gamma, \beta; R, \psi) = p,
$$  \hspace{1cm} (4.32)

$$
\phi_5(\eta; \alpha, \gamma, \beta; R, \psi) = (\bar{\alpha} - i \sin \psi / R) v + \bar{\beta} u,
$$  \hspace{1cm} (4.33)

$$
\phi_6(\eta; \alpha, \gamma, \beta; R, \psi) = (\bar{\alpha} - i \sin \psi / R) v' + \bar{\beta} u',
$$  \hspace{1cm} (4.34)

where $\bar{\alpha} = \alpha \sin \psi, \bar{\beta} = \beta / R$ and the prime denotes differentiation with respect to $\eta$. The perturbation equations become

$$
\begin{align*}
\phi_1 &= \phi_2, \\
\frac{d\phi_2}{dR} &= \frac{1}{R} \left( [\bar{\alpha}^2 + \bar{\beta}^2]_v + i R (\bar{\alpha} U + \bar{\beta} V - \gamma) + [U \sin \psi]_s \right) \phi_1 \\
& \quad + \left[ \frac{W \phi_2}{R} \right]_s + \left( \left( \bar{\alpha} - \left[ \frac{i \sin \psi}{R} \right]_s \right) U' + \bar{\beta} V' \right) \phi_3 \phi_5 \\
& \quad + i \left( \bar{\alpha}^2 + \bar{\beta}^2 - \left[ \frac{i \bar{\alpha}}{R} \right]_s \right) \phi_4 - \left[ \frac{2 V \sin \psi \phi_5}{R} \right]_s,
\end{align*}
$$  \hspace{1cm} (4.36)
\[
\phi_3 = -i\phi_1 - \left[ \frac{\phi_3 \cos \psi}{R} \right]_s, \tag{4.37}
\]
\[
\phi'_4 = \left[ \frac{i W \phi_1}{R} \right]_s - \left[ \frac{i \phi_2}{R} \right]_v - \frac{1}{R} \left( \left[ \ddot{\alpha}^2 + \ddot{\beta}^2 \right]_v + i R \left( \ddot{\alpha} U + \ddot{\beta} V - \gamma \right) + W'_s \right) \phi_3, \tag{4.38}
\]
\[
\phi'_5 = \phi_6, \tag{4.39}
\]
\[
\left[ \frac{\phi'_6}{R} \right]_v = \left[ \frac{2 V \sin \psi \phi_1}{R} \right]_s + \left( \left( \ddot{\alpha} - \left[ \frac{i \sin \psi}{R} \right] \right) V' - \ddot{\beta} U' \right) \phi_3
+ \frac{1}{R} \left( \left[ \ddot{\alpha}^2 + \ddot{\beta}^2 \right]_v + i R \left( \ddot{\alpha} U + \ddot{\beta} V - \gamma \right) + \left[ U \sin \psi \right]_s \right) \phi_5 \tag{4.40}
+ \left[ \frac{\beta \sin \psi \phi_4}{R} \right]_s + \left[ \frac{W \phi_5}{R} \right]_s,
\]

where the subscripts \( v \) and \( s \) indicate which of the \( O(R^{-1}) \) terms arise from the viscous and streamline-curvature effects respectively. Note that with \( \psi = 90^\circ \), (4.35)-(4.40) reduce to the perturbation equations for the rotating-disk boundary layer, and are the same as those used by Lingwood (1995a) apart from the absence of the Coriolis terms that arise from her rotating frame of reference.

By neglecting those terms in (4.35)-(4.40) that arise from streamline curvature, we find the Orr–Sommerfeld equation for the rotating cone in the form
\[
\left( \frac{i}{R} \right) \left( \phi'''_3 \right) - 2 \left( \ddot{\alpha}^2 + \ddot{\beta}^2 \right) \phi''_3 + \left( \ddot{\alpha}^2 + \ddot{\beta}^2 \right) \phi'_3
+ \left( \ddot{\alpha} U + \ddot{\beta} V - \gamma \right) \left( \phi''_3 - \left( \ddot{\alpha}^2 + \ddot{\beta}^2 \right) \phi_3 \right) - \left( \ddot{\alpha} U'' + \ddot{\beta} V'' \right) \phi_3 = 0. \tag{4.41}
\]

Ignoring both the streamline-curvature and viscous terms in the perturbation equations leads to Rayleigh’s equation (4.42). By doing this we assume that viscosity acts in establishing the steady basic flow but has a negligible effect on the instability waves;
\[
\left( \ddot{\alpha} U + \ddot{\beta} V - \gamma \right) \left( \phi''_3 - \left( \ddot{\alpha}^2 + \ddot{\beta}^2 \right) \phi_3 \right) - \left( \ddot{\alpha} U'' + \ddot{\beta} V'' \right) \phi_3 = 0. \tag{4.42}
\]

The Reynolds numbers that characterise the boundary-layer flow over the rotating cone can be formed from a number of different length and velocity scales. The
formulation of this problem as described above is such that the Reynolds number is defined as \( R = x^*\Omega^* \sin \psi \delta^*/\nu^* = r_\alpha \), and is equivalent to the Reynolds number used by Lingwood (1995a) and Malik (1986) on the rotating disk when \( \psi = 90^\circ \). However, the experimental results of Kreith, Ellis & Giesing (1962); Kappesser, Greif & Cornet (1973) and Kobayashi & Izumi (1983) are presented in terms of another Reynolds number, \( Re \). This Reynolds number is based on the local surface radius at the point under analysis, \( x^* \sin \psi \), and the local surface rotation speed at that point, \( x^*\Omega^* \sin \psi \). It is therefore defined as \( Re = x^{*2}\Omega^* \sin^2 \psi /\nu^* \). By using the definition of the boundary-layer thickness, \( \delta^* = (\nu^*/\Omega^*)^{1/2} \), we see that \( Re = R^2 \), and this relationship will enable a comparison between our predictions and the existing experimental data.

A Reynolds number can also be formed that is based on the distance from the tip of the cone, \( x^* \), and local surface speed at a point on the surface, \( x^*\Omega^* \sin \psi \). This local Reynolds number is defined as \( R_X = x^{*2}\Omega^* \sin \psi /\nu^* = R^2 /\sin \psi \), and will be important in the discussion of the absolute instability of the rotating-cone boundary layer in §4.6.

### 4.4 Solution of the perturbation equations

We now discuss the amendments to the methods described in §2.3 that will be used to solve the eigenvalue problem defined by (4.35)–(4.40) with the boundary conditions (2.37). The eigenvalue problem will be solved for certain combinations of values of \( \alpha, \beta \) and \( \gamma \) at each Reynolds number \( R \) and for a particular value of \( \psi \). From these we form the dispersion relation for the rotating cone, \( D_C(\alpha, \beta, \gamma; R, \psi) = 0 \), at each \( \psi \), with the aim of calculating the instability branches.
As in §2.3, we assume that the disturbance quantities tend to zero exponentially as $\eta \to \infty$. An approximate solution at the outer edge of the boundary layer can be found by solving the perturbation equations at the edge of boundary layer,

\begin{align*}
\phi_1' &= \phi_2, \quad & (4.43) \\
\phi_2' &= \left[ \tilde{\alpha}^2 + \tilde{\beta}^2 - i R \gamma \right] \phi_1 + W_\infty \phi_2 + i R \left[ \tilde{\alpha}^2 + \tilde{\beta}^2 - \frac{i \tilde{\alpha}}{R} \right] \phi_1, \quad & (4.44) \\
\phi_3' &= -i \phi_1 - \frac{\phi_3 \cos \psi}{R}, \quad & (4.45) \\
R \phi_4' &= i W_\infty \phi_1 - i \phi_2 - \left[ \tilde{\alpha}^2 + \tilde{\beta}^2 - i R \gamma \right] \phi_3, \quad & (4.46) \\
\phi_5' &= \phi_0, \quad & (4.47) \\
\phi_6' &= \tilde{\beta} \sin \psi \phi_4 + \left[ \tilde{\alpha}^2 + \tilde{\beta}^2 - i R \gamma \right] \phi_5 + W_\infty \phi_6. \quad & (4.48)
\end{align*}

Equations (4.43)–(4.48) arise from the full perturbation equations (4.35)–(4.40) using the mean flow variables at $\eta = 20$, which is denoted by the subscript “$\infty$”. Following the method described in §2.3, (4.43)–(4.48) permit a solution of the form $c_i e^{\kappa_i \eta}$, and substitution of this assumed form enables the numerical solution for $\kappa_j$ for each independent solution defined by $j$. From these we form an approximate solution of the perturbation equations at $\eta = 20$ for each of the transformed variables defined by $i$. Note that since we require an exponentially decaying solution we use only the solutions with $\kappa_j < 0$. Numerically integrating the full perturbation equations from these initial solutions down towards $\eta = 0$ enables the correct eigenvalues to be calculated; details of this process are given in §2.3. Extensive experimentation with the maximum value of $\eta$ chosen has shown that the eigenvalues are independent of the domain as long as it allows the fully developed $U$ and $V$ velocity components to be used, i.e. $\eta > 15$. 
4.5 The convective instability analysis

In this section we solve the perturbation equations (4.35)–(4.40) with the aim of studying the occurrence of convective instabilities. Since we are supposing in the first instance that the flow is not absolutely unstable, it follows that in the Briggs–Bers procedure we can reduce the imaginary part of the frequency down to zero, so that \( \gamma_i = 0 \). To produce the neutral curves for convective instability we insist that the vortices rotate with the cone surface velocity, thereby fixing the ratio \( \gamma_r/\bar{\beta} \), and then \( \alpha \) and \( \bar{\beta} \) are calculated using a spatial analysis. This approach is identical to that described in §2.6, however, unlike on the rotating sphere, experiments on rotating cones have shown that the vortices rotate with the cone surface at all rotation speeds and half-angles. The non-dimensional speed of the surface of the cone is unity, and equating this with the disturbance phase velocity in the same direction, \( \gamma_r/\bar{\beta} \), leads to \( \gamma_r = \bar{\beta} \). This relationship must be satisfied if the vortices are to rotate with the cone.

In §4.5.2 a further approach is taken in which the neutral curves are plotted for each half-angle for various integer \( n \); the ‘global’ neutral curve at a particular \( x_s^* \) is then the envelope of the neutral curves pertaining to each single \( n \). This method enables a prediction of the speed of the spiral vortices with respect to the cone surface rather than making a-priori assumptions about the longitudinal wave speed.

4.5.1 Stationary vortices

We begin by explicitly assuming that the vortices rotate with the surface of the cone, i.e. we fix \( \gamma_r = \bar{\beta} \) and solve the dispersion relation using the methods described in §4.4. At each half-angle we have found that two spatial branches determine the convective instability characteristics of the system. These branches arise from
crossflow and streamline-curvature instability modes and, at each half-angle, are identical to branches 1 & 2 found on the rotating sphere at latitudes $\theta < 66^{\circ}$ (see §2.4.1). For this reason we do not discuss them here.

Figure 4.3 shows the neutral curves of convective instability in the $(R, \alpha_r)$- and $(R, \beta)$-planes for cones with half-angles of $\psi = 20^{\circ}$–$90^{\circ}$ in $10^{\circ}$ increments. The region enclosed within each curve is convectively unstable, and the curve is seen to have two lobes at each half-angle. As with the neutral curves for the rotating-sphere boundary layer discussed in §§2.4 & 3.3, the larger lobe, characterised by higher wavenumbers, is due to crossflow instabilities and the smaller lobe, characterised by smaller wave numbers, is due to streamline-curvature instabilities. As expected, the neutral curves in both wavenumber planes are identical to those calculated by Malik (1986) for the rotating-disk boundary layer when $\psi = 90^{\circ}$.

From figure 4.3 we see that the lower branch of the neutral curves in the $(\alpha_r, R)$-plane is independent of half-angle for sufficiently large $R$. In contrast, the upper branch of each neutral curve depends on half-angle, and decreasing the half-angle from $\psi = 90^{\circ}$ (rotating disk) moves the upper branch higher up the $\alpha_r$-axis. The upper and lower branches of the neutral curves in the $(\beta, R)$-plane depend on the half-angle, and decreasing the half-angle has the effect of moving the neutral curve down the $\beta$-axis.

Table 4.1 gives the critical Reynolds numbers and the non-dimensional location for the onset of each instability mode, $x = R/\sin \psi$. From this data we see that the critical Reynolds numbers decrease with reduced half-angle, and for any given rotation rate, this means that the onset of convective instability occurs at a smaller local radius as the half-angle is reduced. When the onset is expressed in terms of the critical values of $x$, we see that instability is predicted to occur further along
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Figure 4.3: The neutral curves of convective instability for stationary vortices on cones with half-angles $\psi = 20^\circ$–$90^\circ$ (left to right).

the more slender cones for any given rotation rate.

At each half-angle the crossflow mode is predicted to be the most dangerous, and it is interesting to note that the ratio of the critical Reynolds numbers of each mode is roughly constant at each half-angle. This implies that the relative importance of the two instability modes does not change with the half-angle.

Figure 4.4 shows a comparison between the predicted onset of the two instability modes and the observed onset of spiral vortices as measured in three different experiments: Kreith, Ellis & Giesing (1962); Kappesser, Greif & Cornet (1973) and Kobayashi & Izumi (1983). Note the use of the experimental Reynolds number $Re$, which is related to $R$ via $Re = R^2$ as discussed in §4.3.

The predicted onset of the crossflow mode is seen to match Kobayashi & Izumi’s measurements well for half-angles above $\psi = 60^\circ$. However, as the half-angle is reduced, the predicted onset of the crossflow mode deviates from these measurements,
and at $\psi = 30^\circ$, the predictions match the experimental measurements of Kreith et al. and Kappesser et al. reasonably well. For cones with half-angles between $\psi = 30^\circ$–$60^\circ$, the predicted onset of the crossflow mode lies between the three sets of experimental data. Although the predicted onset of convective instability at each half-angle is consistent with the three sets of experimental data, the predictions do not follow any one set for all half-angles and more experiments are required to clarify our predictions.

Note that the measurements conducted by Kappesser et al. are reasonably close to the predicted onset of the streamline-curvature mode when $\psi \geq 60^\circ$, but are close to the predicted onset of the crossflow mode when $\psi \leq 45^\circ$. These observations may suggest that Kappesser et al. were measuring the onset of the streamline-curvature instability mode in their experiments on cones with large $\psi$, and the onset of the crossflow mode on the more slender cones.

Table 4.2 gives the predicted vortex angle at the onset of each instability mode.

<table>
<thead>
<tr>
<th>$\psi$</th>
<th>$R_{\text{cross flow}}$</th>
<th>$R_{\text{slc}}$</th>
<th>$x_{\text{cross flow}}$</th>
<th>$x_{\text{slc}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>165.26</td>
<td>257.76</td>
<td>483.19</td>
<td>753.64</td>
</tr>
<tr>
<td>30</td>
<td>200.71</td>
<td>314.17</td>
<td>401.42</td>
<td>628.34</td>
</tr>
<tr>
<td>40</td>
<td>228.28</td>
<td>357.81</td>
<td>355.14</td>
<td>556.65</td>
</tr>
<tr>
<td>50</td>
<td>249.41</td>
<td>391.85</td>
<td>325.58</td>
<td>511.52</td>
</tr>
<tr>
<td>60</td>
<td>265.49</td>
<td>417.54</td>
<td>306.56</td>
<td>482.13</td>
</tr>
<tr>
<td>70</td>
<td>276.80</td>
<td>435.72</td>
<td>294.56</td>
<td>463.68</td>
</tr>
<tr>
<td>80</td>
<td>283.59</td>
<td>446.82</td>
<td>287.96</td>
<td>453.71</td>
</tr>
<tr>
<td>90</td>
<td>285.36</td>
<td>450.95</td>
<td>285.36</td>
<td>450.95</td>
</tr>
</tbody>
</table>

Table 4.1: The critical values of $R$ and $x$ at the onset of the each instability mode.
Figure 4.4: A comparison between the predicted onset of each instability mode and the experimental measurements of Kreith, Ellis & Giesing (1962); Kappesser, Greif & Cornet (1973) and Kobayashi & Izumi (1983).

and also the predicted number of vortices at the onset of the crossflow mode; the angles and vortex numbers were calculated from (4.27) and (4.28) respectively. The data indicates that both the angle and number of vortices are very much reduced on the more slender cones, which is consistent with the experimental findings of Kobayashi & Izumi (1983). For reasons discussed in §2.4, the values of $n$ are used to show the general trends in the behaviour of the number of vortices, and are not predictions of the actual number. Note that the predicted values of $\epsilon$ and $n$ at $\psi = 90^\circ$ are identical to those calculated by Malik (1986) for the rotating-disk boundary layer.
Table 4.2: The vortex angle $\epsilon$ and number of vortices $n$ at the onset of convective instability.

### 4.5.2 A prediction of the vortex speed

To investigate the vortex speed the approach first described in §2.4.2 is taken, in which we plot the neutral curves defined by $\gamma_i = \alpha_i = 0$ that correspond to each integer $n$ defined by (4.28). The ‘global’ neutral curve at a particular $x^*_i$ would then be the envelope of the neutral curves pertaining to each single $n$. The Reynolds number for the rotating cone depends on two variables, $x^*$ and $\Omega^*$, and in the local analysis we fix $x^* = x^*_i$, varying $R$ therefore corresponds to varying the rotation rate $\Omega^*$. This approach does not require us to fix the longitudinal wave speed $c$, but allows its prediction from the critical values of $\gamma_r$ and $R$ using $c = \gamma_r R / n$. This approach has been taken on cones with half-angles between $\psi = 20^\circ$–$90^\circ$ in increments of $10^\circ$.

Figure 4.5 shows that for half-angles of $\psi = 20^\circ$, $40^\circ$, $60^\circ$ & $80^\circ$, the curves have a single minimum, and this is found to be the case at all half-angles, although they are
Figure 4.5: Critical Reynolds numbers, $R$, at half-angles $\psi = 20^\circ$–$80^\circ$ in $20^\circ$ increments (bottom to top) for neutral curves defined by fixing $n$ at various integer values. 

not shown here. The minimum of each enveloping neutral curve corresponds to the critical Reynolds number of the crossflow lobe, and is identical to those calculated in §4.5.1 when $c$ was fixed at unity. The critical values of $\gamma_r$ and $R$ at the minimum of each curve lead to the prediction that $c \approx 1.0$ as can be seen in table 4.3. This agreement shows that fixing the longitudinal wave speed to $c = 1.0$, as was done in §4.5.1, is correct.

In contrast to the results of similar investigations conducted in §§2.4.2 & 3.3.2 on the rotating-sphere boundary layer, we do not predict slow vortices for the rotating-cone boundary layer. The experimental observations of Kreith, Ellis & Giesing (1962); Kappesser, Greif & Corret (1973) and Kobayashi & Izumi (1983) have shown that the vortices are fixed on the surface of the cone at all half-angles, and our results are consistent with this observation.
\[
\begin{array}{|c|c|c|c|c|}
\hline
\theta & R & \gamma_r & n & c \\
\hline
20 & 165.3 & 0.0475 & 8 & 0.982 \\
40 & 228.3 & 0.0616 & 14 & 1.004 \\
60 & 265.5 & 0.0711 & 19 & 0.993 \\
80 & 283.6 & 0.0773 & 22 & 0.997 \\
\hline
\end{array}
\]

Table 4.3: Critical parameters at half-angles of $\psi = 20^\circ-80^\circ$ for neutral curves defined by fixing $n$ at various integer values.

### 4.6 The absolute instability analysis

In this section we look at the absolute instability of the boundary-layer flow on rotating cones with half-angles of $\psi = 20^\circ-90^\circ$. The Briggs–Bers criterion is applied with fixed $\beta$ to distinguish between convectively and absolutely unstable time-asymptotic responses to an initial boundary-value perturbation, exactly as was done in §2.5 with the only important difference being the physical interpretation of the parameters. To enable the spatio-temporal analysis to be completed, $\alpha$ and $\gamma$ are both complex quantities, while $\beta$ remains real in order to enforce periodicity round the circumference.

As with the analyses of the rotating-sphere boundary layer, the absolute instability of the boundary layer on a rotating cone is determined by pinches between branches 1 & 3, and this is true for cones of all half-angles. The branch structures are identical to those discussed in §2.5 and for this reason are not discussed here.

Pinch-points with $\gamma_i > 0$ have been found for all half-angles, and so the boundary layer on a rotating cone is absolutely unstable for certain values of $R$ and $\beta$ at each half-angle. Figure 4.6 shows the neutral curves of absolute instability in the $(R, \alpha_r)$-, $(R, \alpha_i)$-, $(R, \gamma_r)$- and $(R, \beta)$-planes for each of the half-angles considered. The region
Figure 4.6: Absolute instability neutral curves in the \((R, \alpha_\psi^p), (R, \alpha_\theta^p), (R, \gamma_\psi^p)\) and \((R, \beta)\)-planes for cones with half-angles of \(\psi = 20^\circ - 90^\circ\) in \(10^\circ\) increments (left to right).

enclosed within each curve is absolutely unstable. The figure shows that the onset of absolute instability occurs at a lower Reynolds number for the more slender cones which, for a fixed rotation rate, corresponds to a smaller local radius from the axis of rotation. A better measure of the onset of absolute instability on a rotating cone is the non-dimensional distance along the cone surface from the tip, \(x = R / \sin \psi\), and the critical values of \(R\) and \(x\) at the onset of absolute instability are shown in table 4.4 for each cone. Here we see that, for a fixed rotation rate, the onset of absolute instability occurs further along the more slender cones.

A rotating cone with a half-angle of \(\psi = 90^\circ\) is identical to a rotating disk. In a stationary frame of reference the critical Reynolds number for the onset of absolute instability for this half-angle is \(R = 510.63\), which is identical to that calculated by Lingwood (1995a) for the disk in a rotating frame of reference. This agreement shows that the predicted onset of absolute instability is independent of the frame
Table 4.4: The critical values of $R$ and $x$ for the onset of absolute instability for each cone.

of reference. A comparison between the $\psi = 90^\circ$ neutral curve in figure 4.6 and Lingwood’s neutral curve for the absolute instability of the rotating-disk boundary layer shows that the absolute wavenumbers are also independent of the frame of reference. However, the absolute frequencies are seen to depend on the frame of reference used.

The absolute instability analyses conducted on the rotating-sphere boundary layer in §§2.5 & 3.4 have shown that the local Reynolds number is an important quantity for the onset of absolute instability. By local Reynolds number we mean that based on the distance over the surface of the rotating body from the tip and the local surface velocity at the location under question. The local Reynolds number for the rotating cone was derived in §4.3 to be $R_X = x^2 \Omega^* \sin \psi / \nu^* = R^2 / \sin \psi$.

Figure 4.7 shows a comparison between the predicted onset of absolute instability and experimental measurements for the onset of turbulence reported by Kobayashi
Figure 4.7: A comparison of the predicted critical $R_X$ values for the onset of absolute instability with the transitional values measured by Kobayashi & Izumi (1982). The dashed line shows the transition value on the rotating disk as measured by Lingwood (1996).

& Izumi (1983), both expressed in terms of the local Reynolds number $R_X$. We see that the predicted onset of absolute instability occurs at roughly the same local Reynolds number for all cones, $R_X \approx 2.5 \times 10^5$, which is close to the value calculated for the rotating-sphere boundary layer in §2.5. This means that the boundary layers on the surfaces of rotating cones, disks and spheres (although away from the equator) are all predicted to become absolutely unstable at roughly the same local Reynolds number.

The experimental data in figure 4.7 shows that for cones with $\psi \geq 60^\circ$, the transition point occurs at a local Reynolds number independent of the half-angle and reasonably close to the predicted value of the onset of absolute instability. This close agreement suggests that an absolute instability may well initiate the transition to turbulence for cones with half-angles larger than $60^\circ$. Below $\psi = 60^\circ$ the transitional
Reynolds numbers decrease sharply with decreased half-angle and deviate from the predicted onset of absolute instability. This may indicate that absolute instability does not promote the transition to turbulence for cones with $\psi < 60^\circ$. However, the accuracy of Kobayashi & Izumi's experimental measurements is put into question when we compare their measured onset of turbulence on cones with half-angles close to $\psi = 90^\circ$ with Lingwood's measurement on a rotating disk (Lingwood, 1996). Lingwood's measurement is shown on figure 4.7 by the horizontal dashed line at $\psi = 90^\circ$, and we see that it is very close to our prediction for the onset of absolute instability.

4.7 Conclusion

4.7.1 The validity of the parallel-flow approximation

In the derivation of the governing equations in §4.3, factors

$$h/R = 1 + \eta \cos \psi / R,$$  \hspace{1cm} (4.49)

that multiplied terms in the perturbation equations have been approximated by unity. This approximation is similar to the parallel-flow approximation found in many other boundary-layer investigations, and assumes that $\eta \cos \psi / R \ll 1$. We are now in a position to comment on the validity of this assumption at each half-angle. As shown in figure 4.2, when the boundary layer is fully developed for $\psi = 20^\circ$, $\eta \cos \psi \approx 10$, and this quantity reduces to zero as the half-angle tends to $90^\circ$. This, together with the fact that the critical Reynolds numbers of each instability type decrease with reduced half-angle, means that our approximation is less valid for small half-angles. The convective instability analysis of stationary vortices at
\( \psi = 20^\circ \) shows the critical Reynolds number for the onset of the crossflow mode to be \( R \approx 165 \). This means that the factor (4.49) is approximated by unity with an inaccuracy of about 6%. For the absolute instability calculations at this half-angle the inaccuracy is around 4%. In the light of these estimates, it is in the author’s opinion that the inaccuracy produced by this approximation is not sufficiently large to affect the conclusions of this investigation.

4.7.2 Summary of the results

The convective instability analysis of stationary vortices shows that both crossflow and streamline curvature instabilities occur in the boundary layer over rotating cones at all half-angles. As the half-angle is decreased from \( \psi = 90^\circ \) (rotating disk) the critical Reynolds numbers decrease, and when taking into account the local surface radius, we find that this is equivalent to the instability region moving further along the rotating cone. At all half-angles the ratio of the critical Reynolds numbers for the onset of each convective instability mode is roughly constant, and this implies that the relative importance of streamline curvature on the rotating cone does not change with the half-angle.

The predicted onset of the crossflow mode is seen to match the experimental measurements of Kobayashi & Izumi (1983) well for half-angles above \( \psi = 60^\circ \). However, as the half-angle is reduced the predicted onset of the crossflow mode deviates from these measurements, and at \( \psi = 30^\circ \) the predictions match the experimental measurements of Kreith, Ellis & Giesing (1962) and Kappesser, Greif & Cornet (1973) reasonably well. For cones with half angles between \( \psi = 30^\circ - 60^\circ \), the predicted onset of the crossflow mode lies between the three sets of experimental data, and although the predicted onset of instability is consistent with the three
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sets of experimental data, they do not follow any one set for all half-angles. More experiments are therefore required to clarify our predictions. The measurements conducted by Kappesser et al. are reasonably close to the predicted onset of the streamline-curvature mode when $\psi \geq 60^\circ$, but are close to the predicted onset of the crossflow mode when $\psi \leq 45^\circ$. These observations may suggest that Kappesser et al. were measuring the onset of the streamline-curvature instability mode in their experiments on cones with large $\psi$ and the onset of the crossflow mode on the more slender cones.

The predicted vortex angles and number of vortices at the onset of each instability mode agree with those calculated by Malik (1986) when $\psi = 90^\circ$, and are reduced as the half-angle is decreased. This is consistent with the experimental findings of Kobayashi & Izumi (1983).

Using a different approach in the analysis we have predicted that the vortices rotate with the cone surface at all half-angles. The slow vortices found on the rotating sphere are not predicted to occur on the rotating cone, and these predictions are consistent with all experimental observations of the vortex behaviour on rotating cones.

Expressing the transition points measured by Kobayashi & Izumi (1983) in terms of a local Reynolds number, we find that transition occurs at roughly the same value for cones with $\psi \geq 60^\circ$. We have tried to associate transition to turbulence with the onset of absolute instability, and the predicted onset of the absolute instability is seen to match the experimental values reasonably well at half-angles between $\psi = 60^\circ$–$90^\circ$; for $\psi < 60^\circ$ the discrepancy increases sharply. These comparisons suggest that the transition mechanism for the boundary layer may well be an absolute instability for cones with $\psi > 60^\circ$, and below this half-angle the absolute instability may
be suppressed and other transition mechanisms apply. More experiments will be required in order to investigate the validity of the possible role of absolute instability in the transition, in much the same way as was done by Lingwood (1996) on the rotating disk.

It is interesting to note that the predicted onset of absolute instability occurs at roughly the same local Reynolds number for all cones, \( R_X \approx 2.5 \times 10^5 \), which is close to the value calculated for the rotating-sphere boundary layer in §2.5. This means that the boundary layers on the surfaces of rotating cones, disks and spheres (although away from the equator) are all predicted to become absolutely unstable at roughly the same local Reynolds number.
Chapter 5

The rotating-cone boundary layer in uniform axial flows

This chapter is concerned with the effect of axial flow on the instability mechanisms within the boundary-layer on the surface of rotating cones. Viscous and streamline-curvature effects are included, and local stability analyses are conducted on cones with half-angles of $20^\circ$–$90^\circ$ under various axial flow rates. The formulation is similar to that described in the still fluid analysis presented in chapter 4, and is such that the unsteady perturbation equations are identical to (4.35)–(4.40). This formulation means that the effect of non-zero axial flow is simply to change the basic flow profiles upon which the stability analyses are performed. In contrast to the work on the rotating sphere in axial flow presented in chapter 3, the axial flow parameter used here is based on the local slip velocity at the edge of the boundary layer at the location under consideration and the surface rotation rate at that point. The form of the local slip velocity means that the results cannot be be interpreted in terms of the free-stream axial flow velocity incident on the rotating cone.

The solutions of the steady mean flow and perturbation equations in axial flow
are discussed in §§5.1 & 5.2 respectively. In §5.3 the perturbation equations are solved, and we study the local convective instability of the boundary layer on cones with half-angles between 20°–90° in 10° increments for different values of the local axial flow parameter. The effect of axial flow on the local absolute instability at each half-angle is studied in §5.4.

5.1 The steady mean flow

The formulation of the problem is similar to that described in §4.1, but now the cone rotates in a uniform axial flow. At the edge of the boundary layer the dimensional surface velocity distribution along the cone (i.e. the slip velocity) is $U_\circ^*(x^*)$ and is related to the pressure, $P^*$, by the relation

$$U_\circ^* \frac{\partial U_\circ^*}{\partial x^*} = \frac{1}{\rho^*} \frac{\partial P^*}{\partial x^*}. \tag{5.1}$$

The surface velocity distribution is of the power-law form

$$U_\circ^*(x^*) = C^* x^{*m},$$

which corresponds to potential flow past a cone of half-angle $\psi$, as discussed by Evans (1968). The half-angles which relate to various $m$ are given in table 5.1, and $C^*$ is a constant determined by the free-stream velocity of the axial flow incident on the rotating cone. This inviscid solution of the slip velocity is a good approximation since the boundary-layer separation is parallel to the cone surface.

The steady velocities are non-dimensionalized using the local surface velocity, $x^*\Omega^*\sin\psi$, as

$$U(\eta; \psi) = \frac{U^*}{x^*\Omega^*\sin\psi}, \quad V(\eta; \psi) = \frac{V^*}{x^*\Omega^*\sin\psi}, \quad W(\eta; \psi) = \frac{W^*}{[\nu^*\Omega^*]^{1/2}}, \quad \tag{5.2}$$
Table 5.1: Cone half-angles $\psi$ relating to $m$ in the potential-flow solution.

where $U$, $V$ & $W$ are the non-dimensional velocities in the $x^*$, $\theta$ and $z^*$ directions respectively. The equations that govern the steady mean flow in the boundary layer are stated by Mangler (1945) and are non-dimensionalized using (5.2) as

$$WU' + (U^2 - V^2)\sin \psi = mT_s \Omega_s \sin \psi + U'',$$  
$$WV' + 2UV \sin \psi = V'',$$  
$$W' + 2U \sin \psi = 0,$$

where a prime denotes differentiation with respect to $\eta$. The local axial flow parameter is defined as

$$T_s = \frac{C^* x_s^{*m}}{x_s^* \Omega_s \sin \psi},$$

and is the ratio of the local slip velocity at $x_s^*$ to the speed of the points on the cone surface at that location. Equations (5.3)–(5.5) are based on the usual boundary-layer assumption of large Reynolds number.

The formulation of this problem is such that the local axial flow parameter $T_s$ appears only in the pressure term of the steady-flow equation (5.3). The pressure term is determined completely by the slip velocity, and for this reason the relevant unsteady perturbation equations are identical to (4.35)–(4.40). Details of the deriv-
tion of these perturbation equations can be found in §4.3. This formulation means that the effect of non-zero axial flow is simply to change the steady flow profiles upon which the stability analyses are performed, in much the same way as was done in chapter 3 for the rotating sphere in axial flow. However, it is important to note the distinction between the axial flow parameter used in the rotating sphere analysis and the \textit{local} axial flow parameter used here. The use of $T_s$ means that care must be taken in the interpretation of the results of the stability analyses presented later in this chapter, and this will be discussed in §5.3.1.

Using (5.2) the no-slip boundary condition on the surface of the cone and the condition at the edge of the boundary layer non-dimensionalize to

$$U = W = V - 1 = 0 \quad \text{on} \ \eta = 0,$$

$$V = U - T_s = 0 \quad \text{as} \ \eta \to \infty.$$  \hspace{1cm} (5.6)

With $T_s = 0$, the boundary-layer equations (5.3)–(5.5) and the boundary conditions (5.6) reduce to their zero-axial-flow equivalents given by (4.11)–(4.13) and (4.14) respectively.

Equations (5.3)–(5.5) subject to (5.6) are solved numerically at each $\psi$ and $T_s$ using a fourth-order Runge–Kutta integrator with a Newton–Raphson searching routine; the value of $\psi$ determines $m$ from table 5.1. Figure 5.1 shows the three components of the steady mean flow on a rotating cone with half-angle of $\psi = 70^\circ$ for $T_s = 0.0$–0.25 in increments of 0.05. The steady mean flow profiles for other half-angles behave in a similar manner to those in figure 5.1, and are not shown here.

We note that the behaviour of the $W$-velocity profile is similar to that observed in §3.1, i.e. we see $W$ tending to a constant gradient as $\eta \to \infty$ for non-zero $T_s$. The gradient changes with $T_s$ and can be calculated from (5.5) using the second
Figure 5.1: The steady mean flow velocity profiles for $\psi = 70^\circ$ with $T_s = 0.00-0.25$ in 0.05 increments. For $U$, increasing $T_s$ moves the profiles left to right; for $V$ & $W$, right to left.

condition of (5.6), i.e.

$$\frac{\partial W}{\partial \eta} \sim -2T_s \sin \psi.$$ 

Physically, this behaviour cannot be maintained arbitrarily far from the surface, as it predicts that fluid is entrained into the boundary layer with unbounded speed as $\eta$ is increased, and is of course a consequence of the boundary-layer approximations. Very close to the cone surface, figure 5.1 shows that the flow is independent of the local axial flow parameter.

As with the analysis presented in chapter 3, here the boundary layer will be approximated by the region $0 \leq \eta \leq 20$, and so the $W$-profiles used are those shown in figure 5.1. It is important to note that the region is sufficiently large for the velocity components $U$ and $V$ to be fully developed. In the governing equations, $W$
is scaled on the Reynolds number \( R \), which is necessarily large in the boundary-layer approximation, so that the cramping of the boundary layer should not cause major inaccuracies. In any event, different computational boundary-layer thicknesses were tested in order to check that an outer boundary of \( \eta = 20 \) did indeed give converged unsteady flow results.

## 5.2 Solution of the perturbation equations

We now discuss the amendments to the methods described in §2.3 that will be used to solve the eigenvalue problem defined by (4.35)–(4.40) with the boundary conditions (2.37) in the case of non-zero axial flow. Apart from the form of the perturbation equations at the edge of the boundary layer, this description is identical to that given in §§3.2 & 4.4.

The eigenvalue problem will be solved for certain combinations of values of \( \alpha, \beta \) and \( \gamma \) at each Reynolds number \( R \), and for particular values of \( \psi \) and \( T_s \). From these we form the dispersion relation for the rotating cone in axial flow, \( D_C(\alpha, \beta, \gamma; R, \psi, T_s) = 0 \), at each \( \psi \) and \( T_s \), with the aim of calculating the instability branches.

As in §2.3, we assume that the disturbance quantities tend to zero exponentially as \( \eta \rightarrow \infty \). An approximate solution at the outer edge of the boundary layer can be found by solving the perturbation equations at the edge of boundary layer

\[
\phi_1' = \phi_2, \quad (5.7)
\]

\[
\phi_2' = \left[ \vec{\alpha}^2 + \vec{\beta}^2 - i R \gamma + U_\infty \sin \psi \right] \phi_1 + W_\infty \phi_2 \quad (5.8)
\]

\[
+ i R \left[ \vec{\alpha}^2 + \vec{\beta}^2 - \frac{i \vec{\alpha}}{R} \right] \phi_4,
\]
\begin{equation}
\phi_3' = -i \phi_2' + \frac{\phi_3 \cos \psi}{R}, \quad (5.9)
\end{equation}

\begin{equation}
Re \phi_4' = i W_\infty \phi_1 - i \phi_2 - [\alpha^2 + \beta^2 + i R (\alpha U_\infty - \gamma) + W'_\infty] \phi_3, \quad (5.10)
\end{equation}

\begin{equation}
\phi_5' = \phi_6, \quad (5.11)
\end{equation}

\begin{equation}
\phi_6' = \overline{\beta} \sin \psi \phi_4 + [\alpha^2 + \beta^2 - i R \gamma + U_\infty \sin \psi] \phi_5 + W_\infty \phi_6. \quad (5.12)
\end{equation}

Equations (5.7)–(5.12) arise from the full perturbation equations (4.35)–(4.40) using the mean flow variables at \( \eta = 20 \), which is denoted by the subscript \( \infty \). Note that for non-zero axial flow, both \( U_\infty \) and \( W_\infty \) are non-zero. Following the method described in §2.3, (5.7)–(5.12) permit a solution of the form \( c_i e^{\kappa_j \eta} \), and substitution of this assumed form enables the numerical solution for \( \kappa_j \) for each independent solution defined by \( j \). From these we form an approximate solution of the perturbation equations at \( \eta = 20 \) for each of the transformed variables denoted by \( i \). Note that since we require an exponentially decaying solution we use only the solutions with \( \kappa_j < 0 \). Numerically integrating the full perturbation equations from these initial solutions down towards \( \eta = 0 \) enables the correct eigenvalues to be calculated; details of this process are given in §2.3. Extensive experimentation with the maximum value of \( \eta \) chosen has shown that the eigenvalues are independent of the domain as long as it allows the fully developed \( U \) and \( V \) velocity components to be used, i.e. \( \eta > 15 \).

### 5.3 The convective instability analysis

In this section we solve the perturbation equations (4.35)–(4.40) with the aim of studying the occurrence of convective instabilities in the case of non-zero axial flow over rotating cones. Since we are supposing in the first instance that the flow is not absolutely unstable, it follows that in the Briggs–Bers procedure we can reduce
the imaginary part of the frequency down to zero, so that $\gamma_i = 0$. To investigate convective instability we use the approaches described in §4.5. In §5.3.1 we insist that the vortices rotate with the surface of the cone, i.e. we fix $\gamma_r/\bar{\beta} = 1$, and in §5.3.2 a further approach is taken in which we investigate the vortex speed under increased local axial flow rates.

5.3.1 Stationary vortices

We begin by considering vortices that rotate with the surface of the cone, i.e. we fix $\gamma_r/\bar{\beta} = 1$ and follow the method described in §4.5. Two spatial branches were found that determine the convective instability characteristics for each value of the local axial flow parameter at each half-angle. These branches arise from crossflow and streamline-curvature instability modes, and are identical to branches 1 & 2 found on the rotating sphere at latitudes $\theta < 66^\circ$ (see §2.4.1). For this reason we do not discuss them here.

Figure 5.2 shows the neutral curves in the $(R, \alpha_r)$- and $(R, \bar{\beta})$-planes for a cone with half-angle $\psi = 70^\circ$ for $T_s = 0.05$–$0.25$ in $0.05$ increments. The neutral curves calculated for half-angles between $\psi = 20^\circ$–$90^\circ$ can be found in §A.3. Each curve encloses a region that is convectively unstable. The two lobed structure is seen in the neutral curves for each $T_s$ for all half-angles above $\psi = 20^\circ$. As discussed in §4.5.1, the upper lobe arises from crossflow instabilities and the lower lobe from streamline-curvature instabilities. The two-lobed structure of the neutral curves is found to be exaggerated by the local axial flow for $\psi > 20^\circ$, with the streamline-curvature lobe becoming increasingly important relative to the crossflow lobe with increased $T_s$. The exaggeration of the two-lobed structure is greatest for $\psi = 90^\circ$, but becomes less apparent as the half-angle is reduced. For $\psi = 20^\circ$, increasing $T_s$ from 0 to 0.15
Figure 5.2: Neutral curves for the convective instability of stationary vortices on cone with half-angle $\psi = 70^\circ$ with $T_s = 0.00$–$0.25$ (left to right) in 0.05 increments.

produces a slight exaggeration of the two-lobed structure, but increasing $T_s$ above this value removes the cusp separating the two lobes very rapidly, and for $T_s > 0.20$ the neutral curves are single lobed. In contrast to the single-lobed neutral curves found on the rotating sphere in \S\S\ 2.4.1 & 3.3.1, the single lobe found here is due to the crossflow instability with the streamline-curvature instability defining the shape of the lower branch. The observation that an increased local axial flow parameter increases the strength of the streamline-curvature mode is sensible, as figure 5.1 shows that axial flow increases the amount of fluid entrained into the boundary layer, this would lead to more streamline curvature. However, the reasons for the behaviour of the boundary layer for a half-angle of $\psi = 20^\circ$ are not apparent.

The critical Reynolds number for the onset of the crossflow and streamline-curvature modes at each half-angle are shown in tables 5.2 & 5.3 respectively for $T_s = 0.05$–$0.25$. These results show that axial flow stabilizes the boundary layer to
<table>
<thead>
<tr>
<th>$T_s$</th>
<th>20°</th>
<th>30°</th>
<th>40°</th>
<th>50°</th>
<th>60°</th>
<th>70°</th>
<th>80°</th>
<th>90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>175.70</td>
<td>209.57</td>
<td>238.50</td>
<td>264.47</td>
<td>279.46</td>
<td>292.89</td>
<td>302.03</td>
<td>306.99</td>
</tr>
<tr>
<td>0.1</td>
<td>179.26</td>
<td>217.98</td>
<td>249.24</td>
<td>275.24</td>
<td>296.89</td>
<td>314.55</td>
<td>328.50</td>
<td>338.84</td>
</tr>
<tr>
<td>0.15</td>
<td>185.56</td>
<td>226.54</td>
<td>261.01</td>
<td>291.07</td>
<td>317.67</td>
<td>341.02</td>
<td>361.40</td>
<td>378.91</td>
</tr>
<tr>
<td>0.2</td>
<td>195.65</td>
<td>235.53</td>
<td>273.84</td>
<td>308.84</td>
<td>341.42</td>
<td>371.62</td>
<td>399.67</td>
<td>425.74</td>
</tr>
<tr>
<td>0.25</td>
<td>207.05</td>
<td>244.95</td>
<td>287.71</td>
<td>328.40</td>
<td>367.82</td>
<td>405.80</td>
<td>442.58</td>
<td>478.28</td>
</tr>
</tbody>
</table>

Table 5.2: The critical Reynolds numbers $R$ for the onset of the crossflow mode of convective instability for half-angles $\psi = 20^\circ$–70° and local axial flow parameters $T_s = 0.05$–0.25.

<table>
<thead>
<tr>
<th>$T_s$</th>
<th>20°</th>
<th>30°</th>
<th>40°</th>
<th>50°</th>
<th>60°</th>
<th>70°</th>
<th>80°</th>
<th>90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>260.56</td>
<td>308.24</td>
<td>347.51</td>
<td>380.25</td>
<td>403.63</td>
<td>421.44</td>
<td>432.81</td>
<td>437.95</td>
</tr>
<tr>
<td>0.1</td>
<td>245.36</td>
<td>294.84</td>
<td>335.65</td>
<td>369.29</td>
<td>395.72</td>
<td>416.32</td>
<td>431.22</td>
<td>440.79</td>
</tr>
<tr>
<td>0.15</td>
<td>233.93</td>
<td>285.30</td>
<td>328.24</td>
<td>365.06</td>
<td>394.95</td>
<td>420.21</td>
<td>440.77</td>
<td>457.20</td>
</tr>
<tr>
<td>0.2</td>
<td>-</td>
<td>278.42</td>
<td>324.13</td>
<td>365.91</td>
<td>399.45</td>
<td>430.70</td>
<td>458.53</td>
<td>483.64</td>
</tr>
<tr>
<td>0.25</td>
<td>-</td>
<td>273.62</td>
<td>322.47</td>
<td>367.56</td>
<td>408.07</td>
<td>446.37</td>
<td>482.67</td>
<td>517.83</td>
</tr>
</tbody>
</table>

Table 5.3: The critical Reynolds numbers $R$ for the onset of the streamline-curvature mode of convective instability for half-angles $\psi = 20^\circ$–90° and local axial flow parameters $T_s = 0.05$–0.25, - indicates that the streamline curvature lobe is not seen.
the more dangerous instability mode at each half-angle. If the onset of the more
dangerous instability mode is expressed in terms of \( x = R / \sin \psi \), we find that for
a fixed rotation rate, the increased local axial flow parameter delays the onset of
instability along cones of each half-angle.

As the analysis moves along the cone the local surface radius is increased which
leads to an increased local surface velocity (assuming a fixed rotation rate). The
local slip velocity at the edge of the boundary layer also changes with \( x_s \), but not
necessarily at the same rate as the change in the surface velocity. Keeping \( T_s \)
fixed along the cone in each analysis is therefore not the same as keeping \( C^* \) fixed
and means that the free-stream axial flow incident on the cone is not consistent.
Interpretations of the results in terms of an increasing free-stream velocity for each
cone are therefore not possible. Comparing predictions between cones with different
half-angles but for the same \( T_s \) is also difficult. The onset of instability occurs at a
larger local radii on cones with large half-angles due to the geometry of the problem
(see §4.5), and this corresponds to a larger surface velocity (assuming a fixed rotation
rate). The behaviour of the local slip velocity with \( \psi \) will not balance this, and a
larger free-stream velocity (i.e. different \( C^* \)) is required to maintain a fixed value
of \( T_s \). Note that similar considerations were not required for the rotating-sphere
analysis of chapter 3. This is because the axial flow parameter used there was based
on the free-stream velocity and equatorial rotation speed, i.e. not based on the local
properties of the sphere which change as the location of the analysis moves over the
sphere.

The results obtained in this analysis show that \( T_s = 0.25 \) is not sufficiently large
for the streamline-curvature mode to dominate and become the most dangerous
mode for any half-angle. However, there is no reason to suggest that increasing
the local axial flow parameter above this value should not eventually cause the dominance of the streamline-curvature mode.

Experiments to measure the onset of spiral vortices on cones rotating in axial flow have been conducted by Salzberg & Kezios (1965) and Kobayashi et al. (1983) on cones with a half-angle of $\psi = 15^\circ$ only. As discussed in §4.7.1, the approximations made in this analysis are such that the inaccuracy of our results increases with decreased half-angle, and so analyses on such a slender cone have not been conducted. We are therefore unable to compare our predicted onset of convective instability with experimental data for the onset of the spiral vortices. However, the experimental data does show that the critical Reynolds numbers increase with $T_s$, and our results for each half-angle are consistent with this observation.

Figures 5.3 and 5.4 respectively show the predicted values of the vortex angle, $\epsilon$, and number of vortices, $n$, against $T_s$, calculated from (4.27) and (4.28). For
Figure 5.4: The number of vortices against local axial flow parameter at the onset of each instability mode type for cones with half-angles $\psi = 20^\circ$–$90^\circ$ (bottom to top).

Each half-angle, and for each type of instability mode, both $\epsilon$ and $n$ are predicted to increase with the local axial flow parameter. For reasons discussed in §2.4, the values of $n$ are used to show the general trends in the behaviour of the number of vortices, and are not predictions of the actual number. Note that the predicted values of $\epsilon$ and $n$ at each value of the local axial flow parameter increase with increased half-angle, which is consistent with the zero-axial flow results of §2.4.1. Unfortunately, no experimental data can be found concerning the orientation and number of vortices on cones with $\psi > 15^\circ$ rotating in a non-zero axial flow, and these theoretical predictions cannot be compared with experimental data. However, the results of Kobayashi et al. do show that for $\psi = 15^\circ$, the number and vortex angle increase with $T_i$ and our results are consistent with this observation.
\[
\begin{array}{|c|c|c|c|c|}
\hline
T_s & R & \gamma_r & n & c \\
\hline
0.05 & 292.9 & 0.0874 & 26 & 0.985 \\
0.15 & 341.0 & 0.1222 & 42 & 0.992 \\
0.25 & 405.8 & 0.1639 & 65 & 1.023 \\
\hline
\end{array}
\]

Table 5.4: Critical parameters for a cone with half-angle $\psi = 70^\circ$ with $T_s = 0.05, 0.15 \& 0.25$ for neutral curves defined by fixing $n$ at various integer values.

### 5.3.2 A prediction of the vortex speed

To investigate the effect of axial flow on the vortex speed we follow the method applied to the rotating-cone boundary layer in §4.5.2. Recall that this involves plotting the neutral curves defined by $\gamma_i = a_i = 0$ that correspond to each integer $n$. The ‘global’ neutral curve at a particular $x_s^*$ would then be the envelope of the neutral curves pertaining to each single $n$.

This approach has been taken on cones with half-angles between $\psi = 20^\circ$–$90^\circ$ for $T_s = 0.05$–$0.25$ in increments of $0.05$. Figure 5.5 shows that for $\psi = 70^\circ$, the curves have a single minimum for each value of $T_s$ shown. This is found to be the case for all half-angles and local axial flow parameters, and so it is sufficient to show only this figure. The minimum of each enveloping neutral curve corresponds to the critical Reynolds number of the crossflow lobe in each case, and are identical to those calculated in §5.3.1 when $\gamma_r/\bar{\beta}$ was fixed at unity. The critical values of $\gamma_r$ and $R$ at the minimum of each curve lead to the prediction that $c = \gamma_r R/n \approx 1.0$ as can be seen in table 5.4. This agreement shows that fixing the longitudinal wave speed to $c = 1.0$, as was done in §5.3.1, is correct. As the curves for all $\psi$ and $T_s$ have a single minimum, we do not predict slow vortices for the rotating-cone boundary layer in axial flow. This is consistent with all experimental observations of these
Figure 5.5: Critical Reynolds numbers, $R$, for a cone with half-angle $\psi = 70^\circ$ with $T_i = 0.05, 0.15 & 0.25$ (bottom to top) for neutral curves defined by fixing $n$ at various integer values.

boundary layers, for example Kobayashi et al. (1983) and Kohama (1984a), but is in contrast to the results of figure 3.8 calculated for the rotating-sphere boundary layer in axial flow.

5.4 The absolute instability analysis

In this section we solve the perturbation equations (4.35)-(4.40) subject to the boundary conditions (2.37) with the aim of studying the effect of axial flow on the absolute instability of rotating-cone boundary layers. The Briggs–Bers criterion is applied with fixed $\beta$ to distinguish between convectively and absolutely unstable time-asymptotic responses to an initial boundary-value perturbation, exactly as was done in §4.6. To enable the spatio-temporal analysis to be completed, $\alpha$ and $\gamma$ are both complex quantities, while $\beta$ remains real in order to enforce periodicity round
Table 5.5: The critical Reynolds numbers $R$ for the onset of absolute instability for half-angles $\psi = 20^\circ$–$90^\circ$ and local axial flow parameters $T_s = 0.05$–$0.25$.

The absolute instability of the boundary layer is determined by pinches between branches 1 & 3, and this is true for cones of all half-angles for each value of the local axial flow parameter. The branch structures are identical to those discussed in §2.5, and for this reason we do not discuss them here.

Pinch-points with $\gamma_i > 0$ have been found for all half-angles for each value of the local axial flow parameter, and so the boundary layers on cones rotating in a uniform axial flow are absolutely unstable for certain values of $R$ and $\beta$ for each half-angle. The neutral curves for absolute instability in each case are of a similar shape to those calculated in §4.6 and are not shown here.

The critical Reynolds numbers for the onset of absolute instability for each half-angle are shown in table 5.5 for local axial flow parameters of $T_s = 0.05$–$0.25$ in increments of 0.05. These results show that axial flow stabilizes the boundary layer to absolute instabilities for each half-angle, i.e. the critical Reynolds numbers are increased with increased $T_s$ for each $\psi$.

For reasons discussed in §5.3.1, the use of the local axial flow parameter, $T_s$,
Figure 5.6: Predicted critical $R_X$ values for absolute instability against $T_s$ for $\psi = 20^\circ$–$90^\circ$ in $10^\circ$ increments (bottom to top).

means that the critical local Reynolds numbers, $Re_X$, for cones with different half-angles cannot be directly compared. Therefore a comparison like that shown in figure 3.10 cannot be made between the results for different values of $\psi$. Instead, in figure 5.6 we plot the critical local Reynolds numbers against the local axial flow parameter for each $\psi$. This figure shows that increasing $T_s$ increases the local Reynolds numbers found for each $\psi$. As the use of $T_s$ leads to inconsistent free-stream axial flow rates incident on each cone, it should not be expected that the critical Reynolds numbers are independent of $\psi$ for each non-zero $T_s$, and this is indeed the case in figure 5.6.

Experiments that measure the onset of turbulence on cones rotating in axial flow have been conducted by Salzberg & Kezios (1965) and Kobayashi et al. (1983) on cones with a half-angle of $\psi = 15^\circ$ only. We are therefore unable to compare our predicted onset of absolute instability with experimental data for the onset
of turbulence. However, the experimental data shows that the critical Reynolds numbers increase with the local axial flow parameter, and our results for each half-angle are consistent with this observation.

Although the boundary layers in this analysis have been predicted to become absolutely unstable for each local axial flow parameter, without comparisons with experimental data we are unable to draw any conclusions about whether this mechanism initiates transition to turbulence.

5.5 Conclusion

In this chapter we have followed the formulation of chapter 4 by scaling the boundary-layer variables on the boundary layer thickness and the surface rotation speed at a distance \( x_s^* \) along the cone. This formulation allows the introduction of axial flow into the problem by continuously increasing the local axial flow parameter \( T_s = C^*x_s^{*m}/x_s^*\Omega^* \sin \psi \) from zero. By using this formulation the perturbation equations relevant to the axial flow problem are identical to (4.35)–(4.40). The effect of non-zero axial flow on the analysis is therefore to change the basic flow profiles upon which the analysis is performed. The use of the local axial flow parameter means that the interpretation of the results in terms of a free-stream velocity are not possible.

For stationary vortices we have seen that axial flow stabilises the boundary layer with respect to convective instabilities. This was shown by the increased critical Reynolds numbers for the onset of each instability mode with increased local axial flow at each half-angle. The two-lobed structure of the neutral curves is found to be exaggerated by the local axial flow for \( \psi > 20^\circ \), with the streamline-curvature lobe
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becoming increasingly important relative to the crossflow lobe with increased $T_s$. The exaggeration of the two-lobed structure is greatest for $\psi = 90^\circ$, but becomes less apparent as the half-angle is reduced. For $\psi = 20^\circ$, increasing $T_s$ from 0 to 0.15 produces a slight exaggeration of the two-lobed structure, but increasing $T_s$ above this value removes the streamline-curvature lobe very rapidly, and for $T_s > 0.20$ we see only the crossflow lobe. In contrast to the single-lobed neutral curves found on the rotating sphere in §§2.4.1 & 3.3.1, the single lobe found here is due to the crossflow instability with the streamline-curvature instability defining the shape of the lower branch. For each half-angle, and for both convective instability mode types, both $\epsilon$ and $n$ are predicted to increase with the local axial flow parameter.

To investigate the possible variation of the vortex speed, another approach was taken in which the neutral curves defined by $\gamma_i = \alpha_i = 0$ that correspond to each integer $n$ (the circumferential mode order) were plotted. The global neutral curve at a particular $x^*_i$ is then the envelope of the neutral curves pertaining to each single $n$, and the corresponding vortex speed is predicted from the critical values of $\gamma_r$ and $R$ using $c = \gamma_r R/n$. The experimental observations of Kobayashi et al. (1983) and Kohama (1984a) suggest that the vortices are fixed on the surface of rotating cones of each half-angle for each local axial flow rate, and our predictions of $c$ are consistent with this observation.

We have also seen that axial flow stabilises the boundary layer to absolute instability. This was shown by the critical Reynolds numbers increasing with the local axial flow parameter on cones of each half-angle.

Experiments that measure the onset of spiral vortices and the transition to turbulence on cones rotating in axial flow have been conducted by Kobayashi et al. (1983) and Kohama (1984a) on cones with a half-angle of $\psi = 15^\circ$ only. As discussed in
§4.7.1, the approximations made in this analysis are such that the inaccuracy of our results increases with decreased half-angle, and so analyses on such a slender cone have not been conducted. We are therefore unable to compare our theoretical predictions for the onset of convective and absolute instabilities with experimental measurements for the onset of spiral vortices and turbulence. However, the predictions for each half-angle are consistent with the experimentally observed behaviour of the boundary layer for $\psi = 15^\circ$. Although the boundary layers in this analysis have been predicted to become absolutely unstable for each local axial flow parameter, without comparisons with experimental data we are unable to draw any conclusions about whether this mechanism initiates transition to turbulence.
Chapter 6

A preliminary investigation into linear global modes

Parallel-flow approximations have been used in the analyses of the rotating-sphere and rotating-cone boundary layers presented in chapters 2–5, and so were previously restricted to the local stability characteristics of each flow. However, in this chapter we are concerned with the global behaviour of each boundary-layer type, and calculate leading-order estimates of the linear global mode frequency using an inviscid analysis. We begin by considering the rotating-disk/cone boundary layers in §6.1 where we introduce the techniques that are also used in the slightly more complicated case of the rotating-sphere boundary layer in §6.2.

6.1 The rotating-disk/cone boundary layers

6.1.1 The method

N. Peake (personal communication, 2001) has formulated the inviscid stability problem of the rotating-cone boundary layer in terms of a slow spatial variable
\( X = \varepsilon x \), where \( x \) is the non-dimensional spatial variable along the cone surface and 
\( \varepsilon = (\nu^* / \Omega^*)^{1/2} / a^* \ll 1 \) is a small parameter with \( a^* \) a characteristic length. Details of this formulation can be found in appendix C.4.1 where the variable \( S \) is identical to \( X \) used here. Using Peake’s formulation the non-dimensional Rayleigh equation is derived as

\[
(k \tilde{X} U + V - \tilde{\omega}) \left( \phi_3'' - \left\{ k^2 + \frac{1}{X^2} \right\} \phi_3 \right) - \left( k \tilde{X} U'' + V'' \right) \phi_3 = 0, \tag{6.1}
\]

where a prime denotes differentiation with respect to the wall-normal coordinate \( \eta \). A tilde denotes a scaled variable (as defined in (C.35)) that enables the circumferential wavenumber to be scaled out of the governing stability equation. The wavenumber along the cone is \( k \), and \( \tilde{\omega} \) is the scaled frequency, both are complex quantities; \( \phi_3 \) is the eigenfunction defined by (4.17) and the basic flow quantities \( U \) and \( V \) are identical to those discussed in §4.2.

The long-time behaviour of the response to a disturbance is determined by the uppermost singularity in the \( \tilde{\omega} \)-plane that corresponds to zeros in the dispersion relation. In a strictly parallel-flow problem the pole is determined by the absolute frequency \( \tilde{\omega}^* = \tilde{\omega}(k^*) \), which is defined as the frequency at which two spatial branches in the complex \( k \)-plane coalesce in accordance with the Briggs–Bers criterion (see §2.5). However, if the flow is weakly non-parallel, then there exists a value of \( \tilde{\omega}^* \) for each value of \( \tilde{X} \) on the real axis, and in an analogous way the response becomes singular at some point in the complex \( \tilde{X} \)-plane. According to Monkewitz et al. (1993) this occurs at the saddle point, \( \tilde{X}_s \), where the following conditions are met:

\[
\frac{\partial \tilde{\omega}}{\partial \tilde{X}}(k_s^*, X_s) = 0, \quad \frac{\partial \tilde{\omega}}{\partial k}(k_s^*, X_s) = 0, \quad \tilde{\omega}_s^* = \tilde{\omega}(k_s^*, X_s), \tag{6.2}
\]

and the value of \( \tilde{\omega}_s^* \) gives a leading-order estimate of the global frequency.
The location of the saddle point can be identified by analytically continuing \( \tilde{\omega}^o(\tilde{X}) \) from the real \( \tilde{X} \)-axis into the complex \( \tilde{X} \)-plane using some appropriate approximating function. Here we follow Cooper & Crighton (2000) and use a rational-function approximation in which the calculated values of \( \omega_s(\tilde{X}_s; \psi) \) are approximated by

\[
R_m(\tilde{X}; \psi) = \frac{a_0 + a_1 \tilde{X} + a_2 \tilde{X}^2 + \cdots + a_m \tilde{X}^m}{1 + b_1 \tilde{X} + b_2 \tilde{X}^2 + \cdots + b_m \tilde{X}^m}. \tag{6.3}
\]

As reported by Cooper & Crighton, this type of function tends to be better behaved than, for example, a high-order polynomial when extended into the complex plane and so is to be preferred.

The method used to obtain the best rational-function fit involves a number of steps. A version of Remes algorithm (Ralston & Wilf, 1960) is used to obtain a minimax solution i.e. values of \( a_i \) and \( b_i \) which minimize the maximum deviation \( r_m \), where

\[
r_m = \max |R_m(\tilde{X}; \psi) - \tilde{\omega}^o(\tilde{X}; \psi)|.
\]

For each value of \( m \) there exists a minimax solution and the minimum value of \( r_m \) is selected as giving the best fit. A computer code has been written to calculate the minimax solution that is based on routines given in Press et al. (1992), and an estimate of the error can be made by the root mean square

\[
E_m = \sqrt{\frac{\sum_{j=1}^{N} |R_m(\tilde{X}_j; \psi) - \tilde{\omega}^o(\tilde{X}_j; \psi)|^2}{N}}, \tag{6.4}
\]

with \( N \) the number of data points used along the \( \tilde{X} \)-axis. Using (6.4) we are able to choose the order of the rational function that best approximates the data along the \( \tilde{X} \)-axis. The saddle point is then found by plotting zero contours of the real and imaginary parts of \( \partial \tilde{\omega}^o / \partial \tilde{X} \) in the complex \( \tilde{X} \)-plane obtained from the rational-function approximation. Points where the lines cross identify the values of \( \tilde{X}_s \) at
Figure 6.1: Locus of the absolute frequency $\tilde{\omega}^o$ as $\tilde{X}$ varies for a half-angle of $\psi = 90^\circ$ (i.e. the rotating disk).

which $\partial \tilde{\omega}^o / \partial \tilde{X}$ vanishes exactly, i.e. satisfy the conditions of (6.2).

### 6.1.2 Leading-order estimates of the global frequency

The absolute frequencies $\tilde{\omega}^o(\tilde{X})$ for real $\tilde{X}$ can be found by solving (6.1) for the dispersion relation and applying the Briggs-Bers criterion to calculate the location of the pinch point at each $\tilde{X}$. However, we note that by using the transformation

$$\tilde{\omega} = \frac{\gamma}{\beta}, \quad k = \alpha \sin \psi, \quad \tilde{X} = \frac{1}{\beta},$$  \hspace{1cm} (6.5)

equation (6.1) is transformed into the Rayleigh equation (4.42), i.e. is written in terms of the original variables of chapter 4. It is therefore possible to transform the absolute frequencies calculated by solving the Orr-Sommerfeld equation (4.41) at high Reynolds number into the required $\tilde{\omega}^o(\tilde{X})$.

Figure 6.1 shows $\tilde{\omega}^o(\tilde{X})$ for a half-angle of $\psi = 90^\circ$ (i.e. the rotating disk) that was calculated by transforming data from the solution of (4.41); a Reynolds number
Figure 6.2: Contours of real and imaginary parts of $\partial \omega^0 / \partial \tilde{X} = 0$ in the complex $\tilde{X}$-plane for $\psi = 90^\circ$. Solid line $m = 4$; dashed line $m = 5$; dotted line $m = 6$.

of $R = 15000$ was required to get convergence of the Orr–Sommerfeld results with Reynolds number, i.e. to converge onto the inviscid Rayleigh solution. Using the methods described in §6.1.1, a rational function has been fitted to this data and the problem immersed in the complex $\tilde{X}$-plane. Note that as $\tilde{X} \to \infty$, (6.1) reduces to the Rayleigh equation for a two-dimensional shear flow in the radial direction and is therefore not absolutely unstable. This is consistent with the behaviour of the $\omega_i^0$ curve in figure 6.1.

Figure 6.2 shows zero contours of the real and imaginary parts of $\partial \omega^0 / \partial \tilde{X}$ in the complex $\tilde{X}$-plane obtained from three orders of the rational-function approximation. A comparison of the saddle-point locations and leading-order global frequencies predicted for each $m$ is made in table 6.1, where an indication of the accuracy of

\footnote{This solution has since been confirmed by comparison with the actual numerical solution of (6.1) calculated using the code described in §6.2.}
Table 6.1: Comparison of the saddle-point location and leading-order global frequency at each \( m \) for the rotating-disk boundary layer \((\psi = 90^\circ)\).

the rational-function approximation on the real \( \tilde{X} \)-axis is also given from (6.4). We see that \( m = 4 \) gives the most accurate rational-function fit, and for this reason we consider \( m = 4 \) to give the most accurate prediction of the leading-order global frequency. At all values of \( m \), \( \tilde{\omega}_s^o \) has negative imaginary part and we predict that the rotating-disk boundary layer sustains damped global modes.\(^2\)

Similar calculations have been conducted for half-angles of \( \psi = 20^\circ-80^\circ \) in \( 10^\circ \) increments, and table 6.2 presents the location and leading-order global frequency calculated with the most accurate rational-function approximation at each \( \psi \). Note that in Peake’s inviscid formulation the half-angle enters (6.1) through the \( U \) and \( V \) components of the steady mean flow only, and from figure 4.2 we see that both \( U \) and \( V \) are insensitive to the half-angle. It is therefore of no surprise that the predicted values of \( \tilde{\omega}_s^o \) do not change with \( \psi \), however the actual growth rates may be different. From these results we predict that disturbances in the rotating-cone boundary layer are globally damped for all half-angles.

\(^2\)Although \( \tilde{\omega}_s^o \) is the scaled global growth rate, (6.5) indicates that it is scaled by a positive quantity and so has the same sign as the actual global growth rate.
<table>
<thead>
<tr>
<th>$\psi$</th>
<th>$\bar{X}_3$</th>
<th>$\bar{\omega}'_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20°</td>
<td>5.7812 - 21.9720i</td>
<td>0.7878 - 0.0150i</td>
</tr>
<tr>
<td>30°</td>
<td>5.9505 - 22.0555i</td>
<td>0.7927 - 0.0150i</td>
</tr>
<tr>
<td>40°</td>
<td>6.0673 - 22.1804i</td>
<td>0.7999 - 0.0151i</td>
</tr>
<tr>
<td>50°</td>
<td>6.1650 - 22.2719i</td>
<td>0.8128 - 0.0150i</td>
</tr>
<tr>
<td>60°</td>
<td>6.2505 - 22.3542i</td>
<td>0.8329 - 0.0150i</td>
</tr>
<tr>
<td>70°</td>
<td>6.3654 - 22.4324i</td>
<td>0.8641 - 0.0151i</td>
</tr>
<tr>
<td>80°</td>
<td>6.4624 - 22.5658i</td>
<td>0.8756 - 0.0149i</td>
</tr>
<tr>
<td>90°</td>
<td>6.5783 - 22.6544i</td>
<td>0.8887 - 0.0150i</td>
</tr>
</tbody>
</table>

Table 6.2: Comparison of the saddle-point location and leading-order global frequency for cones of half-angle $\psi$.

### 6.2 The rotating-sphere boundary layer

The rotating-sphere boundary layer has a spatial dependence of the basic state in the latitudinal direction and so should be amenable to a similar type of analysis to that of §6.1. Indeed, Peake (personal communication, 2001) has formulated the inviscid stability problem in terms of a slow spatial variable in the latitudinal direction, $S = \epsilon \theta$, where $\theta$ is the latitudinal angle and $\epsilon = (\nu^*/\Omega^*)^{1/2}/a^* \ll 1$ is a small parameter with $a^*$ the sphere radius. Details of this formulation can be found in appendix C.4.2. Using Peake's formulation the non-dimensional Rayleigh equation is derived as

$$
\left( kU + \frac{\bar{n}}{\sin S} V - \omega \right) \left( \frac{\partial^2 \phi_3}{\partial \eta^2} - \left\{ k^2 + \frac{\bar{n}}{\sin^2 S} \right\} \phi_3 \right) - \left( k \frac{\partial^2 U}{\partial \eta^2} + \frac{\bar{n}}{\sin S} \frac{\partial^2 V}{\partial \eta^2} \right) \phi_3 = 0.
$$

(6.6)
The wavenumber in the latitudinal direction is \( k \), and \( \omega \) is the frequency, both are complex quantities; \( \phi_3 \) is the eigenfunction defined by (2.25) and the basic flow quantities \( U \) and \( V \) are identical to those discussed in §2.1.

Extra complications arise in the analysis of the rotating-sphere boundary layer since the basic flows vary with the slow parameter, this is in contrast to the rotating-cone analysis where a similarity solution exists for each \( \psi \). It is therefore not possible to obtain the absolute frequency by transforming the local solutions calculated in chapter 2. Further, we cannot remove the dependence of the absolute frequency on the ‘slow’ azimuthal wavenumber, \( \bar{n} = en \), by suitable scaling. We therefore require a numerical solution of (6.6) to calculate \( \omega^a(\mathcal{S}; \bar{n}) \), and consider the properties of each global mode defined by \( \bar{n} \).

The homogeneous boundary conditions to (6.6) are given by

\[
\begin{align*}
\phi_3 &= 0 \quad \text{on} \quad \eta = 0, \\
\phi_3 &\rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty.
\end{align*}
\]

(6.7)

For values of \( \mathcal{S} \) between 5°–90° in 0.5° increments the absolute frequency has been calculated by solving the eigenvalue problem defined by (6.6) & (6.7) and then applying the Briggs–Bers criterion. The eigenvalue problem is solved using a Newton–Raphson linear search procedure and a fourth-order Runge–Kutta integrator starting from an analytic solution at the edge of the boundary layer (approximated by \( \eta = 20 \)). The methods used are very similar to those described in §2.2. At each \( \mathcal{S} \) the NAG routine discussed in §2.1 is used to calculate the basic flow quantities \( U(\eta, \mathcal{S}) \) and \( V(\eta, \mathcal{S}) \).

Figure 6.3 shows the calculated \( \omega^a(\mathcal{S}; \bar{n}) \) for \( \bar{n} = 0.05–0.25 \) in increments of 0.05. Note that a region of absolute instability is only predicted to occur for \( \bar{n} \leq 0.2 \), and for \( \bar{n} \leq 0.10 \) the region of absolute instability appears to extend beyond the equator.
Figure 6.3: Locus of the absolute frequency $\omega^*$ as $S$ varies for $\bar{n} = 0.05-0.25$ in increments of 0.05 ($\omega^*_o$: bottom to top; $\omega^*_i$: top to bottom).

Physically, when the sphere rotates in an otherwise still fluid two identical boundary layers are formed over each hemisphere and boundary-layer eruption occurs at the equator where they meet. The absolute instability should not therefore be viewed as extending beyond the equator for those values of $\bar{n}$. Instead we simply interpret this as meaning that rather than having a finite ‘pocket’ of absolute instability (as, for example, when $\bar{n} = 0.15$), we have local absolute instability occurring all the way up to the equator after its onset at some lower latitude. Close to the equator the boundary layer separates from the sphere surface and our predictions are not valid.

Using the methods described in §6.1.1, a rational function has been fitted to each
Figure 6.4: Contours of real and imaginary parts of $\frac{\partial \omega^o}{\partial S} = 0$ in the complex $S$-plane for $\tilde{n} = 0.05$. Solid line $m = 3$; dashed line $m = 4$; dotted line $m = 5$.

$\omega^o$ and the problem immersed in the complex $S$-plane. Figure 6.4 shows zero contours of the real and imaginary parts of $\frac{\partial \omega^o}{\partial S}$ in the complex $S$-plane for $\tilde{n} = 0.05$ obtained from three orders of the rational-function approximation. A comparison of the saddle-point locations and leading-order global frequencies is made in table 6.3, where an indication of the accuracy of the rational-function approximation on the real $S$-axis is also given from (6.4). From this we see that $m = 3$ gives the most accurate rational-function fit and so we consider $m = 3$ to give the most accurate predictions of the leading-order global frequency.

Similar calculations have been conducted for various values of $\tilde{n}$, and table 6.4 presents the location and leading-order global frequency calculated with the most accurate rational-function approximation for each $\tilde{n}$. We see that for $\tilde{n} = 0.05$–0.21 the imaginary part of the leading-order frequency at the saddle is zero, i.e. we predict that the rotating-sphere boundary layer can sustain neutrally stable global
Table 6.3: Comparison of the saddle-point location and leading-order global frequency at each \( m \) for the rotating-sphere boundary layer for \( \tilde{n} = 0.05 \).

modes when there exists a region of absolute instability. However, for \( \tilde{n} > 0.21 \) the region of absolute instability does not occur and we find that disturbances are globally damped. Further investigation of \( \tilde{n} \) between 0.21 and 0.25 shows that the global growth rates become increasingly damped as \( \tilde{n} \) increases. It is known (Huerre & Monkewitz, 1990) that the imaginary part of the global frequency is always less than the maximum value of \( \omega^\phi \) for real \( \mathcal{S} \).

From the definition of \( \epsilon \) given in (C.4) we can write the ‘slow’ azimuthal wavenumber as

\[
\tilde{n} = \frac{1}{a^*} \left( \frac{\nu^*}{\Omega^*} \right)^{1/2} n.
\]

For a fixed azimuthal wavenumber \( n \), decreasing \( \tilde{n} \) is therefore equivalent to increasing the rotation rate of the sphere. The results presented in figure 6.3 and table 6.4 indicate that for low rotation rates the boundary layer is not locally absolutely unstable anywhere and disturbances are globally damped, but as the rotation rate is increased the boundary layer becomes locally absolutely unstable over a finite portion of the sphere and can sustain neutrally stable global modes.
<table>
<thead>
<tr>
<th>$\bar{n}$</th>
<th>$S_5$</th>
<th>$\omega_s^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>$75.6590 + 19.8151i$</td>
<td>$0.1731 + 0.0000i$</td>
</tr>
<tr>
<td>0.10</td>
<td>$72.5895 + 16.1779i$</td>
<td>$0.1810 + 0.0000i$</td>
</tr>
<tr>
<td>0.15</td>
<td>$64.1783 + 6.4278i$</td>
<td>$0.1928 + 0.0000i$</td>
</tr>
<tr>
<td>0.20</td>
<td>$50.7051 + 0.2260i$</td>
<td>$0.1993 + 0.0000i$</td>
</tr>
<tr>
<td>0.21</td>
<td>$48.8351 - 0.9520i$</td>
<td>$0.2008 + 0.0000i$</td>
</tr>
<tr>
<td>0.22</td>
<td>$46.1289 - 2.3249i$</td>
<td>$0.2021 - 0.0074i$</td>
</tr>
<tr>
<td>0.23</td>
<td>$43.9525 - 2.9125i$</td>
<td>$0.2038 - 0.0193i$</td>
</tr>
<tr>
<td>0.24</td>
<td>$41.7206 - 3.8082i$</td>
<td>$0.2063 - 0.0357i$</td>
</tr>
<tr>
<td>0.25</td>
<td>$38.3852 - 4.5016i$</td>
<td>$0.2076 - 0.0456i$</td>
</tr>
</tbody>
</table>

Table 6.4: Comparison of the saddle-point location and leading-order global frequency for each $\bar{n}$. (The zero growth rates are zero to 9 decimal places).

### 6.3 Conclusion

In this chapter we have considered the global behaviour of the rotating-disk, rotating-cone and rotating-sphere boundary layers by taking into account the slowly varying basic state along each body surface. By noting a transformation between Peake’s slow-spatially-varying formulation (see appendix C.4.1) and the local rotating-cone analyses conducted in §4.6, we have determined the absolute frequency for the rotating disk and cone as a function of the slow spatial variable $X$. For the rotating sphere, a numerical solution of the governing equations was computed and the absolute frequency calculated as a function of $S$ for certain values of the scaled azimuthal wavenumber. By immersing each problem in the complex slow-spatial-variable plane by means of a rational-function approximation, the location of saddle points in the absolute frequency have been determined which, in turn, give the
Chapter 6: Linear global modes

leading-order estimate of the global frequency.

For the rotating disk and for each rotating cone, the global mode frequency is found to have a negative imaginary part which indicates that disturbances in the boundary layer are globally damped, and Peake's formulation is such that the scaled global growth rate is independent of the half-angle in the inviscid analysis. The numerical simulations of the Navier–Stokes equations conducted by Davies & Carpenter (2001) suggest that the absolute instability of the rotating-disk boundary layer does not produce a linear amplified global mode; our findings are consistent with this result. For the rotating sphere, the global mode frequency is found to have a zero imaginary part at leading order for values of the azimuthal wavenumber where absolute instability exists (high rotation rates), and negative imaginary part when no absolute instability exists (low rotation rates). This suggests that the rotating-sphere boundary layer may be neutrally stable to global modes when local absolute instability exits, but globally damped otherwise.

Section 6.1 must only be considered as preliminary in the investigation of the global behaviour of the rotating-disk/cone boundary layers. We have studied the global stability of these boundary layers using an inviscid analysis, but the effect of viscosity needs to be studied for comparisons with real flows; this is also true of the rotating-sphere analysis of §6.2. Also, a more satisfying investigation of Peake's inviscid equations would involve a direct solution of the Rayleigh equation with complex $\bar{X}$ so that the analytic continuation of the problem into the complex $\bar{X}$-plane is not required. This is possible for the rotating disk/cone, but is not possible for the rotating sphere since we would require the mean flows for complex $\mathcal{S}$. However, the saddle points for the rotating-sphere analysis were relatively close to the real $\mathcal{S}$-axis and so the rational-function approximation should return accurate results.
In contrast, the rotating-disk/cone saddles are found far away from the real \( \bar{X} \)-axis and the accuracy of the rational-function approximation needs further investigation at these locations. Further, the rotating-sphere analysis lead to predictions of a zero global growth rate, however, it must be remembered that this is only a leading-order estimate of the global frequency, and so \( O(\epsilon) \) correction terms would need to be considered to predict the linear stability. In each case the results of this chapter can be considered as starting points for more sophisticated analyses in the future.
Chapter 7

Conclusions

This thesis consists of a number of chapters each dealing with the stability analysis of a different three-dimensional boundary-layer problem. A summary of the results and conclusions drawn from each investigation are not given in detail here, but can be found at the end of the relevant chapters. Instead, in §7.1 we review the important results and make some general conclusions, and in §7.2 suggestions for further research are made in the light of this thesis.

7.1 Completed work

Disturbances in the incompressible boundary-layer flow over the surface of rotating spheres and rotating cones have been investigated. Using linear-stability theory and the Briggs–Bers criterion, the local convective and absolute instability properties of the boundary layers have been studied at various locations along each body. In each case, analyses have been conducted with the body rotating in an otherwise still fluid and in uniform axial flows. By using a consistent formulation in both situations the resulting perturbation equations are independent of the axial flow rate for
each body, and the effect of non-zero axial flow is to change the basic flow profiles upon which the analyses are performed.

In each analysis the body is considered to rotate in a fixed frame of reference, which is consistent with all previous investigations of the boundary layers over rotating spheres and cones, but is in contrast to the related work of Lingwood (1995a) on the rotating disk. In Lingwood’s work a frame of reference fixed on the rotating disk is used, resulting in Coriolis terms in the mean-flow and perturbation equations. The resulting perturbation equations for the rotating sphere and cone include viscous and streamline-curvature effects and, apart from Coriolis terms, are of a similar form to those used by Lingwood for the rotating disk.

Previous convective instability analyses do exist for some of the three-dimensional boundary layers considered in this thesis, but they are unsatisfactory with the exception of the analysis of cones rotating in still fluid conducted by Kobayashi & Izumi (1983). Taniguchi et al. (1998) consider the convective instability of the boundary-layer flow over a sphere rotating in still fluid, and have reasonable success in predicting the properties of the spiral vortices at locations away from the pole. However, it should be expected that the stability properties of the rotating-sphere boundary layer tend towards those of the rotating disk close to the pole, and the convective instability neutral curves of Taniguchi et al. do not do this. In the convective analysis of the rotating-cone boundary layer in axial flow, Kobayashi (1981) and Kobayashi, Kohama & Kurosawa (1983) only consider cones with half-angle $\psi = 15^\circ$, and it appears that no previous convective instability analyses exist for the rotating-sphere boundary layer in axial flow. To the author’s knowledge no previous absolute instability analyses exist for any of the boundary layers considered here. The work presented in this thesis is therefore original and furthers existing
knowledge of the stability of three-dimensional boundary layers.

Two approaches were taken in the convective instability analyses conducted in this thesis. The first approach assumed that the disturbances rotate with some fixed multiple of the body surface speed, and predictions of the critical Reynolds numbers and the number and orientation of the spiral vortices were made. The second approach enabled a prediction of the onset of the experimentally observed 'slow' vortices.

The results obtained using a stationary vortex assumption for the boundary-layer flow over a sphere rotating in still fluid compare well with existing experimental data for latitudes lower than $\theta = 66^\circ$, and tend to existing results for the rotating disk as the latitude is reduced towards the pole. By assuming the 'slow vortex' speed of $c = 0.76$, the discrepancies at higher latitudes are greatly reduced. By using the second approach in the investigation of this boundary layer, we have shown that the occurrence of the 'slow vortices' may be associated with the first point at which the streamline curvature mode becomes dominant, i.e. $\theta = 66^\circ$. However, we were not able to fix the slower longitudinal wave speed at the higher latitudes from this theory.

The results obtained by assuming stationary vortices in the boundary-layer flow over cones rotating in still fluid compare well with existing experimental data for the onset of spiral vortices for cones of all half-angles. By using the second approach for this family of boundary layers, we have predicted that 'slow vortices' do not exist in the boundary layer on cones of any half-angle, and this is consistent with experimental findings.

It appears that very little experimental data exists for the boundary-layer flow over spheres rotating in axial flow with which the predictions of our convective insta-
Chapter 7: Conclusions

bility analysis could be compared. However Kobayashi & Arai (1990) do report that axial flow increases the Reynolds number at which the changeover between stationary and slow vortices occurs, and our results are consistent with this observation. We were also able to predict that the latitude at which changeover occurs increases slightly as the axial flow rate is increased.

Experiments that measure the onset of spiral vortices on cones rotating in axial flow have been conducted by Kobayashi et al. (1983) and Kohama (1984a) on cones with a half-angle of $\psi = 15^\circ$ only. As discussed in §4.7.1, the approximations made in the cone analysis were such that the inaccuracy of the results increases with decreased half-angle and so analyses on such a slender cone were not conducted. We are therefore unable to compare our theoretical predictions for the onset of convective instabilities with experimental measurements for the onset of spiral vortices. However, the predictions for each half-angle are consistent with the experimentally observed behaviour of the boundary layer for $\psi = 15^\circ$. We also predict that slow vortices do not exist in the rotating-cone boundary layer under any axial flow rate for all half-angles.

Expressing the observed transition points of Kohama & Kobayashi (1983) for spheres rotating in still fluid in terms of a local Reynolds number, we find that transition occurs at roughly the same value at all latitudes up to and including $\theta = 70^\circ$. At $\theta = 80^\circ$ the transition point is slightly lower. We have tried to associate transition with the onset of absolute instability, and the predicted onset of the latitudinal absolute instability is seen to match onto the experimental value well at a latitude of $\theta = 30^\circ$. Beyond this the discrepancy increases but still remains close when below $\theta = 70^\circ$. Similarly, expressing the transition points measured by Kobayashi & Izumi (1983) for cones rotating in still fluid in terms of a local
Reynolds number, we find that transition occurs at roughly the same value for cones with $\psi \geq 60^\circ$. The predicted onset of the absolute instability is seen to match the experimental values reasonably well at half-angles between $\psi = 60^\circ$–90°. For $\psi < 60^\circ$ the discrepancy increases sharply.

The comparisons between the predicted onset of absolute instability and the measured onset of turbulence suggest that the transition mechanism may well be an absolute instability in both types of boundary layer. For the rotating sphere, the absolute instability may well initiate turbulence at latitudes up to $\theta = 70^\circ$, and for the rotating cone the absolute instability may well initiate transition on cones with half-angles above $\psi = 60^\circ$. At other latitudes and for other half-angles the absolute instability may be suppressed and other transition mechanisms apply.

It is interesting to note that the rotating-sphere boundary layer below a latitude of $\theta = 70^\circ$ and the rotating-cone boundary layer for half-angles above $\psi = 60^\circ$ (including the rotating disk case of $\psi = 90^\circ$) all become absolutely unstable at roughly the same local Reynolds number, $R_x \approx 2.5 \times 10^5$. This means that transition to turbulence is predicted to occur at the same local Reynolds number in each of the three-dimensional boundary layers considered here. An obvious extension to this work is therefore to investigate the onset of absolute instability of the boundary layer over more general bodies of revolution. Investigations into the absolute instability of other three-dimensional boundary layers have been conducted, for example Lingwood (1997b) and Taylor & Peake (1998, 1999) are concerned with the boundary-layer flow over a swept wing. However, in this case the pinch points found in the crossflow direction do not lead to an absolutely instability since there is no simultaneous pinching in the streamwise direction.

We have seen that axial flow stabilises each boundary-layer type to absolute in-
stability. This was shown by the critical Reynolds numbers for the onset of absolute instability increasing with axial flow in each boundary layer, and also by the maximum absolute growth rates being reduced with increased axial flow at each latitude in the rotating-sphere analysis.

At each axial flow rate in the rotating-sphere analysis the onset of absolute instability was seen to occur at a roughly constant value as we moved away from the pole. However, the deviation from this constant value as we moved away from the pole was accelerated with increased axial flow. The comparisons made in the still fluid analysis between the onset of absolute instability and the experimentally measured transition locations suggest that if the boundary-layer transition is governed by absolute instability, then it would only be for latitudes where the critical local Reynolds number is approximately independent of latitude. Our results for axial flow therefore indicate that if absolute instability does promote transition in the boundary layer of a rotating sphere then it will be increasingly limited to the region close to the pole. However, for this suggestion to be correct our predicted critical local Reynolds numbers need to be sufficiently close to experimental measurements for the onset of turbulence and, unfortunately, it appears that such data is not currently available for the rotating-sphere boundary layer.

Similarly, although the boundary layers in the analysis of rotating-cone boundary layers in axial flows have been predicted to become absolutely unstable for each local axial flow parameter, without comparisons with experimental data we are unable to draw any conclusions about whether this mechanism initiates transition to turbulence.

A parallel-flow approximation was used in the analyses of the rotating sphere and rotating-cone boundary layers discussed above, and so in each case we were
restricted to the local stability characteristics of each flow. For this reason we have also considered the linear global behaviour of the rotating-disk/cone and rotating-sphere boundary layers by taking into account the slowly varying basic state along each body surface. By noting a transformation between Peake’s slow-spatially-varying formulation and the local rotating-cone analyses conducted in §4.6, we were able to determine the absolute frequency for the rotating disk and cone as a function of the slow spatial variable $\tilde{X}$. For the rotating sphere, a numerical solution of the governing equations was computed and the absolute frequency calculated as a function of $S$ for certain values of the azimuthal wavenumber. By immersing each problem in the complex slow-spatial-variable plane by means of a rational-function approximation, the location of saddle points in the absolute frequency were determined which, in turn, gave the leading-order estimate of the global frequency.

For the rotating disk and for each rotating cone, the global mode frequency was found to have a negative imaginary part which indicates that disturbances in the boundary layer are globally damped. The numerical simulations of the Navier-Stokes equations conducted by Davies & Carpenter (2001) suggest that the absolute instability of the rotating-disk boundary layer does not produce a linear amplified global mode; our findings are consistent with this result. For the rotating sphere, the global mode frequency was found to have a zero imaginary part at leading order for values of the azimuthal wavenumber where absolute instability exists, and negative imaginary part when no absolute instability exists. This suggests that the rotating-sphere boundary layer may be neutrally stable to global modes when local absolute instability exits, but globally damped otherwise.
7.2 Further work

A number of open issues remain after the completion of this thesis. For instance, more experiments will be required in order to investigate the validity of some of the predictions made here, not least the possible role of absolute instability in the transition of the boundary layers studied (in much the same way as was done by Lingwood, 1996). Thorough experimental investigations are particularly required for spheres and families of cones rotating in uniform axial flows, with careful measurements of the vortex properties and critical Reynolds numbers for transition being taken. The location and critical Reynolds numbers for ‘slow’ vortices found in the rotating-sphere boundary layers also need to be measured as a function of the axial flow rate.

Additional effects can also be included in our analysis, such as the behaviour of more general bodies of revolution paying particular attention to the value of the local Reynolds number at the onset of absolute instability.

The global-mode work presented in this thesis must only be considered as preliminary in the investigation of the global behaviour of the boundary layers considered. Throughout we have used an inviscid analysis, but the effect of viscosity needs to be studied for comparisons with real flows. Also, a more satisfying investigation of Peake’s inviscid equations would involve a direct solution of the governing stability equation for the rotating-disk/cone with complex \( \bar{X} \) so that the analytic continuation of the problem into the complex \( \bar{X} \)-plane is not required. The saddle points for the rotating-disk/cone case were found far away from the real \( \bar{X} \)-axis and the accuracy of the rational-function approximation needs further investigation at these locations. Also, the rotating-sphere analysis led to predictions of a zero global growth rate, however, it must be remembered that this is only a leading-order estimate of the
global frequency, and so $O(\epsilon)$ correction terms would need to be considered to predict the linear stability. In each case the results of global-mode work presented here can be considered as starting points for more sophisticated analyses in the future.

Further, the relationship between our linear results and the occurrence of nonlinear global modes, as recently found by Pier & Huerre (2001), needs to be explored. Pier & Huerre describe the appearance of nonlinear fronts at the boundary between convective instability upstream and absolute instability downstream, which of course matches the structure of the linear stability found here on the rotating sphere and cone, and indeed by Lingwood on the rotating disk. The possibility of such a nonlinear wave undergoing some secondary instability, to lead to transition, requires careful study, and work on the rotating-disk boundary layer is well under way in this direction (Pier, 2002).
Appendices
Appendix A

Neutral curves

In this appendix we present the neutral curves of convective and absolute instability that were not shown in the main text of this thesis. In §A.1 we present the neutral curves of absolute instability that were calculated for the rotating sphere in still fluid, and in §A.2 the neutral curves for the convective instability of stationary vortices are presented for the rotating sphere in uniform axial flows. These sets of neutral curves are discussed in §§2.5 & 3.3 respectively. In §A.3 the neutral curves for the convective instability of stationary vortices are presented for rotating cones in uniform axial flows as discussed in §5.3.
A.1 The neutral curves of absolute instability for a rotating sphere in still fluid

Figures A.1–A.8 show the neutral curves of absolute instability at latitudes of \( \theta = 10^\circ–80^\circ \) on a rotating sphere in still fluid.

Figure A.1: Absolute instability neutral curves at \( \theta = 10^\circ \) in the \((R, \alpha_r^\circ)\)-, \((R, \alpha_i^\circ)\)-, \((R, \gamma_r^\circ)\)- and \((R, \beta)\)-planes.
Figure A.2: Absolute instability neutral curves at $\theta = 20^\circ$ in the $(R, \alpha_r^o)$-, $(R, \alpha_i^o)$-, $(R, \gamma_r^o)$- and $(R, \beta)$-planes.

Figure A.3: Absolute instability neutral curves at $\theta = 30^\circ$ in the $(R, \alpha_r^o)$-, $(R, \alpha_i^o)$-, $(R, \gamma_r^o)$- and $(R, \beta)$-planes.
Figure A.4: Absolute instability neutral curves at $\theta = 40^\circ$ in the $(R, \alpha_r^o)$-, $(R, \alpha_i^o)$-, $(R, \gamma_r^o)$- and $(R, \beta)$-planes.

Figure A.5: Absolute instability neutral curves at $\theta = 50^\circ$ in the $(R, \alpha_r^o)$-, $(R, \alpha_i^o)$-, $(R, \gamma_r^o)$- and $(R, \beta)$-planes.
Figure A.6: Absolute instability neutral curves at $\theta = 60^\circ$ in the $(R, \alpha_r^\circ)$-, $(R, \alpha_i^\circ)$-, $(R, \gamma_r^\circ)$- and $(R, \beta)$-planes.

Figure A.7: Absolute instability neutral curves at $\theta = 70^\circ$ in the $(R, \alpha_r^\circ)$-, $(R, \alpha_i^\circ)$-, $(R, \gamma_r^\circ)$- and $(R, \beta)$-planes.
Figure A.8: Absolute instability neutral curves at $\theta = 80^\circ$ in the $(R, \alpha^\circ_r)$-, $(R, \alpha^\circ_i)$-, $(R, \gamma^\circ_r)$- and $(R, \beta)$-planes.
A.2 The neutral curves of convective instability for a rotating sphere in axial flow

Figures A.9–A.15 show the neutral curves for the convective instability of stationary vortices at latitudes of $\theta = 10^\circ - 70^\circ$ on a rotating sphere in axial flow. The axial flow parameter $T$ is defined in §3.1 as $T = U^*_\infty / a^* \Omega^*$. 

Figure A.9: Neutral curves for the convective instability of stationary vortices at $\theta = 10^\circ$ with $T = 0.00-0.25$ (left to right) in 0.05 increments.
Figure A.10: Neutral curves for the convective instability of stationary vortices at $\theta = 20^\circ$
with $T = 0.00$–$0.25$ (left to right) in 0.05 increments.

Figure A.11: Neutral curves for the convective instability of stationary vortices at $\theta = 30^\circ$
with $T = 0.00$–$0.25$ (left to right) in 0.05 increments.
Figure A.12: Neutral curves for the convective instability of stationary vortices at $\theta = 40^\circ$
with $T = 0.00$–$0.25$ (left to right) in 0.05 increments.

Figure A.13: Neutral curves for the convective instability of stationary vortices at $\theta = 50^\circ$
with $T = 0.00$–$0.25$ (left to right) in 0.05 increments.
Figure A.14: Neutral curves for the convective instability of stationary vortices at $\theta = 60^\circ$ with $T = 0.00$–$0.25$ (left to right) in 0.05 increments.

Figure A.15: Neutral curves for the convective instability of stationary vortices at $\theta = 70^\circ$ with $T = 0.00$–$0.25$ (left to right) in 0.05 increments.
A.3  The neutral curves of convective instability for a rotating cone in axial flow

Figures A.16–A.23 show the neutral curves for the convective instability of stationary vortices on cones of half-angle $\psi = 90^\circ$–$20^\circ$ rotating in axial flow. The axial flow parameter $T_s$ and other parameters are defined in chapter 5.

Figure A.16: Neutral curves for the convective instability of stationary vortices on a cone with half angle $\psi = 90^\circ$ with $T = 0.00$–$0.25$ (left to right) in 0.05 increments.
Figure A.17: Neutral curves for the convective instability of stationary vortices on a cone with half angle $\psi = 80^\circ$ with $T = 0.00-0.25$ (left to right) in 0.05 increments.

Figure A.18: Neutral curves for the convective instability of stationary vortices on a cone with half angle $\psi = 70^\circ$ with $T = 0.00-0.25$ (left to right) in 0.05 increments.
Figure A.19: Neutral curves for the convective instability of stationary vortices on a cone with half angle $\psi = 60^\circ$ with $T = 0.00-0.25$ (left to right) in 0.05 increments.

Figure A.20: Neutral curves for the convective instability of stationary vortices on a cone with half angle $\psi = 50^\circ$ with $T = 0.00-0.25$ (left to right) in 0.05 increments.
Figure A.21: Neutral curves for the convective instability of stationary vortices on a cone with half angle $\psi = 40^\circ$ with $T = 0.00-0.25$ (left to right) in 0.05 increments.

Figure A.22: Neutral curves for the convective instability of stationary vortices on a cone with half angle $\psi = 30^\circ$ with $T = 0.00-0.25$ (left to right) in 0.05 increments.
Figure A.23: Neutral curves for the convective instability of stationary vortices on a cone with half angle $\psi = 20^\circ$ with $T = 0.00$–0.25 (left to right) in 0.05 increments.
Appendix B

The series solution of the
boundary-layer equations

The series solution of the boundary-layer equations for the rotating sphere was originally proposed by Howarth (1951) and then developed by Banks (1965). Comparisons with the accurate finite-difference solutions of Manohar (1967) and Banks (1976) have shown it to be accurate up to a latitude of 50°, and in chapters 2 and 3 we use the method to calculate the boundary-layer profiles close to the pole. The series solution method is described here by application to the system of equations (2.6)–(2.9). These equations govern the boundary-layer flow on the surface of a sphere rotating in still fluid, but the method is easily modified to solve the equations for a sphere rotating in an axial flow, as described in §3.1.

We begin by assuming that the non-dimensional mean-flow velocities $U$, $V$ & $W$ can be written as expansions about the pole of the rotating sphere. Mathematically
Appendix B: Series solution of the boundary-layer equations

this is written as

\[ U = \theta F_1 + \theta^3 F_3 + \ldots, \quad (\text{B.1}) \]

\[ V = \theta G_1 + \theta^3 G_3 + \ldots, \quad (\text{B.2}) \]

\[ W = H_1 + \theta^2 H_3 + \ldots, \quad (\text{B.3}) \]

where \( F_n, G_n \) and \( H_n \) are non-dimensional functions of the non-dimensional variable

\[ \eta = (\Omega^*/\nu^*)^{1/2}(r^* - a^*) \] and \( n = 1, 3, 5, \ldots \)

By using the expansions (B.1)-(B.3) in the no-slip boundary condition of (2.9) and equating terms multiplied by powers of \( \theta \), we find that the condition at \( \eta = 0 \) can be written as

\[ F_n(0) = H_n(0) = G_n(0) - \frac{1}{n!}(-1)^{(n-1)/2} = 0. \quad (\text{B.4}) \]

Note that the Maclaurin expansion for \( \sin \theta \) has also been used. In a similar way the quiescent fluid condition of (2.9) leads to

\[ F_n(\infty) = G_n(\infty) = 0. \quad (\text{B.5}) \]

We now substitute the series expansions (B.1)-(B.3) into (2.6) after changing to the spatial variable \( \eta \). Using the Maclaurin expansion for \( \cot \theta \), terms multiplying powers of \( \theta \) can be equated and non-linear ordinary differential equations governing \( F_n, G_n \) and \( H_n \) are found at each order. At first, second, third and fourth order we find equations (B.6), (B.7), (B.8) and (B.9) respectively. Henceforth a prime denotes differentiation with respect to \( \eta \).

\[ H_1 F_1' + F_1'' = F_1^2, \quad (\text{B.6}) \]

\[ 4F_1F_3 + H_1 F_3' + H_3 F_1' - 2G_1 G_3 + \frac{G_1^2}{3} = F_3'', \quad (\text{B.7}) \]
\[ 6F_1F_5 + 3F_3^2 + H_1F_5' + H_3F_3' + H_5F_1' \]
\[ -2G_1G_5 - G_3^2 + \frac{2}{3}G_1G_3 + \frac{1}{45}G_1^2 = F_5'' , \quad (B.8) \]
\[ 8F_1F_7 + 8F_3F_5 + H_1F_7' + H_3F_5' + H_5F_1' - 2G_1G_7 \]
\[ -2G_3G_5 + \frac{1}{3}G_3^2 + \frac{2}{3}G_1G_5 + \frac{2}{45}G_1G_3 + \frac{2}{945}G_1^2 = F_7'' . \quad (B.9) \]

In a similar way, (2.7) leads to \((B.10)-(B.13)\), and (2.8) leads to \((B.14)-(B.17)\) at first, second, third and fourth order respectively

\[ 2F_1G_1 + H_1G_1' = G_1'' , \quad (B.10) \]
\[ 4F_1G_3 + 2F_3G_1 + H_1G_3' + H_3G_1' - \frac{1}{3}F_1G_1 = G_3'' , \quad (B.11) \]
\[ 6F_1G_5 + 4F_3G_3 + 2F_5G_1 + H_1G_5' + H_3G_3' \]
\[ + H_5G_1' - \frac{1}{3}F_1G_3 - \frac{1}{3}F_3G_1 - \frac{1}{45}F_1G_1 = G_5'' , \quad (B.12) \]
\[ 8F_1G_7 + 6F_3G_5 + 4F_5G_3 + 2F_7G_1 + H_1G_7' + H_3G_5' + H_5G_3' \]
\[ + H_7G_1' - \frac{1}{3}F_1G_5 - \frac{1}{3}F_3G_3 - \frac{1}{3}F_5G_1 - \frac{1}{45}F_1G_3 \]
\[ - \frac{1}{45}F_3G_1 - \frac{2}{945}G_1F_1 = G_7'' , \quad (B.13) \]

\[ 2F_1 + H_1' = 0 , \quad (B.14) \]
\[ 4F_3 + H_3' - \frac{1}{3}F_1 = 0 , \quad (B.15) \]
\[ 6F_5 + H_5' - \frac{1}{45}F_1 - \frac{1}{3}F_3 = 0 , \quad (B.16) \]
\[ 8F_7 + H_7' - \frac{2}{945}F_1 - \frac{1}{45}F_3 - \frac{1}{3}F_5 = 0 . \quad (B.17) \]

Incidently, it is interesting to note that the leading order equations \((B.6), (B.10)\) & \((B.14)\) give the coupled non-linear ordinary-differential equations for the steady-mean boundary-layer flow profiles on the rotating disk in a fixed frame of reference (White 1974). Hence we see that the basic boundary-layer flow profiles near to the
\[
\begin{align*}
F'_1(0) &= 0.51023 & G'_1(0) &= -0.61592 \\
F'_3(0) &= -0.22128 & G'_3(0) &= 0.24764 \\
F'_5(0) &= 0.02071 & G'_5(0) &= -0.02568 \\
F'_7(0) &= -0.00189 & G'_7(0) &= 0.00181
\end{align*}
\]

Table B.1: The values of $F'_n(0)$ and $G'_n(0)$ calculated by the shooting routine.

pole of a rotating sphere are very similar to those of the rotating disk. This is to be expected since the sphere is locally flat in the region close to the pole.

The solution of equations (B.6)–(B.17), subject to the boundary conditions (B.4) and (B.5), represents a two-point boundary value problem which is solved using a shooting method. The shooting method is summarised as follows: Values for $F'_n(0)$ and $G'_n(0)$ are guessed and the equations are integrated forwards using a fourth-order Runge–Kutta routine over a suitably large domain. At the outer boundary of the integration domain the calculated values of $F_n$ and $G_n$ are compared with boundary condition (B.5). If they do not match within a preset tolerance a better approximation to the values of $F'_n(0)$ and $G'_n(0)$ are found using a Newton–Raphson procedure. The integration proceeds forwards again and the process continues until the outer boundary condition is satisfied within the required tolerance.

To decide upon the domain size that accurately approximates the infinite domain of the boundary-layer equations, the shooting routine is used over a variety of domain sizes. For each domain size the calculated values of $F'_n(0)$ and $G'_n(0)$ are recorded and the domain size increased for the next calculation. When a domain size is reached on which further increases produce values of $G'_n(0)$ and $F'_n(0)$ that differ in the sixth decimal place only, that domain is considered sufficiently large. A domain of integration between $\eta = 0–20$ has been found to be sufficiently large in all cases.
For the equations relevant to a sphere rotating in still fluid, the values of $G_n'(0)$ and $F_n'(0)$ calculated using this routine match those given by Banks (1965) and are shown in table B.1.
Appendix C

The formulation of the stability problem for an arbitrary body of revolution in terms a slow spatial variable

In this appendix we formulate the stability problem in terms of a slow spatial variable for the boundary layer on an arbitrary body of revolution rotating in still fluid, sections C.2 & C.3 deal with the steady and unsteady equations respectively. This formulation is entirely due to N. Peake (personal communication, 2001), and is not to be considered as work of the present author. However, in chapter 6 linear global-mode analyses of the rotating-disk, rotating-cone and rotating-sphere boundary layers were presented that use this formulation and details of the formulation are required for the understanding of that work. The special cases of the rotating-disk/cone and rotating-sphere boundary layers are described in §§C.4.1 & C.4.2 respectively.
C.1 The geometry and slow spatial variables

Consider the orthogonal coordinate system \((s^*, \phi, \eta^*)\) shown in figure C.1 that is fixed in space with origin located at the tip of the body. The local surface radius of the body is \(r_0^*(s^*)\) as measured from the axis of rotation, and the local surface angle is \(\psi\) as measured from a line parallel to the axis of rotation. The body rotates within this fixed frame of reference with a constant angular frequency \(\Omega^*\), and a star indicates a dimensional quantity.

In this formulation we consider typical length and time scales to be \((\nu^*/\Omega^*)^{1/2}\) and \((\nu^*/\Omega^3)^{1/2}/a^*\) respectively, where \(\nu^*\) is the kinematic viscosity and \(a^*\) is a characteristic length. We non-dimensionalize \(s^*\) and \(r_0^*(s^*)\) with the typical length scale to form

\[
s = \frac{s^*}{(\nu^*/\Omega^*)^{1/2}}, \quad r_0(s) = \frac{r_0^*(s^*)}{(\nu^*/\Omega^3)^{1/2}},
\]

and define non-dimensional spatial variables

\[
S = \frac{s}{a^*}, \quad R_0(S) = \frac{r_0^*}{a^*}.
\]
Appendix C: Arbitrary body of revolution with slow spatial variable

By eliminating \(s^*\) and \(r_o^*(s)\) between (C.1) and (C.2) we find that \(S\) is the slow spatial variable and \(R_o\) the slowly varying surface radius, i.e.

\[
S = \varepsilon s, \quad R_o(S) = \varepsilon r_o(s),
\]

with

\[
\varepsilon = \frac{1}{a^*} \left( \frac{\nu^*}{\Omega^*} \right)^{1/2} \ll 1.
\]

### C.2 The steady flow equations

The equations that govern the steady flow in the boundary layer are stated by Mangler (1945) as

\[
U^* \frac{\partial U^*}{\partial s^*} + W^* \frac{\partial U^*}{\partial \eta^*} - \frac{V^2}{r_o^*} = \nu^* \frac{\partial^2 U^*}{\partial \eta^*^2},
\]

\[
U^* \frac{\partial V^*}{\partial s^*} + W^* \frac{\partial V^*}{\partial \eta^*} - V^* U^* \frac{dr_o^*}{ds^*} = \nu^* \frac{\partial^2 V^*}{\partial \eta^*^2},
\]

\[
\frac{\partial}{\partial s^*} (r_o^* U^*) + \frac{\partial}{\partial \eta^*} (r_o^* W^*) = 0,
\]

where \(U^*, V^*\) and \(W^*\) are the steady-flow components in the \(s^*\), \(\phi\) and \(\eta^*\) directions respectively. Note that there is no pressure term due to the static outer flow. The non-dimensional mean-flow variables are then defined as

\[
U(\eta) = \frac{U^*}{\nu^*/\Omega^*}, \quad V(\eta) = \frac{V^*}{\nu^*/\Omega^*}, \quad W(\eta) = \frac{W^*}{(\nu^*/\Omega^*)^{1/2}}.
\]

where \(\eta = \eta^*/(\nu^*/\Omega^*)^{1/2}\) is the non-dimensional distance from the body surface in the normal direction. Using (C.8) we non-dimensionalize (C.5)–(C.7) to produce

\[
U \frac{\partial U}{\partial S} + W \frac{\partial U}{\partial \eta} - \frac{dR_o}{dS} \frac{V^2}{R_o} = \frac{\partial^2 U}{\partial \eta^2},
\]

\[
U \frac{\partial V}{\partial S} + W \frac{\partial V}{\partial \eta} + \frac{dR_o}{dS} \frac{UV}{R_o} = \frac{\partial^2 V}{\partial \eta^2},
\]

\[
\frac{\partial}{\partial S} (R_o U) + \frac{\partial}{\partial \eta} (R_o W) = 0,
\]
Appendix C: Arbitrary body of revolution with slow spatial variable

where we have also used the definitions (C.1), (C.3) and (C.4) along with the fact that

\[
\frac{dr_s}{dS} \frac{1}{r_s} = \frac{dR_o}{dS} \frac{1}{R_o}.
\]

Equations (C.9)–(C.11) are the non-dimensional boundary-layer equations that govern the steady mean flow over an arbitrary body of revolution with local surface radius \( R_o(S) \) that is rotating in still fluid.

### C.3 The unsteady perturbation equations

We start by considering the coordinate systems defined in figure C.1. Let \( \hat{e}_s^*, \hat{e}_\phi^*, \) and \( \hat{e}_\eta^* \) be unit vectors in the \( s^* \), \( \phi \) and \( \eta^* \) directions respectively, and \( \hat{x}^*, \hat{y}^* \) and \( \hat{z}^* \) be unit vectors in the \( x^* \), \( y^* \) and \( z^* \) directions respectively. We are then able to write the \((s^*, \phi, \eta^*)\)-coordinate system in terms of the cartesian coordinate system:

\[
\hat{e}_s^* = (\cos \phi \hat{x}^* + \sin \phi \hat{y}^*) \sin \psi + \cos \psi \hat{z}^*, \quad (C.12)
\]

\[
\hat{e}_\phi^* = -\sin \phi \hat{x}^* + \cos \phi \hat{y}^*, \quad (C.13)
\]

\[
\hat{e}_\eta^* = (\cos \phi \hat{x}^* + \sin \phi \hat{y}^*) \cos \psi - \sin \psi \hat{z}*. \quad (C.14)
\]

The location vector on the body surface \( \mathbf{r}^* \) is written as

\[
\mathbf{r}^* = (r_o^* + \eta^* \cos \psi)(\cos \phi \mathbf{x}^* + \sin \phi \mathbf{y}^*) + (z_o^* - \eta^* \sin \psi)\mathbf{z}^*,
\]

and from this we find the length scales

\[
h_s^* = \left| \frac{\partial \mathbf{r}^*}{\partial s^*} \right| \approx 1, \quad h_\phi^* = \left| \frac{\partial \mathbf{r}^*}{\partial \phi} \right| = r_o^* + \eta^* \cos \psi, \quad h_\eta^* = \left| \frac{\partial \mathbf{r}^*}{\partial \eta^*} \right| = 1. \quad (C.15)
\]

The \( s^*, \phi \) and \( \eta^* \)-components of the dimensional Euler equations are stated in terms of the length scales (C.15) in (C.16)–(C.18) respectively, and the dimensional
Appendix C: Arbitrary body of revolution with slow spatial variable

The continuity equation is given in (C.19):

\[
\frac{\partial u_s^*}{\partial t^*} + \frac{u_s^* \partial u_s^*}{h_s^* \partial s^*} + \frac{u_\phi^* \partial u_s^*}{h_\phi^* \partial \phi} + \frac{u_\eta^* \partial u_s^*}{h_\eta^* \partial \eta^*} - \frac{u_s^* u_\phi^*}{h_s^* ds^*} - \frac{u_s^{2*}}{h_s^*} \sin \psi = - \frac{1}{h_s^*} \frac{\partial p^*}{\partial s^*}, \tag{C.16}
\]

\[
\frac{\partial u_\phi^*}{\partial t^*} + \frac{u_s^* \partial u_\phi^*}{h_s^* \partial s^*} + \frac{u_\phi^* \partial u_\phi^*}{h_\phi^* \partial \phi} + \frac{u_\eta^* \partial u_\phi^*}{h_\eta^* \partial \eta^*} - \frac{u_\phi^{2*}}{h_\phi^*} \sin \psi + \frac{u_\phi^* u_\eta^*}{h_\phi^*} \cos \psi = - \frac{1}{h_\phi^*} \frac{\partial p^*}{\partial \phi}, \tag{C.17}
\]

\[
\frac{\partial u_\eta^*}{\partial t^*} + \frac{u_s^* \partial u_\eta^*}{h_s^* \partial s^*} + \frac{u_\phi^* \partial u_\eta^*}{h_\phi^* \partial \phi} + \frac{u_\eta^* \partial u_\eta^*}{h_\eta^* \partial \eta^*} + \frac{u_s^{2*}}{h_s^* ds^*} - \frac{u_\eta^{2*}}{h_\eta^*} \cos \psi = - \frac{1}{h_\eta^*} \frac{\partial p^*}{\partial \eta^*}, \tag{C.18}
\]

\[
\frac{1}{h_\phi^* h_\eta^*} \left\{ \frac{\partial}{\partial s^*} \left( h_s^* u_s^* \right) + \frac{\partial}{\partial \phi} \left( h_\phi^* u_\phi^* \right) \right\} + \frac{\partial u_\eta^*}{\partial \eta^*} = 0, \tag{C.19}
\]

where \( u_s^*, u_\phi^*, u_\eta^* \) and \( p^* \) are the dimensional perturbed flow velocities and pressure. These quantities are formed from a basic flow component (denoted by an upper-case quantity) and a perturbing unsteady component (denoted by a lower case hatted quantity):

\[
u_s^* = a^* \Omega^* \left\{ U(S, \eta) + \epsilon \hat{u}_s(S, \eta, \phi, t) \right\}, \tag{C.20}
\]

\[
u_\phi^* = a^* \Omega^* \left\{ V(S, \eta) + \epsilon \hat{u}_\phi(S, \eta, \phi, t) \right\}, \tag{C.21}
\]

\[
u_\eta^* = a^* \Omega^* \left\{ \epsilon W(S, \eta) + \epsilon \hat{u}_\eta(S, \eta, \phi, t) \right\}, \tag{C.22}
\]

\[
p^* = \rho^* a^* \Omega^{23/2} \nu_s^{11/2} \left\{ \epsilon P + \hat{p}(S, \eta, \phi, t) \right\}, \tag{C.23}
\]

where \( \rho^* \) is the dimensional density and \( \epsilon \) is defined in (C.4).

We assume that the perturbing quantities have the normal mode form

\[
(\hat{u}_s(S, \eta, \phi, t), \hat{u}_\phi(S, \eta, \phi, t), \hat{u}_\eta(S, \eta, \phi, t)) = (\bar{u}_s(S, \eta), \bar{u}_\phi(S, \eta), \bar{u}_\eta(S, \eta)) e^{i(kS+n\phi-\omega t)}, \tag{C.23}
\]

where \( k \) and \( n \) are wave numbers in the \( s^* \) and \( \phi \) directions respectively, and \( \omega \) is the frequency. Substitution of (C.20)–(C.23) into (C.16)–(C.19) leads to the non-
dimensional perturbation equations (C.24)–(C.27):

\[-i\omega \bar{u}_s + ik U \bar{u}_s + \frac{i\bar{n}}{R_o} V \bar{u}_s + \bar{u}_\eta \frac{\partial U}{\partial \eta} = -ik \bar{p}, \tag{C.24}\]

\[-i\omega \bar{u}_\phi + ik U \bar{u}_\phi + \frac{i\bar{n}}{R_o} V \bar{u}_\phi + \bar{u}_\eta \frac{\partial V}{\partial \eta} = -i\bar{n} \frac{\partial \bar{p}}{\partial \eta}, \tag{C.25}\]

\[-i\omega \bar{u}_\eta + ik U \bar{u}_\eta + \frac{i\bar{n}}{R_o} V \bar{u}_\eta = -\frac{\partial \bar{p}}{\partial \eta}, \tag{C.26}\]

\[ik \bar{u}_s + \frac{i\bar{n}}{R_o} \bar{u}_\phi + \frac{\partial \bar{u}_\eta}{\partial \eta} = 0, \tag{C.27}\]

where \(\bar{n} = \epsilon n\). By eliminating \(\bar{u}_s\) and \(\bar{u}_\phi\) between (C.24)–(C.27) we can form the Rayleigh equation that governs the inviscid stability problem for the boundary-layer flow over an arbitrary rotating body in still fluid:

\[
\left( k U + \frac{\bar{n}}{R_o} V - \omega \right) \left( \frac{\partial^2 \bar{u}_\eta}{\partial \eta^2} - \left( k^2 + \frac{\bar{n}}{R_o^2} \right) \bar{u}_\eta \right) - \left( k \frac{\partial^2 U}{\partial \eta^2} + \frac{\bar{n}}{R_o} \frac{\partial^2 V}{\partial \eta^2} \right) \bar{u}_\eta = 0. \tag{C.28}\]

### C.4 Special cases

#### C.4.1 The rotating-cone boundary layer

The geometry of the rotating cone is such that \(R_o = S \sin \psi\), where \(S\) is a slow spatial variable along the cone surface and \(\psi\) is the cone half angle. With this particular local surface radius (C.9)–(C.11) become

\[
U \frac{\partial U}{\partial S} + W \frac{\partial U}{\partial \eta} - \frac{V^2}{S} = \frac{\partial^2 U}{\partial \eta^2}, \tag{C.29}\]

\[
U \frac{\partial V}{\partial S} + W \frac{\partial V}{\partial \eta} + \frac{UV}{S} = \frac{\partial^2 V}{\partial \eta^2}, \tag{C.30}\]

\[
\frac{\partial}{\partial \eta} (SU) + \frac{\partial}{\partial \eta} (SW) = 0. \tag{C.31}\]

By using the similarity solution

\[
\bar{U}(\eta) = \frac{U(S, \eta)}{S \sin \psi}, \quad \bar{V}(\eta) = \frac{V(S, \eta)}{S \sin \psi}, \quad \bar{W}(\eta) = W(S, \eta),
\]
the following set of coupled nonlinear ordinary-differential equations is found

\begin{align}
\dddot{U}'' + (\dddot{V}^2 - U^2) \sin \psi - \dot{W} \dddot{U}' = 0, \\
\dddot{V}'' - 2 \sin \psi \dddot{U} \dot{V} - W \dddot{V}' = 0, \\
2 \dddot{U} \sin \psi + \dddot{W}' = 0,
\end{align}

where a prime denotes differentiation with respect to \( \eta \). Note that (C.32)–(C.34) are identical to (4.11)–(4.13).

By using the transformed variables \( U = S \sin \psi \dddot{U}(\eta) \) and \( V = S \sin \psi \dddot{V}(\eta) \) in (C.28) we form the Rayleigh equation that governs the stability problem of the rotating-cone boundary layer:

\[
\left( kS \sin \psi \dddot{U} + \dddot{\bar{n}} \dddot{V} - \bar{\omega} \right) \left( \frac{\partial^2 \dddot{U}}{\partial \eta^2} - \left\{ k^2 + \frac{\dddot{\bar{n}}^2}{S^2 \sin^2 \psi} \right\} \dddot{\bar{U}}_{\eta} \right) \\
- \left( kS \sin \psi \frac{\partial^2 \dddot{U}}{\partial \eta^2} + \dddot{\bar{n}} \frac{\partial \dddot{V}}{\partial \eta} \right) \dddot{\bar{U}}_{\eta} = 0.
\]

Note that use of the transformed parameters

\[
\dddot{\bar{\omega}} = \omega / \dddot{\bar{n}}, \quad \dddot{\bar{S}} = S \sin \psi / \dddot{\bar{n}},
\]

enables the circumferential wavenumber \( \dddot{\bar{n}} \) to be scaled out of the problem:

\[
\left( kS \dddot{U} + \dddot{\bar{V}} - \dddot{\bar{\omega}} \right) \left( \frac{\partial^2 \dddot{U}}{\partial \eta^2} - \left\{ k^2 + \frac{1}{S^2} \right\} \dddot{\bar{U}}_{\eta} \right) - \left( kS \frac{\partial^2 \dddot{U}}{\partial \eta^2} + \dddot{\bar{V}} \frac{\partial \dddot{V}}{\partial \eta} \right) \dddot{\bar{U}}_{\eta} = 0.
\]

### C.4.2 The rotating-sphere boundary layer

The geometry of the rotating sphere is such that \( R_o = \sin S \), and so (C.9)–(C.11) become

\begin{align}
U \frac{\partial U}{\partial S} + W \frac{\partial U}{\partial \eta} - V^2 \cot S = \frac{\partial^2 U}{\partial \eta^2}, \\
U \frac{\partial V}{\partial S} + W \frac{\partial V}{\partial \eta} + UV \cot S = \frac{\partial^2 V}{\partial \eta^2}, \\
\frac{\partial U}{\partial S} + U \cot S + \frac{\partial W}{\partial \eta} = 0.
\end{align}
Note that these equations are identical are (2.6)–(2.8) if $S$ is considered to be the latitudinal angle as defined in figure 2.2. With this particular local surface radius the Rayleigh equation (C.28) is written

$$
\left( kU + \frac{\bar{n}}{\sin S} V - \omega \right) \left( \frac{\partial^2 \bar{u}_\eta}{\partial \eta^2} - \left\{ \frac{k^2}{\sin^2 S} \right\} \bar{u}_\eta \right) - \left( \frac{k}{\sin S} \frac{\partial^2 U}{\partial \eta^2} + \frac{\bar{n}}{\sin S} \frac{\partial^2 V}{\partial \eta^2} \right) \bar{u}_\eta = 0.
$$
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