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On the stability of receding horizon control for continuous-time stochastic systems

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ABSTRACT

We study the stability of receding horizon control for continuous-time non-linear stochastic differential equations. We illustrate the results with a simulation example in which we employ receding horizon control to design an investment strategy to repay a debt.

1. Introduction

In Receding Horizon Control (RHC), the control action, at each time t in $[0, \infty)$, is derived from the solution of an optimal control problem defined over a finite future horizon $[t, t+T]$. The RHC strategy establishes a feedback law which, under certain conditions, can ensure asymptotic stability of the controlled system. This control strategy has been successfully developed over the last twenty years for systems described by deterministic equations. In this context RHC is also well known as Model Predictive Control (MPC) and has proven to be very successful in dealing with non-linear and constrained systems, see e.g. [1–3]. The extension of RHC from deterministic to stochastic systems is the objective of current research. RHC schemes for the control of discrete-time stochastic systems have been proposed recently in [4–7].

In this note, we discuss RHC for systems described by continuous-time non-linear stochastic differential equations (SDEs). In order to study the stability of RHC for continuous-time SDEs, we formulate conditions under which the value function of the associated finite-time optimal control problem can be used as Lyapunov function for the RHC scheme. This is a well established approach for studying the stability of RHC schemes, which here is extended using Lyapunov criteria for stochastic dynamical systems [8]. We illustrate this contribution with a simple example of an optimal investment problem. Optimal investment problems are well suited to be tackled by stochastic control methods, see e.g. [9, 10]. In our example, we design an investment strategy to repay a debt. Having negative wealth due to an initial debt, the investor has the option to increase his/her current debt in order to buy a risky asset. The asymptotic stability of the adopted RHC scheme guarantees that the wealth of the investor tends to zero, so that the initial debt is eventually repaid.

2. Problem statement

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with the natural filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ generated by a standard Wiener process $W : [0, \infty) \times \Omega \to \mathbb{R}^d$ on it. We consider a controlled time-homogeneous SDE for a process $X : [0, \infty) \times \Omega \to \mathbb{R}^n$,

$$dX_t^{0,x_0,u} = b(X_t^{0,x_0,u}, u_t)dt + \sigma(X_t^{0,x_0,u}, u_t)dW_t, \qquad (1)$$
$$X_0^{0,x_0,u} = x_0,$$

where $x_0 \in \mathbb{R}^n$; $b : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times d}$ are continuous functions and satisfy

$$|b(x, u)| + |\sigma(x, u)| \leqslant C(1 + |x| + |u|), \quad \forall (x, u) \in \mathbb{R}^n \times U, \text{ (linear growth)},$$

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and

$$|b(x,u)-b(y,u)|+|\sigma(x,u)-\sigma(y,u)| \leqslant C|x-y|, \quad \forall(x,y,u)\in\mathbb{R}^n\times\mathbb{R}^n\times U, \text{ (Lipschitz)},$$

for some constant $C > 0$; and $u_{(\cdot)}$ is an admissible control process

$$u_{(\cdot)} \in \mathcal{U} := \left\{u : [0,\infty)\times\Omega\to U : \text{ progressively measurable and } \mathbb{E}\int_0^\infty |u_t(\omega)|^2 dt < \infty\right\},$$

with the set $U \subset \mathbb{R}^m$ compact. Here the superscripts of $X^{0,x_0,u}$ mean that the initial value of the process at time 0 is x_0 and the involved control process is $u_{(\cdot)}$. Under above conditions the SDE (1) has a unique adapted continuous square integrable solution X_t, $t \geqslant 0$. In this paper we are concerned with the conditions under which there exists a control process that drives the stochastic system X to the origin $0 \in \mathbb{R}^n$ and guarantees asymptotic stability of the controlled process. Here, the following definition of stability is adopted [8]:

Definition 2.1. Given a stochastic continuous-time process $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$, where $\mathbb{R}_+ := [0,\infty)$, with $X_0 = x_0 \in \mathbb{R}^n$

(S1) The origin is stable almost surely if and only if, for any $\rho > 0$, $\epsilon > 0$, there is a $\delta(\rho,\epsilon) > 0$ such that, if $|x_0| \leqslant \delta(\rho,\epsilon)$,

$$\mathbb{P}\left[\sup_{t\in\mathbb{R}_+}|X_t| \geqslant \epsilon\right] \leqslant \rho.$$

(S1′) An equivalent definition to (S1) is: Let $h(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ be a scalar-valued, nondecreasing, and continuous function of $|x|$. Let $h(0) = 0$, $h(r) > 0$ for $r \neq 0$. Then the origin is stable almost surely if and only if, for any $\rho > 0$, $\lambda > 0$, there is a $\delta(\rho,\lambda) > 0$ such that, for $|x_0| \leqslant \delta(\rho,\lambda)$,

$$\mathbb{P}\left[\sup_{t\in\mathbb{R}_+} h(|X_t|) \geqslant \lambda\right] \leqslant \rho.$$

(S2) The origin is asymptotically stable almost surely if and only if it is stable a.s., and $X_t \to 0$ a.s. for all x_0 in some neighbourhood R of the origin. If $R = \mathbb{R}^n$ then we add 'in the large'.

3. Main results

Let $T > 0$. As a preliminary step we consider the SDE for $X : [t,T]\times\Omega \to \mathbb{R}^n$ starting from the point $x \in \mathbb{R}^n$ at the time $t \in [0,T]$

$$dX_s^{t,x,u} = b(X_s^{t,x,u}, u_s)ds + \sigma(X_s^{t,x,u}, u_s)dW_s, \qquad X_t^{t,x,u} = x. \tag{2}$$

Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ and $g : \mathbb{R}^n \to \mathbb{R}_+$ be continuous nonnegative functions of polynomial growth, that is,

$$|f(x,u)| \leqslant C(1+|x|^p+|u|^q), \quad \forall(x,u)\in\mathbb{R}^n\times U,$$

and

$$|g(x)| \leqslant C(1+|x|^p), \quad \forall x \in \mathbb{R}^n,$$

for some constant $C > 0$ and some $p, q \geqslant 1$. Now we consider the problem of minimizing the following cost functional, $\forall(t,x) \in [0,T]\times\mathbb{R}^n$,

$$J[t,x;T;u_{(\cdot)}] := \mathbb{E}\left[\int_t^T f(X_s^{t,x,u}, u_s)ds + g(X_T^{t,x,u})\right] \tag{3}$$

over the set $\mathcal{U}$ of admissible control processes. We define the value function as

$$v(t,x;T) := \inf_{u_{(\cdot)}\in\mathcal{U}} J[t,x;T;u_{(\cdot)}] = \inf_{u_{(\cdot)}\in\mathcal{U}} \mathbb{E}\left[\int_t^T f(X_s^{t,x,u}, u_s(t,x;T))ds + g(X_T^{t,x,u})\right] \tag{4}$$

and denote $u_s^*(t,x;T)$, $t \leqslant s \leqslant T$, the optimal control process if it exists. In particular, when $t = 0$ we denote $V(x;T) := v(0,x;T)$.

Standard stochastic optimal control theories (see, for instance, [11,12]) about the controlled SDE (1) tell us that the Hamilton–Jacobi–Bellman (HJB) equation for the value function $v(\cdot,\cdot;T)$ is, $\forall(t,x) \in [0,T]\times\mathbb{R}^n$,

$$-\partial_t v(t,x;T) = \inf_{u\in U}\left[\frac{1}{2}\mathrm{tr}[\sigma\sigma^*(x,u)D^2v(t,x;T)] + \langle b(x,u), Dv(t,x;T)\rangle + f(x,u)\right], \tag{5}$$

$$v(T,x;T) = g(x).$$

Hereafter we use the notations

$$\partial_t v := \frac{\partial v}{\partial t}, \qquad Dv = \begin{pmatrix}\frac{\partial v}{\partial x_1}\\ \vdots \\ \frac{\partial v}{\partial x_n}\end{pmatrix}, \quad \text{and}$$

$$D^2 v = \begin{pmatrix} \frac{\partial^2 v}{\partial x_1^2} & \frac{\partial^2 v}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 v}{\partial x_1 \partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial^2 v}{\partial x_n \partial x_1} & \frac{\partial^2 v}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 v}{\partial x_n^2} \end{pmatrix}.$$

Suppose this HJB equation has a unique classical solution (see, e.g. [11] for conditions guaranteeing the existence and uniqueness.) and that the infimum in the equation is attained by $\tilde{u}(t,x;T)$ for every $(t,x) \in [0,T]\times\mathbb{R}^n$, i.e.,

$$-\partial_t v(t,x;T) = \frac{1}{2}\mathrm{tr}[\sigma\sigma^*(x,\tilde{u}(t,x;T))D^2v(t,x;T)] + \langle b(x,\tilde{u}(t,x;T)), Dv(t,x;T)\rangle + f(x,\tilde{u}(t,x;T)),$$

then we construct the optimal control process for the SDE (2) as

$$u_s^*(t,x;T) := \tilde{u}(s, X_s^{t,x,\tilde{u}}; T), \quad t \leqslant s \leqslant T. \tag{6}$$

As a consequence, the value function turns out to be

$$v(t,x;T) = \mathbb{E}\left[\int_t^T f(X_s^{t,x,u^*}, u_s^*)ds + g(X_T^{t,x,u^*})\right] = \mathbb{E}\left[\int_t^T f(X_s^{t,x,\tilde{u}}, \tilde{u}(s, X_s^{t,x,\tilde{u}};T))ds + g(X_T^{t,x,\tilde{u}})\right].$$

Now, for all the states $x \in \mathbb{R}^n$ all the time, we apply the specifically designed feedback law

$$u^c(x;T) := \tilde{u}(0,x;T) \tag{7}$$

to the stochastic system (1). In other words, for the state X_t^{0,x_0,u^c}, at any time $t \geqslant 0$, we apply only the initial optimal control

$$u^c(X_t^{0,x_0,u^c};T) = \tilde{u}(0, X_t^{0,x_0,u^c};T) = u_0^*(0, X_t^{0,x_0,u^c};T)$$

to the system. In particular, for $t = 0$, $u^c(X_0^{0,x_0,u^c};T) = u^c(x_0;T) = \tilde{u}(0,x_0;T)$. We require $u^c(\cdot;T)$ to be continuous and call it the continuous receding horizon control process with the receding horizon T for the controlled SDE (1).

When $t = 0$, using the HJB equation and (7), we obtain

$$-\partial_t v(t, x; T)|_{t=0} - f(x, u^c(x; T)) = \frac{1}{2}\mathrm{tr}[\sigma\sigma^*(x, u^c(x; T))D^2V(x; T)] + \langle b(x, u^c(x; T)), DV(x; T)\rangle.$$

Let us denote

$$\phi(x; T) := f(x, u^c(x; T)) + \partial_t v(t, x; T)|_{t=0} \tag{8}$$

for all $x \in \mathbb{R}^n$. Note that, since (2) is time-homogeneous, we have

$$\begin{aligned} v(t, x; T) &= \inf_{u(\cdot)\in\mathcal{U}} \mathbb{E}\left[\int_t^T f(X_s^{t,x,u}, u_s(t, x; T))ds + g(X_T^{t,x,u})\right] \\ &= \inf_{u(\cdot)\in\mathcal{U}} \mathbb{E}\left[\int_0^{T-t} f(X_{t+r}^{t,x,u}, u_{t+r}(t, x; T))dr + g(X_T^{t,x,u})\right] \\ &= \inf_{u(\cdot)\in\mathcal{U}} \mathbb{E}\left[\int_0^{T-t} f(X_r^{0,x,u}, u_r(0, x; T-t))dr + g(X_{T-t}^{0,x,u})\right], \end{aligned}$$

and

$$v(0, x; H) = V(x; H) = \inf_{u(\cdot)\in\mathcal{U}} \mathbb{E}\left[\int_0^H f(X_r^{0,x,u}, u_r(0, x; H))dr + g(X_H^{0,x,u})\right].$$

Thus

$$\partial_t v(t, x; T)|_{t=0} = -\partial_H v(0, x; H)|_{H=T} = -\partial_H V(x; H)|_{H=T}.$$

Hence, $\phi(x; T)$ can be equivalently expressed as

$$\phi(x; T) = f(x, u^c(x; T)) - \partial_H V(x; H)|_{H=T}. \tag{9}$$

Let us assume that

$$V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}), \tag{A1}$$

i.e. the set of functions $\psi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$, $(x, t) \mapsto \psi(x, t)$ that are twice continuously differentiable in x and once continuously differentiable in t; and that

$$\phi(0; T) = 0 \quad \text{and} \quad \phi(x; T) > 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\}; \tag{A2}$$

and that

$$\liminf_{|x|\to\infty} \phi(x; T) > 0. \tag{A3}$$

Assumption (A1) and the continuity of $u^c(\cdot; T)$ entails that $\phi(\cdot; T)$ is continuous. We now state one of our main results and then discuss assumptions (A1)–(A3) in more detail.

Proposition 3.1 (*Convergence*)**.** *Suppose $b(\cdot, \cdot)$ and $\sigma(\cdot, \cdot)$ are continuous functions of linear growth and Lipschitz, and $f(\cdot, \cdot)$ and $g(\cdot)$ are continuous nonnegative functions of polynomial growth. Suppose the HJB equation (5) has a unique classical solution and $u^c(\cdot; T)$ is continuous. Under the assumptions (A1), (A2), and (A3), we have that almost all the trajectories of the stochastic system (1) driven by the continuous receding horizon control $u^c(\cdot; T)$ defined in (7) converge to the origin.*

Proof. Here we denote $X_t := X_t^{0,x_0,u^c}$ for simplicity. By Itô's formula and the Hamilton–Jacobi–Bellman equation we have

$$\begin{aligned} dV(X_t; T) &= \langle DV(X_t; T), dX_t\rangle + \frac{1}{2}\mathrm{tr}[\sigma\sigma^*(X_t, u^c(X_t; T))D^2V(X_t; T)]dt \\ &= \langle DV(X_t; T), b(X_t, u^c(X_t; T))\rangle dt + \langle DV(X_t; T), \sigma(X_t, u^c(X_t; T))dW_t\rangle \\ &\quad + \frac{1}{2}\mathrm{tr}[\sigma\sigma^*(X_t, u^c(X_t; T))D^2V(X_t; T)]dt \\ &= -\phi(X_t; T)dt + \langle DV(X_t; T), \sigma(X_t, u^c(X_t; T))dW_t\rangle. \end{aligned}$$

By the continuity of $DV(\cdot; T)$, $\sigma(\cdot, \cdot)$, $u^c(\cdot; T)$, and X_t the process

$$\xi_t := \int_0^t \langle DV(X_r; T), \sigma(X_r, u^c(X_r; T))dW_r\rangle$$

is a continuous local martingale w.r.t. $\mathcal{F}_t$, see e.g. [13, pp. 36 and 146]. In other words, there exists an increasing sequence of $\mathcal{F}_t$-stopping times $\{\tau_k\}_{k\in\mathbb{N}}$ such that $\tau_k \to \infty$ a.s. as $k \to \infty$, and $\xi_{t\wedge\tau_k}$ is an $\mathcal{F}_t$-martingale for all k. Hence, for $0 \leqslant s \leqslant t < \infty$ and all k, $\mathbb{E}[\xi_{t\wedge\tau_k}|\mathcal{F}_s] - \xi_{s\wedge\tau_k} = 0$. Again by the continuity of DV, σ, u^c, X_t and the fact that X_t is normally distributed with finite expectation there exists a constant $C > 0$, dependent on s and t but independent of k, such that

$$\sup_{k\in\mathbb{N}} \mathbb{E}|\xi_{t\wedge\tau_k} - \xi_{s\wedge\tau_k}|^2 = \sup_{k\in\mathbb{N}} \mathbb{E}\int_{s\wedge\tau_k}^{t\wedge\tau_k} |\sigma^*(X_r, u^c(X_r; T))DV(X_r; T)|^2\, dr \leqslant C < \infty,$$

by which we learn that $\{\xi_{t\wedge\tau_k} - \xi_{s\wedge\tau_k}\}_{k\in\mathbb{N}}$ is uniformly integrable; see [14, Theorem C.3 on p. 311]. Thus by Vitali convergence theorem (see e.g. [14, Theorem C.4 on p. 312]) it follows that

$$\mathbb{E}[\xi_t|\mathcal{F}_s] - \xi_s = \lim_{k\to\infty} (\mathbb{E}[\xi_{t\wedge\tau_k}|\mathcal{F}_s] - \xi_{s\wedge\tau_k}) = 0.$$

Therefore by assumption (A2) we get that, for $0 \leqslant s \leqslant t < \infty$,

$$\mathbb{E}[V(X_t; T)|\mathcal{F}_s] - V(X_s; T) = -\mathbb{E}\left[\int_s^t \phi(X_r; T)dr \,\middle|\, \mathcal{F}_s\right] \leqslant 0. \tag{10}$$

In particular when $s = 0$, $\mathbb{E}[V(X_t; T)] \leqslant V(X_0; T) = V(x_0; T) < \infty$. Hence, $\{V(X_t; T), \mathcal{F}_t\}_{0\leqslant t<\infty}$ is a nonnegative supermartingale. Let us recall the supermartingale convergence theorem, see, for instance, [13, p. 18]: Suppose $\{M_t, \mathcal{F}_t\}_{0\leqslant t<\infty}$ is a right-continuous, nonnegative supermartingale. Then $M_\infty(\omega) := \lim_{t\to\infty} M_t(\omega)$ exists for $\mathbb{P}$-a.e. $\omega \in \Omega$, and $\{M_t, \mathcal{F}_t\}_{0\leqslant t\leqslant\infty}$ is a supermartingale. Hence, there exists a random variable V_∞, integrable, such that $V(X_t; T) \to V_\infty$, a.s.. Now, Eq. (10) implies particularly that

$$\mathbb{E}[V_\infty] - V(x_0; T) = -\int_0^\infty \mathbb{E}[\phi(X_t; T)]\, dt, \tag{11}$$

which in turn gives $\lim_{t\to\infty} \mathbb{E}[\phi(X_t; T)] = 0$. Therefore we obtain that

$$\lim_{t\to\infty} \phi(X_t; T) = 0, \text{ a.s.},$$

because $\phi \geqslant 0$. By assumption (A3) we learn that X_t does not go to infinity but travels into the set $\Phi := \{x \in \mathbb{R}^n : \phi(x; T) = 0\}$. Again from assumption (A2) we know that $\Phi = \{0\}$ and it is closed. Thus by the continuity of $\phi(\cdot; T)$ it is implied that almost all the trajectories of the process X_t converge to the origin in $\mathbb{R}^n$. □

In (A1), continuous twice differentiability of V is required for the applicability of Itô's formula. This assumption will be relaxed in future work (see Section 5).

In order to illustrate assumptions (A2) and (A3) note that $\partial_H V(x; H)|_{H=T}$ in (9) is the rate at which the optimal cost increases with increments in the horizon T. Now, one sufficient condition for $\phi(0; T) = 0$ in assumption (A2) is

$$b(0, u) = 0 \quad \text{and} \quad \sigma(0, u) = 0, \quad \forall u \in U; \tag{A2.1}$$

and

$$f(0, u) = 0, \quad \forall u \in U \text{ and } g(0) = 0. \tag{A2.2}$$

Note that (A2.1) requires that the state, at the origin, is not affected by the noise. This is an unavoidable assumption for obtaining asymptotic stability in the sense of Definition 2.1. If this assumption is not met, then one has to resort to other notions of stability such as, for example, mean square boundedness, see e.g. [6,7,15]. In light of (9) we obtain that a sufficient condition for $\phi(x;T) > 0,\ \forall x \in \mathbb{R}^n \setminus \{0\}$ in assumption (A2) is

$$\begin{cases} f(x,u) = 0 \quad \text{and} \quad \partial_H V(x;H)|_{H=T} < 0, \\ \quad \forall x \in \mathbb{R}^n \setminus \{0\},\ \forall u \in U;\ \text{or} \\ f(x,u) > 0 \quad \text{and} \quad \partial_H V(x;H)|_{H=T} \leqslant 0, \\ \quad \forall x \in \mathbb{R}^n \setminus \{0\},\ \forall u \in U. \end{cases} \tag{A2.3}$$

Here, we recover the condition of monotonic decrease of the value function with the length of the horizon, which is a well-studied condition for the stability of receding horizon control schemes in a deterministic setting, see e.g. [3,16,17].

Finally, a sufficient condition for (A3) is

$$\begin{cases} \liminf_{|x|\to\infty} f(x,u) = 0, \quad \forall u \in U \text{ and} \\ \quad \limsup_{|x|\to\infty} \partial_H V(x;H)|_{H=T} < 0;\ \text{or} \\ \liminf_{|x|\to\infty} f(x,u) > 0, \quad \forall u \in U \text{ and} \\ \quad \partial_H V(x;H)|_{H=T} \leqslant 0,\ \forall x \in \mathbb{R}^n \setminus \{0\};\ \text{or} \\ \lim_{|x|\to\infty} f(x,u) = +\infty, \quad \forall u \in U \text{ and} \\ \quad \exists B \text{ s.t. } \partial_H V(x;H)|_{H=T} \leqslant B,\ \forall x \in \mathbb{R}^n \setminus \{0\}. \end{cases} \tag{A3.1}$$

Under additional assumptions, asymptotic stability according to Definition 2.1 can be obtained. Let us assume in addition that for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that

$$V(x;T) < \epsilon \quad \text{if } |x| \leqslant \delta(\epsilon); \tag{A4}$$

and that there exists a continuous, nondecreasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying

$$h(0) = 0; \qquad h(\rho) > 0, \quad \forall \rho \neq 0,$$

such that

$$V(x;T) \geqslant h(|x|), \quad \forall x \in \mathbb{R}^n. \tag{A5}$$

Corollary 3.2 (*Asymptotic Stability*)**.** *Under the assumptions of Proposition 3.1, and assumptions (A4), and (A5), we have that the origin in $\mathbb{R}^n$ is asymptotically stable.*

Proof. Let us recall the supermartingale inequality, see, [13, p. 13]: Suppose $\{M_t, \mathcal{F}_t\}_{0\leqslant t<\infty}$ is a supermartingale whose every path is right-continuous. Let $\lambda > 0$ and $[a,b]$ be a subinterval of $[0,\infty)$. Then, we have

$$\mathbb{P}\left[\sup_{a\leqslant t\leqslant b} M_t \geqslant \lambda\right] \leqslant \frac{\mathbb{E}[M_a^+]}{\lambda}.$$

Hence, for any $\lambda > 0$, we have

$$\mathbb{P}\left[\sup_{0\leqslant t<\infty} V(X_t;T) \geqslant \lambda\right] \leqslant \frac{\mathbb{E}[V(x_0;T)]}{\lambda} = \frac{V(x_0;T)}{\lambda}.$$

Using assumption (A4), we obtain that for any $\lambda > 0$, $\rho > 0$, there exists $\delta(\rho,\lambda) > 0$ such that

$$\mathbb{P}\left[\sup_{0\leqslant t<\infty} V(X_t;T) \geqslant \lambda\right] \leqslant \rho \quad \text{if } |x_0| < \delta(\rho,\lambda).$$

Then, by assumption (A5), we immediately obtain

$$\mathbb{P}\left[\sup_{0\leqslant t<\infty} h(X_t) \geqslant \lambda\right] \leqslant \mathbb{P}\left[\sup_{0\leqslant t<\infty} V(X_t;T) \geqslant \lambda\right] \leqslant \rho,$$

which entails the asymptotic stability of the origin in $\mathbb{R}^n$. □

In the following section, we present a simple illustrative example in which assumptions (A1)–(A5) can be verified explicitly.

4. Example: repayment of a debt

In this example we consider a variant of the Merton's portfolio problem, see e.g. [11,12]. Suppose in a complete financial market there are only two assets, one asset being risk free such as, for instance, a bank deposit or a bond, and the other one being a risky asset such as, for instance, a stock. The assets obey the following price process, $t \geqslant 0$,

$$\begin{aligned} dP_t^1 &= rP_t^1 dt, \\ dP_t^2 &= bP_t^2 dt + \sigma P_t^2 dW_t, \end{aligned} \tag{12}$$

where $b > r > 0$, and $\sigma \neq 0$ are given constants. Here r is the risk free interest rate for the bank deposit, b is the drift, or average, rate of the stock's return, and σ is the volatility of the stock's return. Suppose an agent's total wealth at time t is X_t, comprising the risk free part $\Pi_t^1 P_t^1$ and the risky part $\Pi_t^2 P_t^2$, where Π_t^1 and Π_t^2 are the quantity of the assets, respectively. The control variable u_t is the portion of the total wealth X_t invested by the agent on the risky asset at time t. In a continuous-time setting, it is assumed that the allocation of wealth takes place instantaneously. Hence, for $t \geqslant 0$, we have:

$$\begin{aligned} X_t &= \Pi_t^1 P_t^1 + \Pi_t^2 P_t^2, \\ \Pi_t^1 P_t^1 &= (1-u_t)X_t, \\ \Pi_t^2 P_t^2 &= u_t X_t. \end{aligned} \tag{13}$$

We assume that the investment obeys the self financing condition. That is, starting from an initial wealth x_0, the agent can only sell or buy these two assets but is not allowed to borrow money from outside or consume his or her wealth. Written down as a differential equation, this is $P_t^1 d\Pi_t^1 + P_t^2 d\Pi_t^2 = 0$. Using (12) and (13) we obtain

$$\begin{aligned} dX_t &= \Pi_t^1 dP_t^1 + \Pi_t^2 dP_t^2 + P_t^1 d\Pi_t^1 + P_t^2 d\Pi_t^2 \\ &= \Pi_t^1 dP_t^1 + \Pi_t^2 dP_t^2 \\ &= [r + (b-r)u_t]X_t dt + \sigma u_t X_t dW_t. \end{aligned}$$

Therefore, the wealth satisfies the stochastic differential equation

$$\begin{aligned} dX_t^{0,x_0,u} &= [r + (b-r)u_t]X_t^{0,x_0,u} dt + \sigma u_t X_t^{0,x_0,u} dW_t, \\ X_0^{0,x_0,u} &= x_0, \end{aligned} \tag{14}$$

which is called the wealth process.

In this example, we assume that the agent's initial wealth is negative and his or her aim is to repay any debt eventually. Hence, we have $x_0 < 0$ and investigate the asymptotic stability of the origin $0 \in \mathbb{R}$. In this case, the wealth process will always be negative until the time it reaches zero. However, note that $X_t < 0$ does not necessarily mean that the agent has only debt without any money to invest or stock to sell. If negative values of u_t in (13) are allowed then the investor is able to increase his or her debt in order to buy stocks. Let us explain the financial meanings of all possible values of u_t when $X_t < 0$:

- If $u_t = 0$, then $\Pi_t^1 P_t^1 = X_t$ and $\Pi_t^2 P_t^2 = 0$. In this case, the investor just owns a debt with the bank equal to his or her negative wealth.
- If $u_t < 0$, then $\Pi_t^1 P_t^1 < X_t$ and $\Pi_t^2 P_t^2 > 0$. In this case, the investor is borrowing additional funds from the bank and is using it to buy stocks. The investor owns a debt with the bank equal to $(1-u_t)X_t$ and a positive quantity of stocks whose value is $u_t X_t$.
- If $u_t > 0$, then $\Pi_t^1 P_t^1 > X_t$ and $\Pi_t^2 P_t^2 < 0$. In this case, the investor is borrowing stocks (short selling) and is using this additional wealth to reduce his or her debt with the bank. However, in general, this is not desirable because a debt in stocks is more risky than a debt with the bank.

Note that if $\Pi_t^1 P_t^1 < 0$ (i.e. when the investor has a debt with the bank) then r is the rate at which interests are payed to the bank when owing debt. Here, in order to simplify the exposition of the problem, we assume that r is the same whether $\Pi_t^1 P_t^1 < 0$ or $\Pi_t^1 P_t^1 > 0$. However, it will be shown that this issue is immaterial. In fact, the derived control process will be constantly negative until the wealth reaches zero. In turn, this means that $\Pi_t^1 P_t^1 < 0$ and, therefore, r will not change meaning throughout. Finally, we consider the constraints $u_t \in [c_1, c_2]$ with $c_1 < 0$ and $c_2 \geqslant 0$. The constraint c_1 means that the investor is not allowed to increase the debt with the bank by more than $(1+|c_1|)$ times his or her current (negative) wealth. Similarly, c_2 is a constraint on borrowing stocks. If short selling is not allowed then $c_2 = 0$.

4.1. Running cost

We consider cost function (3) with

$$f(x,u) := |x|^\beta, \quad \forall x \in \mathbb{R}, \quad \text{and} \quad g(x) := 0, \quad \forall x \in \mathbb{R}, \tag{15}$$

with $\beta > 2$ being a given constant. Aiming to drive the process from $x_0 < 0$ to 0, we are not quite concerned about the behaviour of the process after the hitting time of the origin. But for technical reasons, i.e. consistency with the theory, we define the function f on the whole real line $\mathbb{R}$.

To start with, we consider the problem with restriction $x \in (-\infty, 0]$. In this case, the value function $v(t,x;T)$ satisfies the HJB equation, $(t,x) \in [0,T] \times (-\infty, 0]$,

$$\begin{aligned} -\partial_t v(t,x;T) \;&=\; \inf_{u\in[c_1,c_2]} \left[\frac{1}{2}(\sigma u x)^2 D^2 v(t,x;T) \right. \\ &\left. \quad + (r + (b-r)u)xDv(t,x;T) + (-x)^\beta \right], \end{aligned} \tag{16}$$

$$v(T,x;T) = 0.$$

If $D^2 v(t,x;T) > 0$ then a necessary condition for $\tilde{u}$ to be a minimizer is

$$\sigma^2 \tilde{u} x^2 D^2 v(t,x;T) + (b-r)xDv(t,x;T) = 0,$$

that is,

$$\tilde{u} = -\frac{(b-r)Dv(t,x;T)}{\sigma^2 x D^2 v(t,x;T)}. \tag{17}$$

In order to solve the HJB equation (16), we try to find a value function in the form

$$v(t,x;T) = (-x)^\beta w(t), \quad x \leqslant 0, \tag{18}$$

with w to be determined. For $x \leqslant 0$, by substituting (18) into (17), we obtain

$$\tilde{u} = -\frac{(b-r)(-\beta)(-x)^{\beta-1}}{\sigma^2 x \beta(\beta-1)(-x)^{\beta-2}} = -\frac{b-r}{(\beta-1)\sigma^2}. \tag{19}$$

Note that the so-obtained $\tilde{u}$ has the same expression as in the classical Merton's problem (although here we assumed $\beta > 2$ instead of $\beta < 1$), see e.g. [11, pp. 160–161], or [12, pp. 168–169]. Using (18) and (19), the HJB equation becomes

$$\begin{aligned} -(-x)^\beta w'(t) \;&=\; \frac{\beta(b-r)^2}{2(\beta-1)\sigma^2} x^2 (-x)^{\beta-2} w(t) \\ &\quad - \left(\beta r - \frac{\beta(b-r)^2}{(\beta-1)\sigma^2}\right)(-x)^{\beta-1} x w(t) + (-x)^\beta \end{aligned}$$

that is,

$$w'(t) = \left[\frac{\beta(b-r)^2}{2(\beta-1)\sigma^2} - \beta r\right] w(t) - 1 \tag{20}$$

and we obtain that the solution is

$$w(t) = \frac{1}{\eta}\left(1 - e^{\eta(t-T)}\right), \tag{21}$$

where

$$\eta := \frac{\beta(b-r)^2}{2(\beta-1)\sigma^2} - \beta r. \tag{22}$$

Hence, the value function (18) is actually given by

$$v(t,x;T) = (-x)^\beta \frac{1}{\eta}\left(1 - e^{\eta(t-T)}\right), \quad x \leqslant 0. \tag{23}$$

Here we assume $\eta \neq 0$. Note that whether $\eta < 0$ or $\eta > 0$ it always holds that $w(t) > 0$. Thus $D^2 v = \beta(\beta-1)(-x)^{\beta-2} w(t) > 0$ as we expected. (However, in the next subsection, we will narrow the requirement to be $\eta > 0$.)

Eventually, we obtain that the corresponding receding horizon control process is

$$u^c(X_t;T) = \tilde{u}(0, X_t; T) = -\frac{b-r}{(\beta-1)\sigma^2}. \tag{24}$$

The stochastic system (14) under the receding horizon control process (24) becomes

$$\begin{aligned} dX_t^{0,x_0,u} \;&=\; \left[r - \frac{(b-r)^2}{(\beta-1)\sigma^2}\right] X_t^{0,x_0,u} dt \\ &\quad - \frac{b-r}{(\beta-1)\sigma} X_t^{0,x_0,u} dW_t, \end{aligned} \tag{25}$$

$$X_0^{0,x_0,u} = x_0.$$

In this case the solution turns out to be a geometric Brownian motion that can be written down explicitly (see e.g.[13, pp. 349–50])

$$\begin{aligned} X_t^{0,x_0,u} \;&=\; x_0 \exp\left\{\left[r - \frac{(2\beta-1)(b-r)^2}{2(\beta-1)^2\sigma^2}\right] t \right. \\ &\left. \quad - \frac{b-r}{(\beta-1)\sigma} W_t \right\}. \end{aligned} \tag{26}$$

The asymptotic properties near the origin can be seen directly from the explicit solution (26). However, in the following subsection we will use Proposition 3.1 and Corollary 3.2 to assess the stability of the system. Note that the derived control policy (24) is in fact a constant process with negative value. This implies that the agent is always advised to borrow money from the bank and buy stocks with it. He or she will repay the debt in the end when his or her total wealth reaches zero by making profit from investment in stocks.

4.2. Verification of assumptions

Here we verify when assumptions (A1)–(A5) are met. Note that in the previous subsection we considered the HJB equation with restriction $x \in (-\infty, 0]$, but the solution for $x \in (0, \infty)$ follows analogous steps. In particular, the value function is

$$v(t,x;T) = \frac{x^\beta}{\eta}\left(1 - e^{\eta(t-T)}\right), \quad x > 0. \tag{27}$$

Thus, combining (23) and (27) yields to

$$v(t,x;T) = \frac{|x|^\beta}{\eta}\left(1 - e^{\eta(t-T)}\right), \quad \forall x \in \mathbb{R}. \tag{28}$$

Note that, for $v(0,x;T)$ given by (28) we have

$$V(x;T) = v(0,x;T) = \frac{|x|^\beta}{\eta}\left(1 - e^{-\eta T}\right), \quad \forall x \in \mathbb{R},$$

which is finite for all $(x, T) \in \mathbb{R} \times \mathbb{R}_+$. In addition, since $\beta > 2$, we learn that

$$D^2V(x; T) = \frac{\beta(\beta-1)|x|^{\beta-2}}{\eta}\left(1 - e^{-\eta T}\right), \quad \forall x \in \mathbb{R},$$

is continuous. This, combined with the continuity of

$$\partial_T V(x; T) = |x|^\beta e^{-\eta T}, \quad \forall x \in \mathbb{R},$$

ensures that $V \in C^{2,1}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R})$; thus assumption (A1) is satisfied. For

$$\begin{aligned}\phi(x) &= \partial_t v(t, x; T)|_{t=0} + f(x, u^c(x; T)) \\ &= |x|^\beta \left(1 - e^{-\eta T}\right), \quad \forall x \in \mathbb{R},\end{aligned}$$

to be positive we require $\eta > 0$. Hence assumption (A2) is satisfied when $\eta > 0$. Since $\phi(x) \to +\infty$ when $x \to \pm\infty$ we know assumption (A3) is met. In light of the continuity of V we know that for any $\epsilon > 0$ there is $\delta > 0$ such that $V(x; T) < \epsilon$ for $|x| < \delta$; thus assumption (A4) is satisfied. Since $V(x; T) = V(|x|; T)$ then if we choose $h(\rho) := V(\rho; T)$ for $\rho \geqslant 0$ then it holds that $V(x; T) \geqslant h(|x|)$ and $h(0) = 0$, $h(\rho) > 0$ for $\rho > 0$. So assumption (A5) is satisfied.

In conclusion, for given r, b and σ, we obtain that assumptions (A1)–(A5) are satisfied for all choices of β such that:

$$2 < \beta < 1 + \frac{1}{2}\frac{1}{r}\frac{(b-r)^2}{\sigma^2}, \tag{29}$$

where the inequality on the right-hand side corresponds to the condition $\eta > 0$. Thus, a necessary condition to have a stabilizing control process is that the right-hand side of the above inequality is greater that the left-hand side; that is $(b-r)^2 > 2r\sigma^2$. Finally, it is easy to see that the constraint $u \in [c_1, c_2]$ can be met provided that it is possible to choose

$$\beta > 1 + \frac{(b-r)}{|c_1|\sigma^2}, \tag{30}$$

which, taking into account (29), can be done if $|c_1| > 2r/(b-r)$.

4.3. Terminal cost

It is also possible to consider a cost function in the form

$$f(x, u) := 0, \quad \forall x \in \mathbb{R}, \quad \text{and} \quad g(x) := |x|^\beta, \quad \forall x \in \mathbb{R}. \tag{31}$$

The derived controlled process for $x \leqslant 0$ turns out to be the same as in (24). However, in this case, the value function is given by

$$v(t, x; T) = |x|^\beta e^{\eta(t-T)}, \quad \forall x \in \mathbb{R}, \tag{32}$$

where η is given again by (22). Through similar steps, one eventually obtains the same stability conditions of the previous case. However, note that in this case

$$\partial_T V(x; T) = -\eta|x|^\beta e^{-\eta T} \leqslant 0, \quad \forall x \in \mathbb{R}.$$

Hence, the sufficient condition (A2.3) is satisfied, while this is not true in the case of cost function (15) where (A2) is satisfied but (A2.3) is not.

4.4. Numerical illustration

We present a simulation example where: $r = 0.03$, $b = 0.1$, $\sigma = 0.15$, and $x_0 = -100$. Here we illustrate the behaviour of the wealth process under the receding horizon control process (24) for three different choices of β in cost function (15): $\beta = 2.1$, $\beta = 4.5$ and $\beta = 8$. The corresponding Monte Carlo simulations are displayed in Figs. 1–3 respectively. Note that, according to (29), for the given values of r, b and σ, we have that the wealth process is

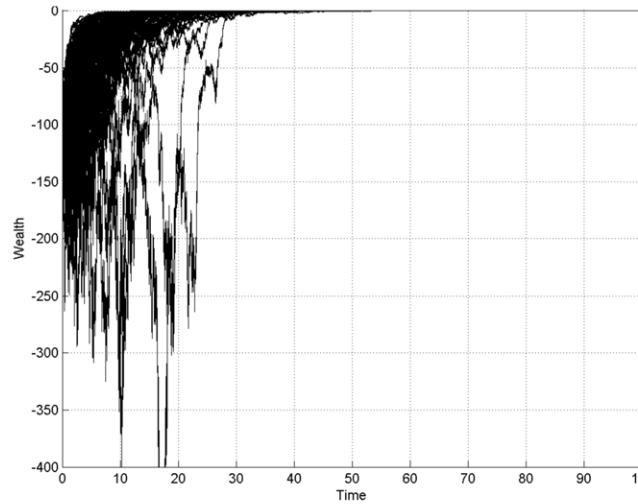

Fig. 1. Wealth process for $\beta = 2.1$ (100 simulations).

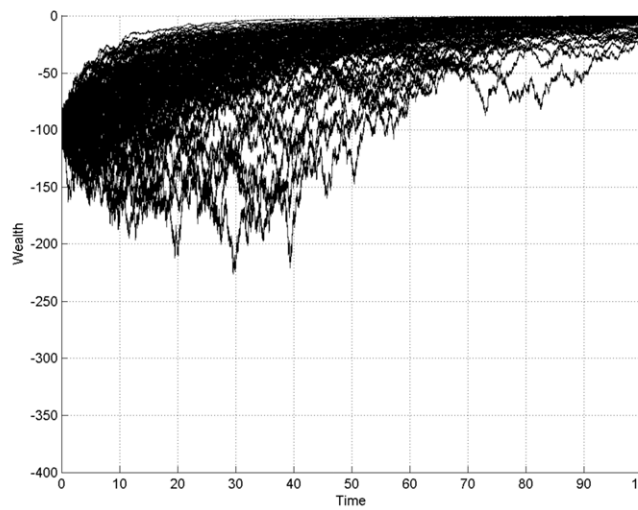

Fig. 2. Wealth process for $\beta = 4.5$ (100 simulations).

asymptotically stable for $\beta \in (2, 4.6)$. By inspecting the figures, it can be seen that for $\beta = 2.1$ the wealth process is clearly asymptotically stable but there is a significant risk of a large initial undershoot. For $\beta = 4.5$ the process converges much slower but the risk of a initial undershoot is reduced. For $\beta = 7.8$ the wealth process is not asymptotically stable (note the different time scale in the figure).

5. Conclusions

In this note, we have discussed the RHC strategy for systems described by continuous-time SDEs. We have obtained conditions on the associated finite-horizon optimal control problem which guarantee the asymptotic stability of the RHC law. We have shown that these conditions recall their deterministic counterpart. We have illustrated the results with a simple example in which the RHC law can be obtained explicitly.

In current work, we are addressing the problem of implementation in more realistic applications. For this purpose, it will be necessary to formulate conditions on the control problem (3) which can guarantee that assumptions (A1)–(A5) hold true and which can be imposed or verified easily. For problems where the dimension of the state space is not prohibitive, it is possible to solve the associated finite-horizon optimal control problem with numerical methods [18–20]. The other possibility is to approach the finite-horizon

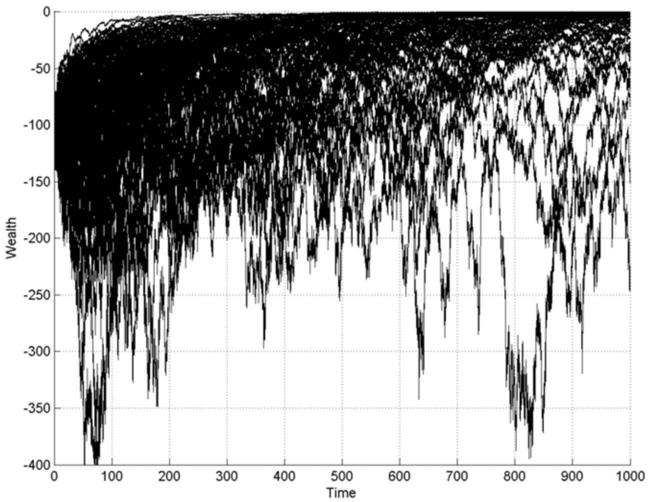

Fig. 3. Wealth process for $\beta = 7.8$ (100 simulations).

problem (3) by direct on-line optimization. In this case, one considers a parameterized class of feedback policies and iteratively optimizes the parameter of the feedback policy, at regular time intervals, based on the value of the current state. In this context simulation-based optimization methods have shown to be promising tools, see e.g. [21–23].

In addition, as well known, HJB equations generally do not admit classical solutions and the notion of viscosity solution is currently the standard framework for this subfield [12,24]. Our results can be generalized under this setting without essential difficulty, where assumption (A1) can be relaxed to be V being only continuous. Finally, the target of the controlled process can be generalized from a single point, the origin, to any compact set. Notions of stability can be defined analogously to (S1–S2) in Definition 2.1 with the function h being replaced by a distance function. This can be applied to more interesting and practical examples.

Finally, the reader is referred to [25] for a related approach being developed in the field of robotic navigation.

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