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Using manipulatives in the foundations of arithmetic

Main report

Rose Griffiths, Jenni Back and Sue Gifford

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“ Carefully chosen practical activities and resources ... have two principal benefits: they aid conceptual understanding and make learning more interesting. Too few of the schools used these resources well

Office for Standards in Education
(Ofsted) 2012: 27

Executive Summary

Manipulatives are objects that can be handled and moved and are used to develop learners' understanding of a mathematical situation.

We wanted to provide guidance for teachers of 3 to 9 year olds in the use of manipulatives to support children's deep and robust understanding of the number system through problem solving, reasoning, investigation, talk and explanation. In order to do this we focused on the early development of number sense, which is explored further in the literature review.

The project comprised three phases:

1. A review of the research literature related to the role of manipulatives in the teaching and learning of arithmetic. We also examined current practice in the use of manipulatives in the teaching of arithmetic through a questionnaire and focus group interviews.
2. The development of guidance on the use of manipulatives in the teaching of arithmetic, building upon work with teachers and children as well as the evidence from research.
3. Dissemination of the findings through the writing of a book for teachers, *Making Numbers*, and related presentations and professional development with teachers.

Findings

The survey of over 450 teachers, together with focus group interviews, suggested that:

- manipulatives were seen as most appropriate for younger children and those perceived as lower attainers, creating an association with age which risks stigmatizing older children perceived as low attainers.
- teachers' choice of manipulatives was subject to disparate factors rather than pedagogical principles

- teachers would welcome and needed guidance on the use of a range of manipulatives, in order to teach mathematics more effectively to all children.

The literature review and our work with teachers and children indicated key pedagogic principles for the effective use of manipulatives. These include:

- careful matching of both manipulatives and activities to the mathematical focus
- the identification and assessment of children's prerequisite understanding
- familiarisation through play
- the teaching of protocols and vocabulary related to the manipulatives
- activities involving comparison, equivalence, analysis and generalisation
- discussion, requiring children to use manipulatives to justify reasoning
- requiring children to visualize
- linking manipulatives to abstract symbols
- encouraging children's own recording on paper
- creating an inclusive mathematics learning community.

These pedagogical strategies help young children to generalize understanding to larger numbers, as well as giving teachers insights into children's understanding and thinking processes.

We identified three key aspects of number sense as the result of our review of the literature and our work with teachers and children:

- **Counting**: knowing the number names in order, forwards and backwards; understanding how to count objects, events or actions in ones, and also in twos, fives, tens, tens and ones, and so on.

- **Comparing**: having a feel for the relative sizes of numbers; putting numbers in order; estimating.
- **Composition**: understanding how each number can be made up in different ways by addition, subtraction, multiplication and division; knowing how our number system uses groups of hundreds, tens and ones.

Guidance

In order to reach the greatest number of teachers in a way that was readily accessible and engaging, a book for teachers was produced that adhered to the following principles:

- comprehensively illustrated with colour photographs and diagrams that linked carefully with the text, reducing the need for written explanations;
- clearly organized so that teachers could easily find items relevant to the children they were working with;
- including explanation about children's learning, pedagogical strategies, practical ideas for activities, examples of how children might use them and with links to further research.

Since a major focus is the different ways that a number can be made and represented, we chose *Making Numbers* as the title for our book. Each chapter considers a range of numbers: 0 to 12, 9 to 20, 15 to 50 and, finally, 25 to 200 and beyond. These ranges overlap in order to emphasise that children's understanding of number does not exist within discrete ranges.

We made four short films advising teachers and four animations for use in the classroom with children. The films and animations are available free on the OUP website.

Introduction

Manipulatives are objects that can be handled and moved. These range from everyday items such as buttons and shells, to commercially produced resources like Cuisenaire rods and Dienes' blocks. Research in teaching arithmetic to young children suggests that manipulatives can support mathematical development. However, the most effective ways to use manipulatives may not be clearly understood by teachers, and so they are often under-used.

This study aimed to develop research-informed guidance on the use of manipulatives in the teaching of arithmetic, for teachers of children aged three to nine years old. The project ran for two years (Sept 2014 – Oct 2016), during which we conducted a literature review, a survey of current practice, developed exemplars of good practice through observation and small-scale teaching investigations, and began the dissemination of findings. The resulting guidance, *Making Numbers*, is published by Oxford University Press as a fully illustrated book for teachers. It is accompanied on the OUP website by animations for children and by short videos offering advice on using the book and animations. For more information about *Making Numbers* see Oxford University Press, www.oxfordprimary.co.uk. The four animations and four films are available free at www.oxfordowl.co.uk.

This report summarises the main findings from the survey and the literature review. An accompanying report, *Using manipulatives in the foundations of arithmetic: Examples for teachers*, outlines the resulting guidance for teachers.

Background

The National Curriculum in England (DfE 2013) requires learners 'to move fluently between representations of mathematical ideas' (DfE 2013:3). However, in the absence of guidance, it is not immediately obvious to mathematics educators how we should help learners to do this. Teachers of younger children have traditionally begun by using practical resources or manipulatives, including everyday objects, manufactured toys, counters or mathematically structured apparatus, of which there is a wide variety currently marketed. Particularly in relation to arithmetic, there is a lack of published consensus about their relative merits or use. Although government inspectors advocate practical resources they have also criticized schools for the way they use them:

'Carefully chosen practical activities and resources ... have two principal benefits: they aid conceptual understanding and make learning more interesting. Too few of the schools used these resources well.' (Office for Standards in Education (Ofsted) 2012: 27)

More recently, the English government has been influenced by high performing jurisdictions, such as Singapore. Brown (2014) pointed out their curriculum was based on that of the UK in the 1950s and that practical work has become fashionable there, commenting that this factor seemed to have escaped some politicians. Current Singaporean influences in the UK, as evidenced by the government funded National Centre for Excellence in Teaching Mathematics (NCETM), include the use of Singapore textbooks, which advocate a Brunerian concrete- pictorial- abstract approach (Hoong, Kin and Pien, 2015). The government also advocates the Shanghai approach to teaching maths, which does not include manipulatives. Current policies compete with recommendations from previous policy, which supported the use of manipulatives such as beadstrings, influenced by Dutch research (Beishuizen, 2010). In addition, marketing aimed at schools promotes particular manipulatives, such as Numicon (Wing, 2001), which revives Stern's (1949) number plates based on a ten-frame image. This plethora of confusing recommendations underlined the need for research and guidance for teachers about the effective use of a range of manipulatives.

Our definition of manipulatives is: 'objects that can be handled and moved and are used to develop learners' understanding of a mathematical situation'. This includes both everyday and structured materials, used with a pedagogical intention, and reflects recent literature (Swan & Marshall, 2010; Carbonneau, Marley & Selig, 2013). In line with other studies, we include fingers but exclude measuring tools and calculators. We

also exclude virtual manipulatives, as being worthy of a separate study.

As stated in our original aims, we wanted to help teachers to support children's deep and robust understanding of the number system through problem solving, reasoning, investigation, talk and explanation. Our focus on arithmetic arose from its centrality in primary school curricula and the tensions that many primary school teachers experience between 'telling' children how to follow arithmetic procedures and supporting the children to understand and make sense of them for themselves. This led to the adoption of the idea of developing number sense as the key focus of primary arithmetic. We therefore investigated the learning and teaching of basic arithmetic for children aged 3 to 9, focusing on developing thinking and reasoning through the effective use of manipulatives.

The three phases of the project comprised a review of current and historical research and practice; the development of detailed exemplars of good practice in collaboration with teachers and children; and the dissemination of our exemplars and findings. The initial review of research and practice fed into the development of the guidance which aimed to support teachers to use manipulatives:

- in order to develop children's deep and robust understanding of the number system through a focus on number sense and mathematical thinking, including problem solving, reasoning and investigation.
- in a repertoire of materials and related images, used alongside each other in a rich multi-representational environment.
- in a progression from concrete to iconic and symbolic representation.

These aims were met in the resulting book, demonstrating the repertoire through illustrations of a variety of resources used in different combinations. It draws on evidence from research and includes examples of developing mathematical thinking through problem solving, pattern spotting and reasoning. However, we found from researching both theory and practice that effective teaching is more complex than simply presenting a linear progression in representation from concrete via pictorial to symbolic.

Review of current and historical research and practice

In order to discover what guidance was needed, the project included:

A survey, including a questionnaire and group interviews, to discover:

- current practice
- teachers' views on what guidance was needed.

A review of the literature regarding:

- the history of manipulative use
- theories of learning about manipulatives and representation
- studies of the effectiveness of manipulative use.

Survey of current practice

The survey used both quantitative and qualitative approaches. A questionnaire survey was conducted with approximately 450 teachers of 3 to 11 year olds in 35 English rural, suburban and urban schools in the Midlands, London and the South-East, representing a range of contexts. This was an opportunistic sample, from the geographical areas in which the researchers were based. Since we were working or in partnership with most of the schools, the teachers might have received more mathematics professional development than the general population of teachers. However, our relationships with the schools meant that we had an unusually high response rate, since headteachers asked staff to complete questionnaires in a staff meeting. Teachers were asked to report which manipulatives they had used in the previous month, according to age range and their understanding of the pupils' level of attainment. (While not wishing to label children, for the sake of brevity in communication, children perceived as attaining at these levels were referred to as lower, middle or higher attainers).

The responses showed that most teachers used manipulatives, but usage varied with children's age and their perceived attainment level. 95% of teachers of 5 to 7 year olds used them with middle attainers, compared with 70% for 9 to 11 year olds. Only 45% of teachers of older

children used them with higher attainers, but 92% used them with lower attainers. From a list of common manipulatives, the most frequently used across all the age groups were counters and interlocking cubes, followed by number plates (e.g. Numicon), Dienes' hundreds, tens and ones apparatus, money, and beadstrings. Some choices were consistent with government policies: however Numicon was used by 50% of teachers, which may be due to commercial marketing.

Focus group interviews were carried out to shed light on the reasons for choosing and using different manipulatives. They took place in the three locations in England and involved 20 teachers of children aged 3 to 9 from 13 primary schools. Focus groups offered the opportunity to gather information in a non-threatening environment and the discussion provoked richer responses from the teachers than we had been able to gather in the questionnaires. Once again the samples were opportunistic, predominantly being a subset of teachers who answered the questionnaire, and can be taken as indicative rather than definitive. The interviews suggested that teachers' selection of manipulatives was almost accidental, depending on what was available or had been encountered, and also identified teachers' lack of confidence about when and how to use manipulatives. Some teachers expressed reluctance to use manipulatives in case children became dependent on them. Some thought that

manipulatives were inappropriate for teaching about larger numbers: for one reception teacher, this meant using a number line rather than objects for numbers over 10. There were also concerns that children would be confused if their teacher provided more than one sort of manipulative.

Additionally, we consulted 22 academics, teacher educators and researchers, during a sequence of three workshops at an international conference in Prague, 2015 (SEMT at Charles University). Drawing upon their experience across nine countries, this group confirmed a greater use of manipulatives with their youngest children, but with less distinction according to attainment level. In many countries, the materials used were largely traditional, such as sticks and beans, and four countries used the abacus.

The survey and focus group interviews suggested overall that:

- manipulatives were seen as most appropriate for younger children and those perceived as lower attainers, creating an association with age which risks stigmatizing older children perceived as low attainers.
- teachers' choice of manipulatives was subject to disparate factors rather than pedagogical principles.

The findings confirmed teachers' need for current guidance on the use of a range of manipulatives, in order to teach mathematics more effectively to all children.

AGE RANGE	LOWER ATTAINERS	MIDDLE ATTAINERS	HIGHER ATTAINERS	NUMBER OF CLASSES
3–5	98	98	95	87
5–7	98	95	87	152
7–9	93	80	52	114
9–11	92	70	45	131
ALL	94	85	68	484

Table 1: Teachers reported use of manipulatives in the previous 4 weeks. Results by attainment group are given as percentages

Literature review

The methodological approach of the literature review was to consider studies and theories from a range of perspectives, in order to gain insights into factors affecting the pedagogical use of manipulatives. This included material relating to the history of pedagogy, cognitive and social constructivist theories of learning and empirical studies of teaching.

In the section that follows we offer a brief resume of our findings. For more detail, see the full literature review on the University of Leicester School of Education website.

Principles of manipulative use

The historical review highlighted shifts in the focus of western primary mathematics, moving from understanding number structures towards calculation proficiency over the period from the early twentieth century to the present. At times this has led to the use of manipulatives being dislocated from their original pedagogies, as with Dienes' (1969) multibase apparatus, to the disuse and rediscovery of manipulatives, such as Stern's number plates, and to the invention of new manipulatives, such as the Dutch rekenrek, which uses grouping in fives and has two rows of ten beads on a frame with groups of five in two different colours. Manipulative use has also been influenced by technological advances such as the development of plastic materials, resulting in the production of interlocking cubes.

A meta analysis (Carbonneau, Marley and Selig, 2013) of a large number of quantitative empirical studies of the effectiveness of manipulatives has pointed out that evidence has tended to be inconclusive and contradictory, possibly because they can only control for crude factors, such as length of interventions or the amount of 'instructional guidance'. However, like qualitative studies, their findings indicated the importance of a range of pedagogical processes, which were not well understood by the teachers involved.

Mathematical learning theories imply that using manipulatives can

increase conceptual understanding. Neuroscientifically, understanding is defined as networked representations, so that the more connections there are in different modes, the deeper the understanding (Goswami & Bryant 2007). This offers a neurologically based endorsement of multi-sensory teaching. Goldin (2002) identified internal representations as including imagistic (visual-spatial, tactile-kinaesthetic, auditory-rhythmic), verbal, written and affective modes. Potentially, manipulatives support representations and memory in all these modes.

Actively linking these internal representations through talking, drawing and writing can help strengthen networks of understanding: according to Hiebert and Carpenter (1992), making connections between external representations helps make connections between internal representations. This also suggests that comparing and contrasting representations of the same mathematical relations using different manipulatives can develop understanding. Vergnaud (1987) pointed out that all representations of a concept are necessarily partial, portraying some aspects and ignoring others: for instance, colour rods link to continuous models of numbers, whereas number plates emphasise odd and even properties.

However, the partiality of representations might be exploited to stimulate discussion of similarities and differences, as in a 'pedagogy for understanding through representation' suggested by Harries, Bamby and Suggate (2008).

Qualitative empirical studies also identify major issues, such as how individual children make sense of particular

manipulatives and the complexity of factors affecting this process. Sometimes difficulties arise because children do not have the prerequisite understandings for what is being represented, for instance part-whole relationships or units of 10. There is disagreement about whether the perceptual richness of everyday objects helps children to make connections between mathematics and their experience or whether the 'noise' obscures a mathematical focus (Uttal, Scudder & DeLoache, 1997; Carbonneau, Marley & Selig, 2013).

There may also be a developmental factor, with some models being too complex for younger children to access, as Bamby, Bolden, Raine & Thompson (2011) found with the multiplicative representation of the array. However, Papic, Mulligan and Mitchelmore (2011) found that individual young children varied in their awareness of pattern and mathematical structures and that this variation linked to mathematical achievement. They reported a successful intervention to develop pattern awareness with four year olds, which suggests that building on young children's strengths with visuo-spatial memory helps to develop pre-algebraic understanding. This echoes Gattegno's (1954) approach with colour rods with young children, in prioritising algebraic relations before numbers.

A major issue in all areas of the literature is how manipulatives may be used to develop understanding of generalized abstract relationships in the extended number system. While this development is clearly outlined in the pedagogical approaches of educationalists such as Dienes (1969) and Gattegno (1954), these may not be currently understood by

many teachers. This process is currently summarized as moving from concrete to abstract, but research suggests this is not a simple or one-way process. Bruner's (1966) enactive, iconic and symbolic modes of representation are currently recommended in England as the Singaporean Concrete–Pictorial–Abstract (CPA) progression, which has come to be interpreted as teachers presenting examples in different modes, in a linear sequence. This neither reflects original intentions for an active curriculum nor Bruner's non-linear pedagogy, according to Hoong, Kin and Pien (2015).

Mason and Johnston-Wilder (2006) interpret Bruner's sequence as providing active 'worlds of experience' for learners. They suggested a sequence of MGA:

- Manipulating objects or examples,
- Getting a sense of relationships, using visualization and 'schematic diagramatisation', and
- Articulating in verbal or symbolic form.

For this, teaching objectives need to focus on conceptual understanding and generalization: for example, children might find pairs of numbers which make a given total and develop generalisations about commutativity and compensation. This approach supports the linking of representations through discussion and children's own diagrammatic and symbolic recording, and also identifies the role of visualizing in moving away from manipulatives to more abstract thinking involving prediction and generalization.

The MGA approach also does not necessitate a one-way sequence, as CPA implies. We found this approach was exemplified in the work of Goutard (1964), who developed Gattegno's (1954) work with Cuisenaire rods in Canada: she identified three similar phases of conceptual learning as empirical (exploration), systematization, and mastery of structures. Goutard's work is currently being developed in England by teachers such as Ainsworth (2013), who also encourages children's 'free

writing', with interesting results in terms of abstract and symbolic generalization. Such an approach demonstrates the potential of manipulatives to develop abstract thinking, including algebraic and proportional reasoning.

The literature review indicated key teaching processes that need to accompany effective manipulative use. These include:

- careful matching of both manipulatives and activities to the mathematical focus
- the identification and assessment of children's prerequisite understanding
- familiarisation through play
- the teaching of protocols and vocabulary related to the manipulatives
- activities involving comparison, equivalence, analysis and generalisation
- discussion, requiring children to use manipulatives to justify reasoning
- requiring children to visualize
- linking manipulatives to abstract symbols
- encouraging children's own recording on paper
- creating an inclusive mathematics learning community.

These pedagogical strategies seem to help young children to generalize understanding to the extended number system, as well as giving teachers insights to children's understanding and thinking processes. Finally, for all children to engage in the kind of exploratory activity that encourages articulation, reasoning and conjectures, a respectful classroom ethos is required which gives all children positive expectations for their learning of mathematics.

Manipulatives are important for all as they offer a vehicle for explanation of complex mathematical ideas and offer children ways of modelling them that enable teachers to identify their level of understanding and assess obstacles to their mathematical development. They can be just as useful as a tool for supporting higher level mathematical thinking and

reasoning as an adjunct to performing calculations. The teachers we worked with appreciated seeing this additional aspect of their value.

Progression of ideas in arithmetic

We had identified number sense as our main focus and we drew on international literature to identify key ideas, including the review by Back, Sayers & Andrews (2013). We identified learning trajectories drawing on the work of Clements & Sarama (2009), Nunes & Bryant (2009) and others. In our study, for teachers of this age range, we identified three 'big ideas': counting, comparing and the composition of numbers. These are described in more detail later in this report, but are summarized here:

- Counting: knowing the number names in order, forwards and backwards; understanding how to count objects, events or actions in ones, and also in twos, fives, tens, tens and ones, and so on.
- Comparing: having a feel for the relative sizes of numbers; putting numbers in order; estimating.
- Composition: understanding how each number can be made up in different ways by addition, subtraction, multiplication and division; knowing how our number system uses groups of hundreds, tens and ones.

As we had identified as key principles teachers' knowledge of children's development of mathematical ideas and how to assess them, we structured the book in terms of the progression of these 'big ideas'.

The development of guidance for teachers

The development of exemplars of good practice was undertaken following the principles of design-based research (Swan 2006), based on:

- findings of the literature review
- observation of recommended good practice
- small scale teaching investigations
- activity trials within a series of professional development sessions.

The development of guidance for teachers was informed by the literature on the professional development of teachers, noting that a programme should address both the subject matter and knowledge required by the teacher, and ideas about how children learn that subject matter (Borko, 2004; Garet, Porter, Desimone, Birman and Yoon, 2001).

Swan (2006) articulates the principles of design-based research which focus on changing classroom practices through the systematic design of tasks and the advocacy of specific pedagogic approaches. We used an analogous approach: the literature review and survey provided several useful starting points, such as the need to match manipulatives and activities closely to children's current and intended skills, experience and understanding. The development process was also supported by observation of good practice and small-scale teaching investigations with teachers in schools. We trialled activities with linked pedagogy through a series of professional development sessions, in a model advocated in the Researching Effective CPD in Mathematics Project (NCETM, 2009). Four-session courses, focusing on the use of manipulatives in relation to curriculum areas, were carried out with four groups of teachers, involving over 100 teachers altogether. Time was allowed in between sessions for the teachers to trial activities in school and to consider what content knowledge and which pedagogical issues were important and helpful to them.

We used feedback from these courses to inform the development of our

advice and the activities presented in our guidance. Teachers' reports drew attention to the range of issues that needed to be considered. These echoed pedagogical findings from the literature review and included:

- practical considerations such as acquiring suitable manipulatives and storing them in ways that were accessible to children,
- choosing and devising suitable activities;
- knowing key aspects of children's possible learning trajectories for a particular concept, so that they could more easily assess children's understanding;
- encouraging discussion;
- questioning and challenging children
- finding ways of modelling productive activity

Manipulatives can be very useful as a tool for supporting higher level mathematical thinking and reasoning, as an adjunct to performing calculations, and the teachers we worked with appreciated seeing this additional aspect of their value.

The development process also pointed to several key ideas about number, which we have examined in depth, such as counting large numbers, part-whole relationships, unitizing and place value. A major focus is the different ways that a number can be made and represented, leading to deeper understanding of number relationships and flexibility in calculation: hence the title of our book, *Making Numbers*. Activities have consequently been designed to investigate equivalences, patterns and problem solving, to develop discussion about similarities and differences, and to encourage children's active linking of different representations through talking, drawing, writing and the use of symbols.

We had experimented with two less common forms of providing professional development during 2014 and 2015: an exhibition of classroom activities, and an

interactive community display. Both of these provided ideas that contributed

to this project. Both also demonstrated that teachers gain from watching children experimenting and directing their own work.

(a) The Counting Crazy exhibition.

With funding from a local organization, Leicester's School of Education and a group of schools and nurseries provided an exhibition of activities and advice on counting. It was open for three weeks for educational practitioners, parents and carers to visit, bringing children with them to try things out if they wished. The activities included games, role-play spaces and collections of items to count and organise. Teachers appreciated the opportunity to try something out straight away with children; they were interested in watching what children chose to do and how they used particular items; and they were able to discuss how they might repeat activities in their own classrooms with other visitors.

(b) Ten Thousand Hedgehogs exhibition

This co-operative project engaged over 40 schools, nurseries and children's organisations across the city of Leicester in making salt-dough hedgehogs, initially for an interactive exhibition in a city museum for schools and families to visit, and then 'on tour' visiting a succession of schools for a week at a time. Children could rearrange the hedgehogs, count them, and estimate how many they thought there were. Many teachers were surprised at children's interest and success in using the hedgehog manipulatives productively – for example, counting groups of between three and several hundred, depending on their age and level of skill. Given large numbers of interesting items, children's motivation to count and compare numbers was evident.

Both exhibitions seemed to help teachers appreciate the value of being able to visualize ten, one hundred or one thousand items. Many commented that

the hedgehog display was the first time that they had actually thought about what ten thousand items would look like. We had provided pens and paper for teachers to make notes at both of these exhibitions, but few people used them. Instead, most took photographs on their phones – reinforcing our existing feeling that any materials we produced for teacher development would need to be very visual.

Developing the book

In order to reach the greatest number of teachers in a way that was readily accessible and longer-lasting, a book for teachers provided the best solution. We therefore planned a book that conveyed messages through illustrations as well as through text, and adhered to several principles about the kind of book we would like to provide for teachers, to improve the likelihood of our work having a positive impact. The book would need to be accessible and thought-provoking:

- comprehensively illustrated with colour photographs and diagrams that linked carefully with the text –reducing the need for written explanations;
- clearly organized so that teachers could easily find items relevant to the children they were working with;
- including explanation about children’s learning, pedagogical strategies, practical ideas for activities, examples of how children might use them and with key links to further research.

Since a major focus of this project is the different ways that a number can be made and represented, the book was titled *Making Numbers*, and was published by Oxford University Press.

Whilst for our focus on number sense, we identified the themes of counting, comparing and composition, we decided that it would be more useful for teachers to have each chapter considering these three overlapping ideas together, using a specific range of numbers. In order

to emphasise that children’s number understanding does not exist within discrete ranges, we chose to overlap these, as numbers 0 to 12, 9 to 20, 15 to 50 and finally 25 to 200 and beyond.

We made four short films for teachers, taking advantage of film to show key aspects of moving items when exploring counting, comparing and composition, and to outline ways of using the animations. These are also available free on the OUP website.

Using manipulatives in the foundations of arithmetic:

Examples for teachers

The second part of this report is provided separately, and provides examples for teachers with illustrations from *Making Numbers*. These are organised within three components of number sense: counting, comparison and composition.



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@uniofleicester · /uniofleicester

For more information

Education

University of Leicester,
21 University Road,
Leicester, LE1 7RF, UK

t: +44 (0)116 252 3688

e: EducationDept@le.ac.uk

w: www.le.ac.uk/education



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