# A value function that explains the magnitude and sign effects.

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**Abstract**

Two of the anomalies of the exponentially discounted utility model are the 'magnitude effect' (larger magnitudes are discounted less) and the 'sign effect' (a loss is discounted less than a gain of the same magnitude). The literature has followed Loewenstein and Prelec (1992) in attributing the magnitude effect to the increasing elasticity of the value function and the sign effect to a higher elasticity for losses as compared to gains. We provide a simple, tractable, functional form that has these two properties, which we call the simple increasing elasticity value function (SIE). These functional forms underpin the main explanation of the magnitude and sign effects and may aid applications and further theoretical development.

*Keywords: Anomalies of the exponentially discounted utility model, the magnitude effect, the sign effect, SIE value functions.*

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## 1. Introduction

The standard model of intertemporal choice, exponentially discounted utility (EDU), is apparently contradicted by a large body of empirical evidence. See, for example, Loewenstein and Prelec (1992) (henceforth 'LP') and, for surveys, see Frederick et al. (2002) and Manzini and Mariotti (2008). Moreover, it appears that these anomalies are not simply mistakes; see Frederick et al. (2002), section 4.3.

Following LP, the subsequent literature has explained the *magnitude effect* (larger magnitudes are discounted less) through increasing elasticity of the utility function (IE). To date, the IE condition remains the main, if not the only, explanation of the magnitude effect. In this paper, we show that several popular classes of utility functions violate the IE condition. These include CARA (constant absolute risk aversion), CRRA (constant relative risk aversion), HARA (hyperbolic absolute risk aversion), logarithmic and quadratic.

We develop a scheme for generating utility functions that exhibit the IE property as required to explain the magnitude effect. We call the simplest class that has this property, the class of *simple increasing elasticity* utility functions (SIE). Each member of this class is formed by a product of a HARA function and a CRRA function and, therefore, is quite tractable. A particularly attractive feature of the SIE class of utility functions is that it is compatible with any theory where preferences are separable in time and outcomes.[1]

For intertemporal choice theories based on prospect theory (Kahneman and Tversky, 1979 and Tversky and Kahneman, 1992), rather than standard utility theory, the *sign effect* (a loss is discounted less than a gain of an equal magnitude) becomes important. Again following LP (1992), the literature explains the sign effect through a value function with higher elasticity for losses than for gains. We show that members of the SIE class of functions can also satisfy this property.

## 2. Formulation

LP describe four anomalies, all with good empirical support. We reproduce two below.

1. *Magnitude effect.* Thaler (1981) reported that subjects were, on average, indifferent between receiving \$15 immediately and \$60 one year later (an implied discount rate of 139% per annum[2]). They were also indifferent between receiving \$3000 immediately

[1] This includes, for instance, the following attempts to develop models that provide a better explanation of economic behavior over time such as Phelps and Pollak (1968), Loewenstein and Prelec (1992), Laibson (1997), Read (2001), Rubinstein (2003), Manzini and Mariotti (2006), Scholten and Read (2006a,b), Halevy (2007) and Ok and Masatlioglu (2007).

[2] The estimated discount factor $D^e$ equals the ratio of current to future reward times the ratio of marginal utilities of these rewards. Assuming that the marginal utilities are approximately the same,

and receiving \$4000 one year later (an implied discount rate of 29% per annum). This is a refutation of EDU because the implied discount rate is magnitude dependent and is too high.

2. *Sign effect* (or *gain-loss asymmetry*). Subjects in a study by Loewenstein (1988) were, on average, indifferent between receiving \$10 immediately and receiving \$21 one year later (an implied discount rate of 74% per annum). They were also indifferent between loosing \$10 immediately and losing \$15 dollars one year later (an implied discount rate of 40.5% per annum). Note that this is a refutation of EDU for two reasons. First, the implied discount rates are different, second, they are both too high (even allowing for capital market imperfections and liquidity constraints).

### 2.1. Prospect theory

We follow LP in taking prospect theory (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992) as our underlying decision theory. We take $v$ to be the value function introduced by Kahneman and Tversky (1979). Thus $v$ satisfies:

$$v : (-\infty, \infty) \to (-\infty, \infty) \text{ is continuous, strictly increasing (monotonicity).} \tag{2.1}$$

$$v(0) = 0 \text{ (reference dependence) and is twice differentiable except at 0.} \tag{2.2}$$

$$\text{For } x > 0\text{: } -v(-x) > v(x) \text{ (loss aversion).} \tag{2.3}$$

Following LP, we define the *elasticity* of $v$, $\epsilon_v(x)$, by

$$\epsilon_v(x) = \frac{x}{v}\frac{dv}{dx},\ x \neq 0. \tag{2.4}$$

### 2.2. Preferences

Let $\varphi : [0, \infty) \to (0, 1]$ be a strictly decreasing function with $\varphi(0) = 1$ and $\varphi(t) \to 0$ as $t \to \infty$. Then we call $\varphi$ a *discount function*. If, in addition, $\varphi$ is continuous, then we call $\varphi$ a *continuous discount function*.

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_+$ the set of non-negative reals. An *outcome* is an ordered pair $(x, t)$, $x \in \mathbb{R}$, $t \in \mathbb{R}_+$. $x$ may be interpreted as a monetary reward or an increment in wealth or consumption, etc. (a gain if $x > 0$, a loss if $x < 0$). $t$ is to be interpreted as the time at which $x$ is received. $v(x)$ is the value of $x$ at the time it is received. $v(x)\varphi(t)$ is the value of $x$ received at time $t$ discounted back to time

the ratio of rewards is simply used to approximate $D^e$. Thus, in this case, $D^e = \frac{15}{60} = 0.25$. Assuming continuous compounding, $D^e \approx D = e^{-\theta}$, where $\theta$ is the discount rate. Taking logs on both sides, $\theta = -\ln D^e$, which in this case is $-\ln(0.25) = 1.39$, as claimed. The same method is used to report the other discount rates in experiments, below.

0. $v(x)\varphi(t)[\varphi(s)]^{-1}$ is the value of $x$ received at time $t$ and discounted back to time $s$. $(y,t)$ is *equivalent* to $(x,s)$ if $v(x)\varphi(s) = v(y)\varphi(t)$. $(y,t)$ is *preferred* to $(x,s)$ if $v(x)\varphi(s) \leq v(y)\varphi(t)$. $(y,t)$ is *strictly preferred* to $(x,s)$ if $v(x)\varphi(s) < v(y)\varphi(t)$.

### 2.3. Assumptions and consequences

We introduce two assumptions from LP, followed by two theorems, also from LP.

**A1** *Magnitude effect.* If $0 < x < y$, $v(x) = v(y)\varphi(t)$ and $a > 1$, then $v(ax) < v(ay)\varphi(t)$. If $y < x < 0$, $v(x) = v(y)\varphi(t)$ and $a > 1$, then $v(ax) > v(ay)\varphi(t)$.

**A2** *Gain-loss asymmetry.* If $0 < x < y$ and $v(x) = v(y)\varphi(t)$, then $v(-x) > v(-y)\varphi(t)$.

**Proposition 1** *(LP, p584): For a continuous discount function, A1 implies that the value function is*
*(a) subproportional:* $(0 < x < y$ *or* $y < x < 0) \Rightarrow \frac{v(x)}{v(y)} > \frac{v(ax)}{v(ay)}$, *for* $a > 1$, *and*
*(b) more elastic for outcomes of larger absolute magnitude:* $(0 < x < y$ *or* $y < x < 0) \Rightarrow \epsilon_v(x) < \epsilon_v(y)$.

**Proposition 2** *(LP, p583): For a continuous discount function, A2 implies the following:*
*(a) Losses are discounted less heavily than gains in the following sense:* $0 < x < y \Rightarrow \frac{v(x)}{v(y)} > \frac{v(-x)}{v(-y)}$.
*(b) The value function is more elastic for losses than for gains:* $x > 0 \Rightarrow \epsilon_v(-x) > \epsilon_v(x)$.

We now add the standard assumption from prospect theory that the value function is strictly concave for gains and strictly convex for losses (Kahneman and Tversky, 1979).

**A3** *Declining sensitivity.* For $x > 0$, $v''(x) < 0$ (strict concavity for gains). For $x < 0$, $v''(x) > 0$ (strict convexity for losses).

Combining A3 with Proposition 1 we get:

**Proposition 3** *: A1 and A3 imply that* $0 < \epsilon_v < 1$.

## 3. Decreasing elasticity of HARA utility functions

We consider several popular classes of value functions including CRRA (constant relative risk aversion), HARA (hyperbolic absolute risk aversion), CARA (constant absolute risk aversion), logarithmic and quadratic[3]. Proposition 4, below, shows that each member of this family exhibits constant or declining elasticity, contradicting LP's Proposition 1.

[3] The latter three classes of functions are also regarded as members of the HARA family.

1. Constant relative risk aversion functions (CRRA)[4]

$$v(x) = \frac{x^{1-\gamma}}{1-\gamma},\ 0<\gamma<1;\ \epsilon_v(x) = \frac{xv'(x)}{v(x)} = 1-\gamma. \tag{3.1}$$

2. Hyperbolic absolute risk aversion functions (HARA)[5]

$$\begin{aligned} v(x) &= \frac{\gamma}{1-\gamma}\left[\left(\mu+\frac{\theta x}{\gamma}\right)^{1-\gamma} - \mu^{1-\gamma}\right],\ \theta>0,\ \mu>0,\ 0<\gamma<1,\ x\geq 0, \\ \epsilon_v(x) &= \frac{xv'(x)}{v(x)} = \frac{(1-\gamma)\frac{\theta x}{\gamma}\left(\mu+\frac{\theta x}{\gamma}\right)^{-\gamma}}{\left(\mu+\frac{\theta x}{\gamma}\right)^{1-\gamma} - \mu^{1-\gamma}}. \end{aligned} \tag{3.2}$$

3. Constant absolute risk aversion functions (CARA)

$$v(x) = 1-e^{-\theta x},\ \theta>0,\ x\geq 0;\ \epsilon_v(x) = \frac{1}{\sum_{n=1}^{\infty}\frac{(\theta x)^{n-1}}{n!}}. \tag{3.3}$$

4. Logarithmic functions

$$v(x) = \ln(1+\theta x),\ \theta>0,\ x\geq 0;\ \epsilon_v(x) = \frac{\theta x}{(1+\theta x)\ln(1+\theta x)}. \tag{3.4}$$

5. Quadratic functions

$$v(x) = \frac{1}{2}\mu^2 - \frac{1}{2}(\mu-\theta x)^2,\ \theta>0,\ 0\leq x<\frac{\mu}{\theta};\ \epsilon_v(x) = \frac{2\theta x(\mu-\theta x)^2}{\mu^2-(\mu-\theta x)^2}. \tag{3.5}$$

Proposition 4 shows that none of the common types of utility functions can be used to operationalize the LP explanation of the magnitude effect.

**Proposition 4** : *For members of the CRRA class of value functions (3.1), $\epsilon_v(x)$ is constant. For members of the HARA (3.2), CARA (3.3), logarithmic (3.4) and quadratic (3.5) classes of functions, $\epsilon_v(x)$ is declining. Hence these families violates Proposition 1.*

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[4] The general restriction is that $\gamma\neq 1$. However, we need the stronger restriction, $0<\gamma<1$, in order to satisfy Proposition 3.

[5] The general restrictions are $\theta>0$, $\left(\mu+\frac{\theta x}{\gamma}\right)^{1-\gamma}>0$, $\gamma\neq 1$. Since we allow $x\in[0,\infty)$, the restriction $\left(\mu+\frac{\theta x}{\gamma}\right)^{1-\gamma}>0$ implies that $\mu>0$ and $\gamma>0$. We then also need $\gamma<1$ in order to satisfy Proposition 3.

Note that, traditionally, the HARA class is defined by $v(x) = \frac{\gamma}{1-\gamma}\left(\mu+\frac{\theta x}{\gamma}\right)^{1-\gamma}$, and that $\epsilon_v(x) = (1-\gamma)\left(1+\frac{\gamma\mu}{\theta x}\right)^{-1}$, which is *increasing* in $x$, as required by Proposition 1. While an additive constant, of course, makes no difference in expected utility theory; its absence here would violate the assumption $v(0)=0$. However, including the constant $-\frac{\gamma}{1-\gamma}\mu^{1-\gamma}$, to make $v(0)=0$, results in $\epsilon_v(x)$ *decreasing* with $x$, as will be shown by Proposition 4, and, hence, violates Proposition 1.

## 4. Increasing elasticity utility functions

In this section we, first, provide a simple tractable functional form for the value function that is compatible with Propositions 1,2 and 3; the *simple increasing elasticity* (SIE) value function. Second, we provide a scheme for generating further such functions.

Choose a function, $h(x)$, satisfying:

$$0 < h(x) < 1,\ h'(x) > 0, \tag{4.1}$$

then solve the following differential equation for $v(x)$:

$$\frac{x}{v}\frac{dv}{dx} = h(x). \tag{4.2}$$

This method only yields candidate value functions, which then have to be verified. For example, choose

$$h(x) = \frac{ax}{b+x} + c,\ x \geq 0,\ a > 0,\ b > 0,\ c > 0,\ a + c \leq 1. \tag{4.3}$$

Substituting from (4.3) into (4.2), separating variables, then integrating, gives:

$$\begin{aligned} \frac{x}{v}\frac{dv}{dx} &= \frac{ax}{b+x} + c, \\ \int \frac{dv}{v} &= a\int \frac{dx}{b+x} + c\int \frac{dx}{x}, \\ \ln v &= a\ln(b+x) + c\ln x + \ln K, \\ v(x) &= K(b+x)^a x^c. \end{aligned} \tag{4.4}$$

Choosing $a = 1-\gamma$, $b = \frac{\gamma\mu}{\theta}$, $c = \sigma$ and $K = \frac{\gamma}{1-\gamma}\left(\frac{\theta}{\gamma}\right)^{1-\gamma}$ produces a candidate value function for gains:

$$v(x) = \frac{\gamma x^\sigma}{1-\gamma}\left(\mu + \frac{\theta}{\gamma}x\right)^{1-\gamma},\ x \geq 0, \tag{4.5}$$

The restrictions $a > 0$, $b > 0$, $c > 0$, $a + c \leq 1$ give $0 < \sigma \leq \gamma < 1$ and $\mu/\theta > 0$. To ensure that $v' > 0$, take $\theta > 0$. Hence, $\mu > 0$. Putting all these together gives the candidate value function

$$\begin{aligned} v(x) &= \frac{\gamma x^\sigma}{1-\gamma}\left(\mu + \frac{\theta_+}{\gamma}x\right)^{1-\gamma},\ x \geq 0, \\ v(x) &= -\lambda\frac{\gamma(-x)^\sigma}{1-\gamma}\left(\mu - \frac{\theta_-}{\gamma}x\right)^{1-\gamma},\ x < 0, \\ \mu &> 0,\ \theta_- > \theta_+ > 0,\ \lambda \geq 1,\ 0 < \sigma \leq \gamma < 1. \end{aligned} \tag{4.6}$$

It may be interesting to note that (4.6) is a product of a CRRA function, $x^{\sigma}$, and a HARA function, $\frac{\gamma}{1-\gamma}\left(\mu+\frac{\theta_{\pm}}{\gamma}x\right)^{1-\gamma}$.

Proposition 5, below, establishes that the value function (4.6) has all the desirable properties.

**Proposition 5** *: From (4.6) it follows that*
*(a)* $v:(-\infty,\infty)\rightarrow(-\infty,\infty)$ *is continuous,* $v(0)=0$ *(reference dependence),* $v$ *is* $C^{\infty}$*except at* $x=0$ *and, for* $x>0$, $-v(-x)>v(x)$ *(loss aversion).*
*(b)* $\epsilon_v(x)=\sigma+\frac{1-\gamma}{1+\frac{\gamma\mu}{\theta_+ x}}>0$, $\epsilon_v'(x)>0$, $x>0$.
*(c)* $\epsilon_v(x)=\sigma+\frac{1-\gamma}{1-\frac{\gamma\mu}{\theta_- x}}>0$, $\epsilon_v'(x)<0$, $x<0$.
*(d)* $x>0\Rightarrow\epsilon_v(x)<\epsilon_v(-x)$.

**Remark 1** *(The sign effect): The restriction* $\theta_->\theta_+>0$ *(along with the other restrictions) guarantees that* $x>0\Rightarrow\epsilon_v(x)<\epsilon_v(-x)$*, as required by Proposition 2(b).*[6]

**Corollary 1** *: From (b) and (c) of Proposition 5, we get that* $\epsilon_v(x)\rightarrow\sigma$ *as* $x\downarrow 0$ *and as* $x\uparrow 0$*. Hence we can define* $\epsilon_v(x)$ *as a continuous function for all* $x\in(-\infty,\infty)$ *as follows.* $\epsilon_v(0)=\sigma$, $\epsilon_v(x)=\sigma+\frac{1-\gamma}{1+\frac{\gamma\mu}{\theta_{\pm}|x|}}>0$ *for* $x\neq 0$*. Note that* $\epsilon_v(x)$ *is increasing in* $|x|$ *and* $\epsilon_v(x)\rightarrow\sigma+1-\gamma\leq 1$*, as* $|x|\rightarrow\infty$*.*

**Remark 2** *(SIE value function): In the light of Corollary 1, we may call the value function (4.6) a simple increasing elasticity (SIE) value function.*

# 5. Appendix: Proofs

The proofs of Propositions 1 and 2 are more detailed versions of those in Loewenstein and Prelec (1992). We first establish some preliminary results in Lemmas 1 and 2.

**Lemma 1** *: A continuous discount function is onto* $(0,1]$.

**Proof of Lemma 1**: Let $\varphi(t):[0,\infty)\rightarrow(0,1]$ be a continuous discount function. Let $0<x\leq 1$. If $x=1$, then $\varphi(0)=x$. Suppose $0<x<1$. Hence, $x<\varphi(0)$. Since $\varphi(t)\rightarrow 0$ as $t\rightarrow\infty$, it follows that $\varphi(s)<x$ for some $s\in[0,\infty)$. Hence, $\varphi(s)<x<\varphi(0)$. Since $\varphi$ is continuous it follows, from the intermediate value theorem, that $x=\varphi(t)$, for some $t\in(0,s)\subset[0,\infty)$. □

**Lemma 2** *: Let* $\varphi$ *be a continuous discount function. If* $0<x\leq y$*, or if* $y\leq x<0$*, then* $v(x)=v(y)\varphi(t)$ *for some* $t\in[0,\infty)$.

[6] One may wonder whether it is possible to allow the other parameters to take different values in the domains of gains and losses. The answer is no, as can be easily shown.

**Proof of Lemma 2**: Let $\varphi(t)$ be a continuous discount function and $r \geq 0$. Suppose $0 < x \leq y$. From (2.1) and (2.2), it follows that $0 < v(x) \leq v(y)$ and, hence, $0 < \frac{v(x)}{v(y)} \leq 1$. Since $\varphi(t) : [0, \infty) \to (0, 1]$ is onto, it follows that $\frac{v(x)}{v(y)} = \varphi(t)$ for some $t \in [0, \infty)$. A similar argument applies if $y \leq x < 0$. $\square$

**Proof of Proposition 1:** Start by assuming

$$0 < x < y. \tag{5.1}$$

By Lemma 2, there is a time, $t$, such that the consumer is indifferent between receiving the increment $x$ now and receiving the increment $y$, $t$-periods from now. Then, letting $v$ be the value function and $\varphi$ the discount function, we get

$$v(x) = v(y)\varphi(t), \text{ for some } t \in [0, \infty). \tag{5.2}$$

Let

$$a > 1, \tag{5.3}$$

then the *magnitude effect*, assumption A1, predicts that

$$v(ax) < v(ay)\varphi(t), \tag{5.4}$$

(5.2) gives

$$\frac{v(x)}{v(y)} = \varphi(t). \tag{5.5}$$

Since $y, a$ are positive, it follows that $ay$, and hence, $v(ay)$ are also positive. Hence, (5.4) gives

$$\frac{v(ax)}{v(ay)} < \varphi(t), \tag{5.6}$$

(5.5) and (5.6) give[7]

$$\frac{v(ay)}{v(ax)} > \frac{v(y)}{v(x)}. \tag{5.7}$$

Since $a > 0, x > 0$ and $y > 0$, and since $v(0) = 0$ and $v$ is strictly increasing, it follows that $v(x), v(y), v(ax)$ and $v(ay)$ are all positive. Let $\widetilde{x} = \ln x$, $\widetilde{y} = \ln y$, $\widetilde{a} = \ln a$. Then, since $x < y$ and $a > 1$, it follows that $\widetilde{x} < \widetilde{y}$, $\widetilde{a} > 0$. Let $\widetilde{v}(\widetilde{x}) = \ln v\left(e^{\widetilde{x}}\right)$, then (5.7) gives $\ln v(ay) - \ln v(ax) > \ln v(y) - \ln v(x)$, which leads to

$$\widetilde{v}(\widetilde{y} + \widetilde{a}) - \widetilde{v}(\widetilde{x} + \widetilde{a}) - [\widetilde{v}(\widetilde{y}) - \widetilde{v}(\widetilde{x})] > 0. \tag{5.8}$$

Take $\delta x > 0$, $\widetilde{a} = \delta x$, $\widetilde{y} = \widetilde{x} + \delta x$, then (5.8) gives

$$\frac{\widetilde{v}(\widetilde{x} + 2\delta x) - \widetilde{v}(\widetilde{x} + \delta x) - [\widetilde{v}(\widetilde{x} + \delta x) - \widetilde{v}(\widetilde{x})]}{(\delta x)^2} > 0. \tag{5.9}$$

[7] In the course of their proof, LP derive, incorrectly, the formula (LP (18) p583): $\frac{v(x)}{v(y)} < \frac{v(ax)}{v(ay)}, 0 < x < y; a > 1$ (the first $<$ should be $>$)

Now

$$\left[\frac{d\widetilde{v}}{dx}\right]_{x=\widetilde{x}} = \lim_{\delta x\to 0}\frac{\widetilde{v}\left(\widetilde{x}+\delta x\right)-v\left(\widetilde{x}\right)}{\delta x},$$

$$\left[\frac{d\widetilde{v}}{dx}\right]_{x=\widetilde{x}+\delta x} = \lim_{\delta x\to 0}\frac{\widetilde{v}\left(\widetilde{x}+2\delta x\right)-\widetilde{v}\left(\widetilde{x}+\delta x\right)}{\delta x},$$

$$\left[\frac{d^2\widetilde{v}}{dx^2}\right]_{x=\widetilde{x}} = \lim_{\delta x\to 0}\frac{\left[\frac{d\widetilde{v}}{dx}\right]_{x=\widetilde{x}+\delta x}-\left[\frac{d\widetilde{v}}{dx}\right]_{x=\widetilde{x}}}{\delta x}.$$

Hence,

$$\frac{d}{dx}\left(\epsilon\left(x\right)\right) = \left[\frac{d^2\widetilde{v}}{dx^2}\right]_{x=\widetilde{x}} = \lim_{\delta x\to 0}\frac{\widetilde{v}\left(\widetilde{x}+2\delta x\right)-\widetilde{v}\left(\widetilde{x}+\delta x\right)-\left[\widetilde{v}\left(\widetilde{x}+\delta x\right)-\widetilde{v}\left(\widetilde{x}\right)\right]}{\left(\delta x\right)^2}. \tag{5.10}$$

Since the limit of a converging sequence of positive numbers in non-negative, we get, from (5.9) and (5.10):

$$\frac{d}{dx}\left(\epsilon\left(x\right)\right)\geq 0.$$

If $\epsilon\left(x\right)$ were constant on some non-empty open interval, then the value function would take the form $v\left(x\right)=cx^{\gamma}$ on that interval and subproportionality, (5.7), would be violated. Hence $\epsilon'\left(x\right)>0$ almost everywhere. Thus $\epsilon\left(x\right)$ increases with $x$.

Now consider the case $y<x<0$. Then (5.7) still holds. But now we define $\widetilde{x}=\ln\left(-x\right)$, $\widetilde{y}=\ln\left(-y\right)$ and $\widetilde{v}\left(\widetilde{x}\right)=-\ln\left(-v\left(-e^{\widetilde{x}}\right)\right)$. As before, (5.8) holds and $v'\left(x\right)>0$ almost everywhere. Thus $\epsilon\left(x\right)$ increases with $x$.

It then follows that the value function is more elastic for outcomes that are larger in absolute magnitude. □

**Proof of Proposition 2:** Suppose $0<x<y$. By Lemma 2, $v\left(x\right)=v\left(y\right)\varphi\left(t\right)$ for some $t\in[0,\infty)$. Hence, $\frac{v(x)}{v(y)}=\varphi\left(t\right)$. By assumption A2, $v\left(-x\right)>v\left(-y\right)\varphi\left(t\right)$. Since $-y<0$, it follows that $v\left(-y\right)<0$ and, hence, $\frac{v(-x)}{v(-y)}<\varphi\left(t\right)$. Then, $\frac{v(-x)}{v(-y)}<\frac{v(x)}{v(y)}$. Taking logs gives $\ln v\left(y\right)-\ln v\left(x\right)<\ln\left(-v\left(-y\right)\right)-\ln\left(-v\left(-x\right)\right)$ and, hence, $\frac{\ln v(y)-\ln v(x)}{\ln y-\ln x}<\frac{\ln(-v(-y))-\ln(-v(-x))}{\ln y-\ln x}$. Letting $\widetilde{x}=\ln x$ and $\widetilde{y}=\ln y$, we get $\frac{\widetilde{v}_+(\widetilde{y})-\widetilde{v}_+(\widetilde{x})}{\widetilde{y}-\widetilde{x}}<\frac{\widetilde{v}_-(\widetilde{y})-\widetilde{v}_-(\widetilde{x})}{\widetilde{y}-\widetilde{x}}$. Take limits as $\widetilde{y}\to\widetilde{x}$, to get $\frac{d\widetilde{v}_+}{d\widetilde{x}}\leq\frac{d\widetilde{v}_-}{d\widetilde{x}}$, from which it follows that $\epsilon_v\left(x\right)\leq\epsilon_v\left(-x\right)$. □

**Proof of Proposition 3**: That $\epsilon_v>0$, follows from (2.3) and (2.4). Also from (2.4) we get:

$$v''\left(x\right)=\frac{v\left(x\right)}{x}\left[\epsilon_v'-\frac{\epsilon_v\left(1-\epsilon_v\right)}{x}\right]. \tag{5.11}$$

If $x>0$ then $v(x)>0$, $v''\left(x\right)<0$, $\epsilon_v'\left(x\right)\geq 0$. From (5.11) it follows that, necessarily, $\epsilon_v<1$. If $x<0$ then $v(x)<0$, $v''\left(x\right)>0$, $\epsilon_v'\left(x\right)\leq 0$. From (5.11), it follows that, again, $\epsilon_v<1$. □

**Proof of Proposition 4**: The result is obvious for the CRRA and CARA classes, from (3.1) and (3.3), respectively. We shall concentrate on giving the proof for the HARA class (3.2). For the remaining two classes: the logarithmic (3.4) and the quadratic (3.5), the proof is similar but easier and, so, will be omitted. Let $f(y) = \ln y - \gamma \ln(\mu + y) - \ln\left[(\mu + y)^{1-\gamma} - \mu^{1-\gamma}\right]$, $y = \frac{\theta x}{\gamma}$, $y \geq 0$. Then, from (3.2), $\epsilon_v(x) = (1-\gamma)\, e^{f\left(\frac{\theta x}{\gamma}\right)}$. Hence, $\epsilon_v$ is decreasing if, and only if, $f(y)$ is decreasing. Let $g(y) = \mu^{2-\gamma} + (1-\gamma)\,\mu^{1-\gamma} y - \mu(\mu + y)^{1-\gamma}$, then it is straightforward to show that $f'(y) < 0$ if, and only if, $g(y) > 0$. Simple calculations show that $g(0) = 0$, $g'(0) = 0$ and $g''(y) = \gamma\mu(1-\gamma)(\mu + y)^{-\gamma-1} > 0$. Hence, $g(y) > 0$ for $y > 0$. Hence $f$ and, thus, also $\epsilon_v$, is decreasing. $\square$

**Proof of Proposition 5**: Follows from (4.6) by direct calculation. $\square$

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