PRICING GROWTH-RATE RISK[1]

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[1] Presented at Spectral and Curbature Methods in Finance and Econometrics

VALUATION DYNAMICS

- Two channels of uncertainty: growth and discounting
  - Exposure of the alternative consumption profiles or cash flows to underlying shocks
  - Prices or imputed compensations of alternative exposures to shocks

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- Use Markov formulations and martingale methods

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- Example economy - Campbell-Cochrane

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- X is the solution to $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$ where W is an $\{\mathcal{F}_t\}$ Brownian motion.

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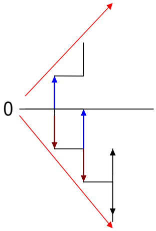
- Process A is nonstationary and can grow linearly.
- Sums of additive functionals are additive. Add the parameters.

MULTIPLICATIVE FUNCTIONAL

Exponentiate an additive functional.

- Process M can grow or decay exponentially.
- Products of multiplicative functionals are multiplicative.
- Examples: stochastic growth, G, and stochastic discount, S, functionals and their product SG.

ILLUSTRATION



$Y_0=0$

$Y_1 = Y_0 +/- 1$

$Y_2 = Y_1 +/- 1$

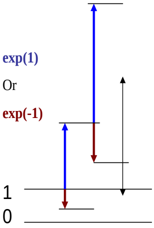
…

Range of Y grows + or -

A Simple Discrete-Time *Additive* Functional Y

$k(X_t, X_{t-1}) = +/- 1$

Note: X's can be temporally dependent



A Simple *Multiplicative* Functional M made from exponentiating Y

$Y_t = \exp[k(X_0,X_1)]\exp[k(X_1, X_2)]\ldots\exp[k(X_{t-1},X_t)]$

$Y_0 = 0$ $M_0 = \exp(0) = 1$

$Y_1 = Y_0 +/- 1$ $M_1 = \exp(0)\exp(+/- 1)$

$Y_2 = Y_1 +/- 1$ $M_2 = \exp(0)\exp(+/-1)\exp(+/- 1)$

Range of M is non-negative.

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MULTIPLICATIVE FACTORIZATION

Hansen-Scheinkman (Econometrica, 2009)

$$M_t = \exp(\eta t)\hat{M}_t \left[\frac{\hat{e}(X_t)}{\hat{e}(X_0)}\right]$$

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- Reminiscent of a permanent-transitory decomposition from time series.
  However, note these key differences:
- Not unique and co-dependence between components matters.

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- My use differs from the construction of risk neutral measure in mathematical finance.
  - η is constant even though short-term interest rates and growth rates can fluctuate.
  - When valuing medium or long-term cash-flows fluctuations in future short-term interest rates under a risk neutral measure are adjusting for risk. This leads me to prefer a decomposition with a constant growth or interest rate.
  - I will incorporate G into the construction of the martingale $\hat{M}$ and hence study growth and discounting simultaneously.

FROBENIUS-PERRON THEORY/ MARTINGALES

- Solve,

$$\mathbb{M}_t e(x) = E\left[M_t e(X_t) | X_0 = x\right] = \exp(\eta t) e(x)$$

  where e is strictly positive. Eigenvalue problem.

- Construct martingale

$$\hat{M}_t = \exp(-\eta t) M_t \left[\frac{e(X_t)}{e(X_0)}\right].$$

- For $\hat{e} = 1/e$

$$M_t = \exp(\rho t) \hat{M}_t \left[\frac{\hat{e}(X_t)}{\hat{e}(X_0)}\right].$$

LOCALIZATION IN TIME

$$\lim_{t \downarrow 0} \frac{E\left[M_t e(X_t) | X_0 = x\right] - \exp(\eta t) e(x)}{t} = 0,$$

which gives an equation in e and η to be solved.

$$\mathbb{B} e = \eta e,$$

where

$$\lim_{t \downarrow 0} \frac{E\left[M_t e(X_t) - e(x) | X_0 = x\right]}{t} = \mathbb{B} e$$

DIFFUSION DYNAMICS

Let

$$A_t = \int_0^t \beta(X_u)du + \int_0^t \gamma(X_u) \cdot dW_u$$

where

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t.$$

Let $M_t = \exp(A_t)$.

DIFFUSION DYNAMICS CONTINUED

Basic equation:

$$\mathbb{B}e = \eta e$$

$e > 0$.

It is convenient to express the corresponding eigenvalue equation in terms of $\log e$:

$$\begin{aligned}\rho \quad = \quad & \left(\beta + \frac{1}{2}|\gamma|^2\right) + \left(\sigma\gamma' + \mu\right) \cdot \frac{\partial \log e}{\partial x} + \frac{1}{2}\text{trace}\left(\sigma\sigma' \frac{\partial^2 \log e}{\partial x \partial x'}\right) \\ & + \frac{1}{2}\left(\frac{\partial \log e}{\partial x}\right)' \sigma\sigma' \left(\frac{\partial \log e}{\partial x}\right)\end{aligned}$$

Typically has multiple solutions. Multiple martingale representations.

MULTIPLICATIVE MARTINGALES

Recall the factorization:

$$M_t = \exp(\rho t)\hat{M}_t \left[\frac{\hat{e}(X_t)}{\hat{e}(X_0)} \right].$$

for $\hat{e} = 1/e$

Change of measure associated with $\hat{M}$.

- preserves Markov structure
- at most one is stochastically stable - Hansen-Scheinkman (Econometrica, 2009)

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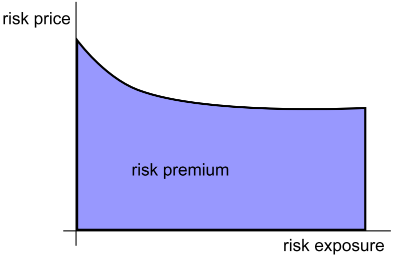
A PREVIEW OF THINGS TO COME

- Risk premium: reward for holding cash-flows exposed to non-diversifiable shocks. Exposure and price contributions.
  - lognormal model: elasticities are constant.
  - More generally there is a nonlinear pricing relationship. Compute marginal prices.



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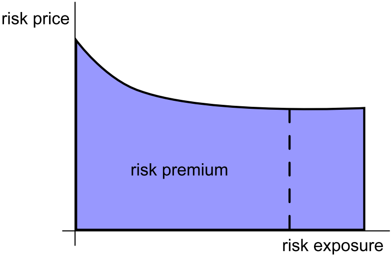

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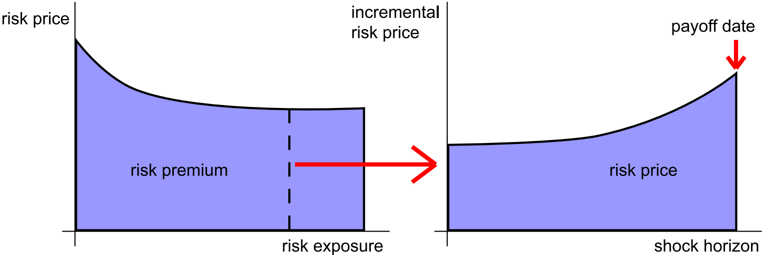
- Price exposure over an entire time horizon of an asset payoff.
- Decompose risk price into the contributions of shocks exposures at alternative points in time between date zero and date t.



SHOCK ELASTICITIES AND STOCHASTIC GROWTH

We use a perturbation approach similar to methods used in sensitivity analyses for option prices (the so-called Greeks). We use this approach to characterize how shocks alter the stochastic dynamics and valuation.

See Fourni-Lasry-Lebuchoux-Lions (Finance and Stochastics, 2001)

DIFFUSION DYNAMICS CONTINUED

State dynamics given by:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t.$$

To represent growth use multiplicative functionals parameterized as $M_t = \exp(A_t)$ where:

$$A_t = \int_0^t \beta(X_u)du + \int_0^t \gamma(X_u) \cdot dW_u$$

The local mean of M_t is

$$M_t \left[\beta(X_t) + \frac{|\gamma(X_t)|^2}{2} \right].$$

The multiplicative functional is a local martingale if its local mean is zero:

$$\beta(X_t) + \frac{|\gamma(X_t)|^2}{2} = 0.$$

PERTURBATIONS

Parameterize a perturbation to M as $MH(\epsilon)$ using: $[\beta_h(x,\epsilon), \epsilon\gamma_d(x)]$ where $\beta_h(x,0) = 0$ and where γ_h defines a direction of risk exposure.

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$$\log H_t(\epsilon) = \int_0^t \beta_h(X_u,\epsilon)du + \epsilon \int_0^t \gamma_d(X_u)dW_u.$$

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- Construct the additive functional:

$$D_t = \int_0^t \beta_d(X_u)du + \int_0^t \gamma_d(X_u) \cdot dW_u$$

where

$$\beta_d(x) = \frac{\partial}{\partial \epsilon}\beta_h(x,\epsilon)|_{\epsilon=0}.$$

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  where

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We use this additive functional to represent the derivative:

$$\frac{d}{d\epsilon} \log E\left[M_t H_t(\epsilon) | X_0 = x\right]|_{\epsilon=0} = \frac{E\left(M_t D_t | X_0 = x\right)}{E\left(M_t | X_0 = x\right)}$$

OBSERVATIONS

$$\frac{d}{d\epsilon} \log E\left[M_t H_t(\epsilon) | X_0 = x\right] |_{\epsilon=0} = \frac{E\left(M_t D_t | X_0 = x\right)}{E\left(M_t | X_0 = x\right)} \tag{1}$$

- The perturbation $H(\epsilon)$ applies to all instants in time between $u = 0$ and $u = t$.
- The left side of formula (1) is an elasticity for appropriate constructions of γ_d.

A BETTER REPRESENTATION

Recall the factorization:

$$M_t = \exp(\eta t)\hat{M}_t \left[\frac{e(X_0)}{e(X_t)} \right]$$

where $\hat{M}$ is a multiplicative martingale. The drift for dW_t under the $\hat{\cdot}$ probability measure is:

$$\gamma_m + \alpha$$

where α is the exposure of $\log e(x)$ to dW_t:

$$\alpha = \sigma' \left[\frac{\partial \log e}{\partial x} \right].$$

Using this change of measure

$$\frac{d}{d\epsilon} \log E\left[M_t H_t(\epsilon) | X_0 = x \right] |_{\epsilon=0} = \frac{\hat{E}\left[\hat{e}(X_t) D_t | X_0 = x \right]}{\hat{E}\left[\hat{e}(X_t) | X_0 = x \right]}$$

where $\hat{e} = \frac{1}{e}$.

AN IMPLIED DECOMPOSITION

Recall

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Consider a martingale perturbation ($\beta_d = 0$) and compute

$$\phi(x, v) = \left(\frac{\partial}{\partial x} \log \hat{E}[\hat{e}(X_v) | X_0 = x] \right)' \sigma(x).$$

The function ϕ gives state-dependent moving-average coefficients for the process $\{\log \hat{e}(X_t)\}$. Nonlinear Wold representation:

$$\log \hat{e}(X_t) = \int_0^t \phi(X_u, t-u) \cdot d\hat{W}_u + \hat{E}\left[\log \hat{e}(X_t) | X_0 = x\right].$$

AN ADDITIVE DECOMPOSITION CONTINUED

With $\beta_d = 0$, we have

$$\frac{d}{d\epsilon} \log E\left[M_t H_t(\epsilon) | X_0 = x\right] =$$
$$\frac{\hat{E}\left[\hat{e}(X_t) \int_0^t \left(\gamma_d(X_u) \cdot \left[\gamma_m(X_u) + \phi(X_u, t-u) - \phi(X_u, 0)\right]\right) du | X_0 = x\right]}{\hat{E}\left[\hat{e}(X_t) | X_0 = x\right]}$$

Two steps:

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Limit:

$$\lim_{t \to \infty} \frac{1}{t} \frac{d}{d\epsilon} \log E\left[M_t H_t(\epsilon) | X_0 = x\right] = \hat{E}\left(\gamma_d(X_t) \cdot \left[\gamma_m(X_t) - \phi(x, 0)\right]\right)$$

SHOCK-EXPOSURE ELASTICITIES

Recall

$$\frac{d}{d\epsilon}\log E\left[M_t H_t(\epsilon)|X_0 = x\right] =$$

$$\frac{\hat{E}\left[\hat{e}(X_t)\int_0^t \left(\gamma_d(X_u)\cdot\left[\gamma_m(X_u)+\phi(X_u,t-u)-\phi(X_u,0)\right]\right)du|X_0 = x\right]}{\hat{E}\left[\hat{e}(X_t)|X_0 = x\right]}$$

Date zero contribution is the **shock-exposure elasticity function**:

$$\epsilon(x,t) = \gamma_d(x)\cdot\left[\gamma_m(x)+\phi(x,t)-\phi(x,0)\right].$$

Measures the impact of a shock at date zero on the process M at alternative dates t given alternative initial states x. Apply to $M = G$.

Limiting contribution

$$\frac{\hat{E}[\hat{e}(X_{\tau+t})\epsilon(X_\tau,t)]}{\hat{E}[\hat{e}(X_{\tau+t})]}.$$

depends only on t.

DIGRESSION ON IMPULSE-RESPONSE FUNCTIONS

Shock elasticity is a nonlinear counterpart to an impulse-response function

1. Coincides with the commonly used impulse-response function for $A = \log M$ when M is log-normal. See Sims (Econometrica, 1980).
2. Related experiment in Gallant-Tauchen-Rossi (Econometrica, 1993), Koop-Pesaran-Potter (Journal of Econometrics, 1996) and Gourieroux-Jasiack (Annales d'Économie et de Statistique, 2000) - either set future shocks to zero or compute conditional expectations while adjusting the initial condition.
3. Related to Clark-Ocone-Haussman formula and the Malliavin derivative.
4. Related to the Wu (PNAS, 2005) characterization of dependence - coupling - add an independent shock at future dates.

Our approach is to localize the exposure. Useful in quantifying risk exposure of a macroeconomic growth or cash flow.

RISK PRICE ELASTICITIES

Three multiplicative functionals.

1. S with a corresponding (β_s, γ_s) is the stochastic discount factor functional;
2. G with a corresponding (β_g, γ_g) is the stochastic growth functional;
3. $H(\epsilon)$ is a parameterized perturbation.

Logarithm of the expected cash flow return is:

$$\log E\left[G_t \underset{\substack{\uparrow \\ \text{expected payoff}}}{H_t(\epsilon)} \middle| X_0 = x\right] \quad - \quad \log E\left[S_t G_t \underset{\substack{\uparrow \\ \text{expected price}}}{H_t(\epsilon)} \middle| X_0 = x\right]$$

We proceed as before to construct price elasticities by computing a derivative and localizing the exposure of the shock.

SHOCK-PRICE ELASTICITIES

Recall:

$$\underset{\text{expected payoff}}{\log E\left[G_t \overset{\uparrow}{H_t}(\epsilon)|X_0 = x\right]} \quad - \quad \underset{\text{expected price}}{\log E\left[S_t G_t \overset{\uparrow}{H_t}(\epsilon)|X_0 = x\right]}$$

Suppose that $GH(\epsilon)$ is a martingale for all ϵ. First term is zero. Focus on price dynamics. Apply our previous analysis with SG playing the role of M and $H(\epsilon)$ constructed to satisfy the martingale restriction.

- **Shock-price elasticity function**

$$\pi(x,t) = -\gamma_d(x) \cdot [\gamma_s(x) - \phi(x,0) + \phi(x,t)]$$

  Measures the impact on the price of exposure to a shock at date zero to growth functional G at date t.

- Limiting contribution:

$$\frac{\hat{E}[\hat{e}(X_{\tau+t})\pi(X_\tau, t)]}{\hat{E}[\hat{e}(X_{\tau+t})]}.$$

FOUR RELATED CONCEPTS

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- Familiar concept of a local risk price:

  $$\pi(x,0) = -\gamma_d(x) \cdot \gamma_s(x).$$

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- Limiting counterpart:

$$\frac{\hat{E}[\hat{e}(X_{\tau+t})\pi(X_\tau, t)]}{\hat{E}[\hat{e}(X_{\tau+t})]}.$$

INCLUDING TRANSIENT CASH-FLOW COMPONENTS

Cash flow:

$$G_t H_t(\epsilon) \left[\frac{f(X_t)}{f(X_0)} \right]$$

for $f > 0$. Expected payoff component is now relevant.

Pricing the next instant shock as it alters the future growth functional is invariant to this change.

Shock-price elasticity function remains the same. Evaluating contributions to value from shocks in the future will be be altered.

GROWTH-RATE RISK PRICES

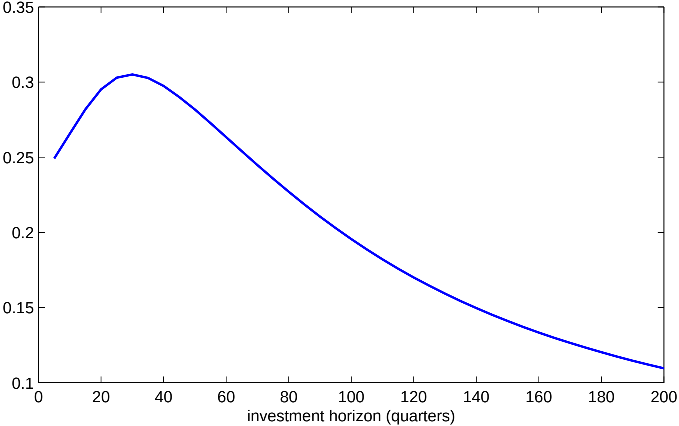

FIGURE: Campbell-Cochrane model

SHOCK PRICE ELASTICITIES

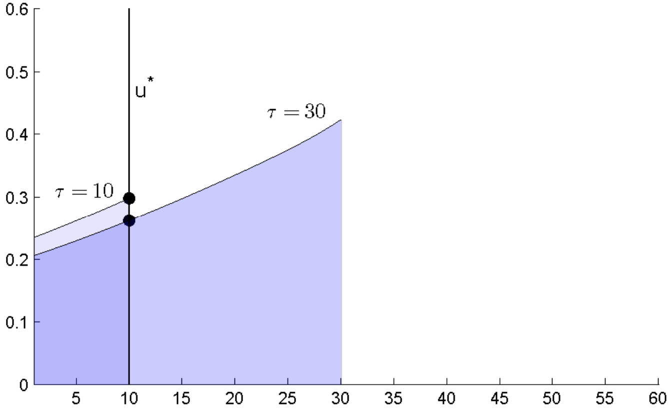

FIGURE: Campbell - Cochrane model, τ is the investment horizon

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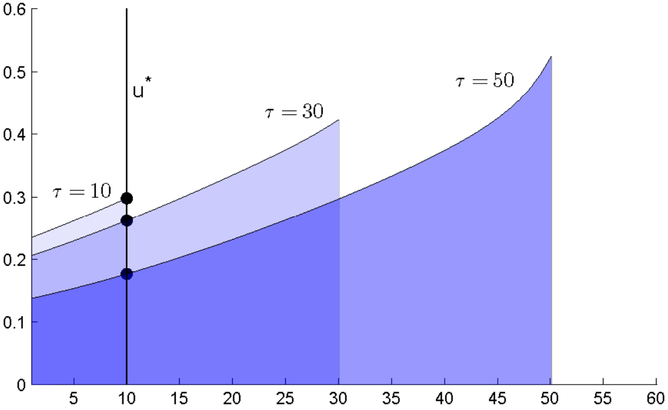

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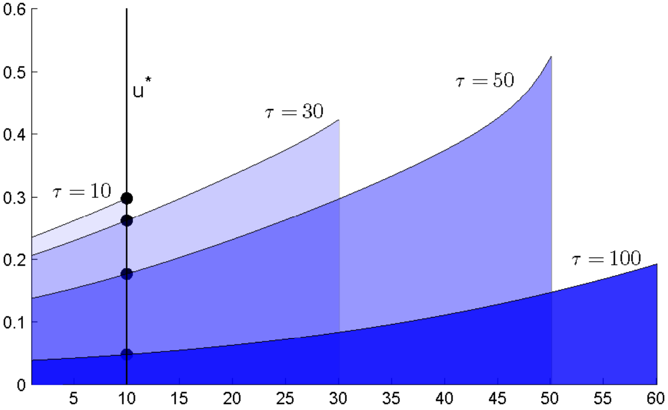

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INSTANTANEOUS SHOCK PRICE ELASTICITIES AND THEIR LIMIT

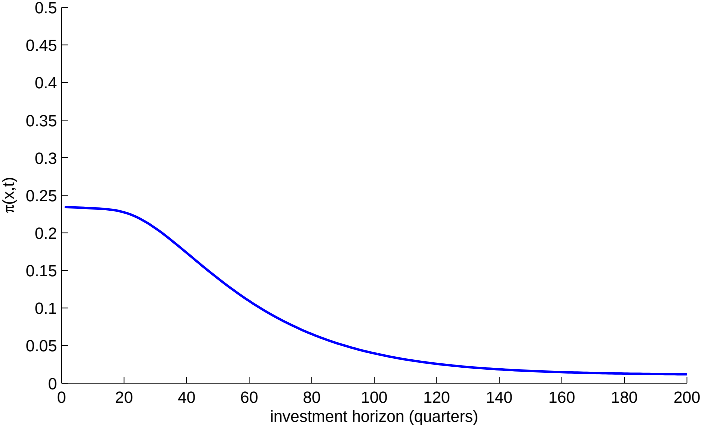

FIGURE: blue - median state, red - two quartiles, black - limit curve

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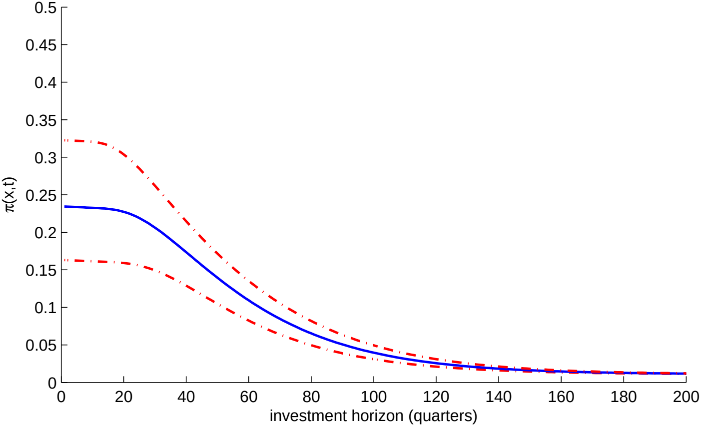

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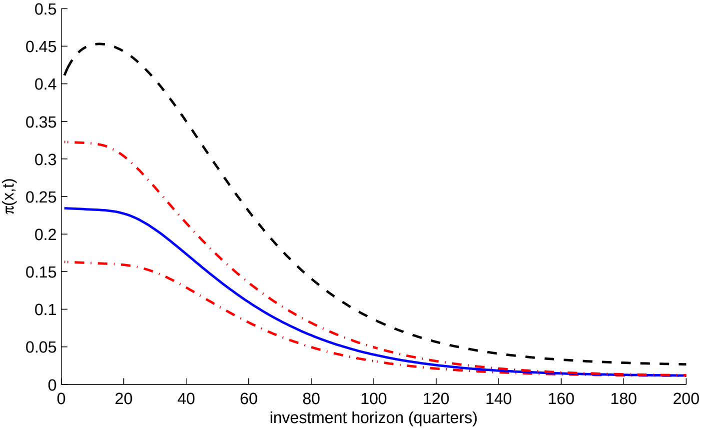

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CONCLUSION

- Study discounting and growth simultaneously.
- Apply a change of probability measure - distinct from the risk neutral measure but better suited for characterizing risk dynamics.
- Infer shock exposure and shock price dynamics by computing the nonlinear moving average or Wold-style decomposition for a particular function of the Markov state under the change in measure.

WHAT STILL NEEDS TO BE DONE?

- Big shocks or jump risk;
- Inferential methods;
- Robustness checks;
- Pass along these same challenges to investors within the model;