Wishart spectral dynamics

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Statement of Purposes

Wishart pricing technology is appealing for many applications, Wishart efficient estimation is still under debate. The price of modeling flexibility is potentially paid in terms of estimation complexity and lack of identifiability.

- We focus on plain vanilla index option prices as provided in OptionMetrics database,
- Estimation shows a competitive pricing error using *one* set of parameters for *five years* of daily data and with diffusive processes
- Dynamic factors are identified in terms of observable portfolios: level, term structure slope and stochastic leverage.

Plan of the talk

Financial Motivation	Preliminary PCA analysis of the IV surface
Model	Framework, structure
Identification	Spectral representation
Estimation	Est. procedure, Computational Remarks
Results	Pricing performance, a Stylized Model
Conclusion	Conclusions and Perspectives

Empirical Stylized Facts

PCA of implied volatility surface

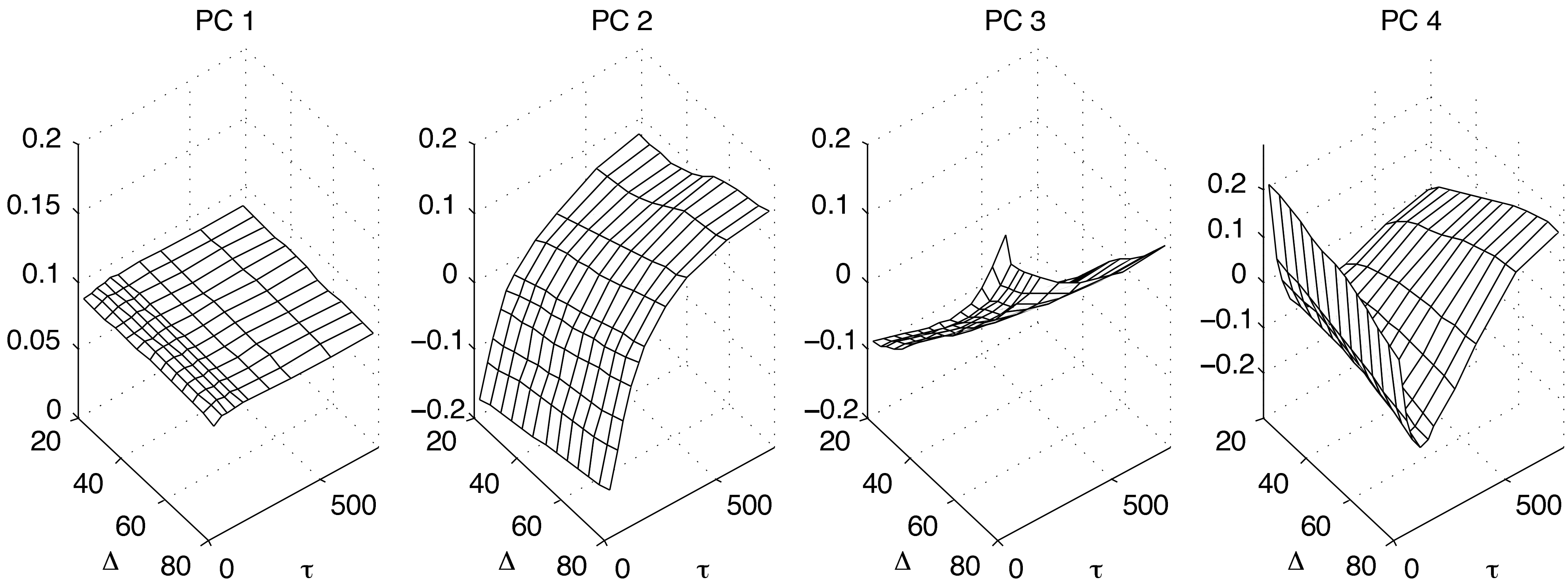
Data S&P 500 call options
– 1996-2007/06 for explorative ananlysis
– 2000/01-2004/12 for estimation

Method Plain PCA vectorized implied volatility matrix
Grid of synthetic options from OptionMetrics
– Delta range = 20%, 25%, . . . , 80%
– maturities = 1,2,3,4,5,6,9,12,18,24 months

Sample	l	PC_1	PC_2	PC_3	PC_4	n
Full	2	95.46	3.32	0.67	0.17	2844
Low vola	3	93.24	4.65	0.98	0.46	1422
High vola	3	87.57	9.72	1.42	0.45	1422

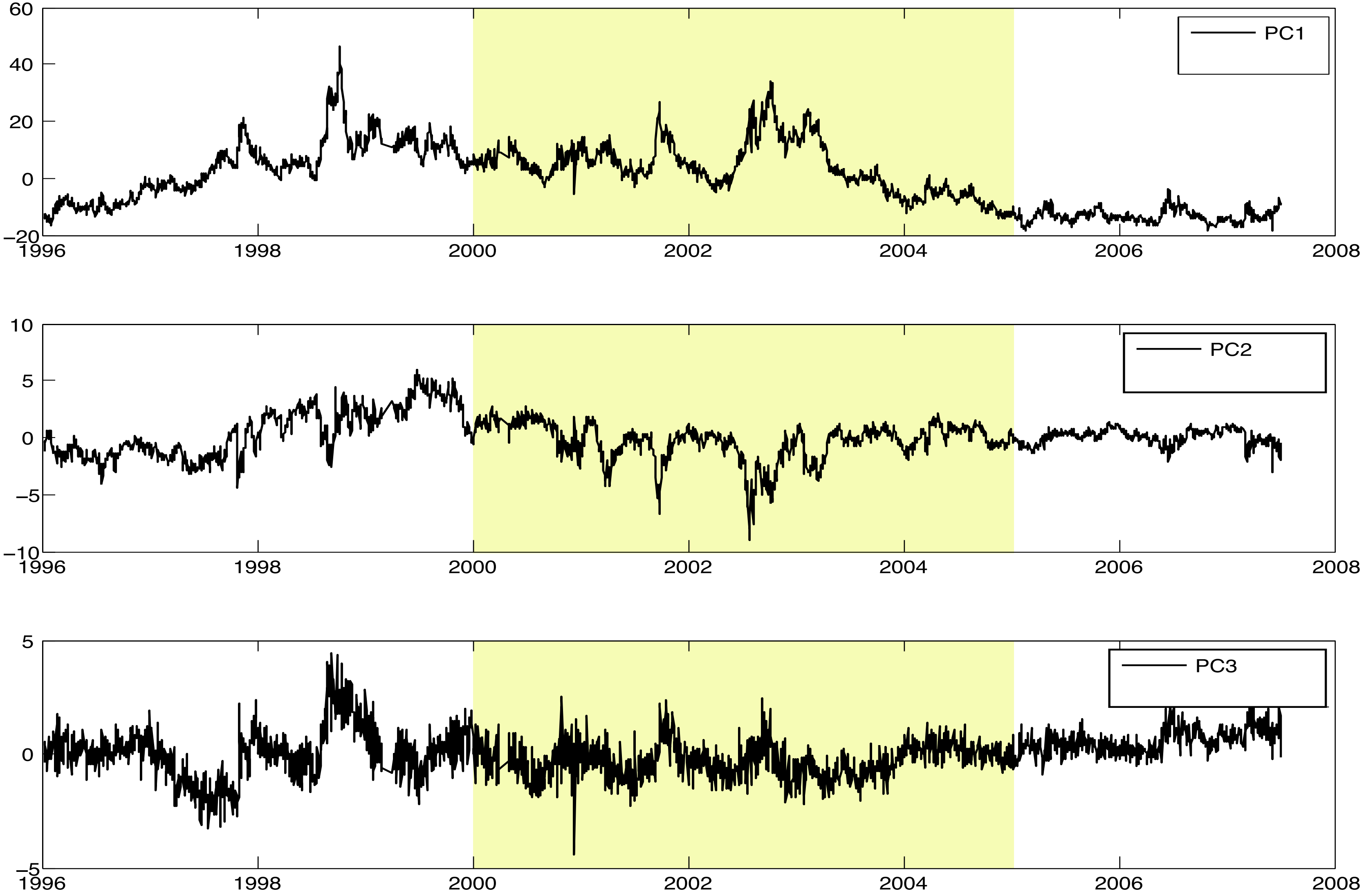
No. of factors; Fraction explained

Eigenvectors



Level | Term structure | Risk reversal | TS hump

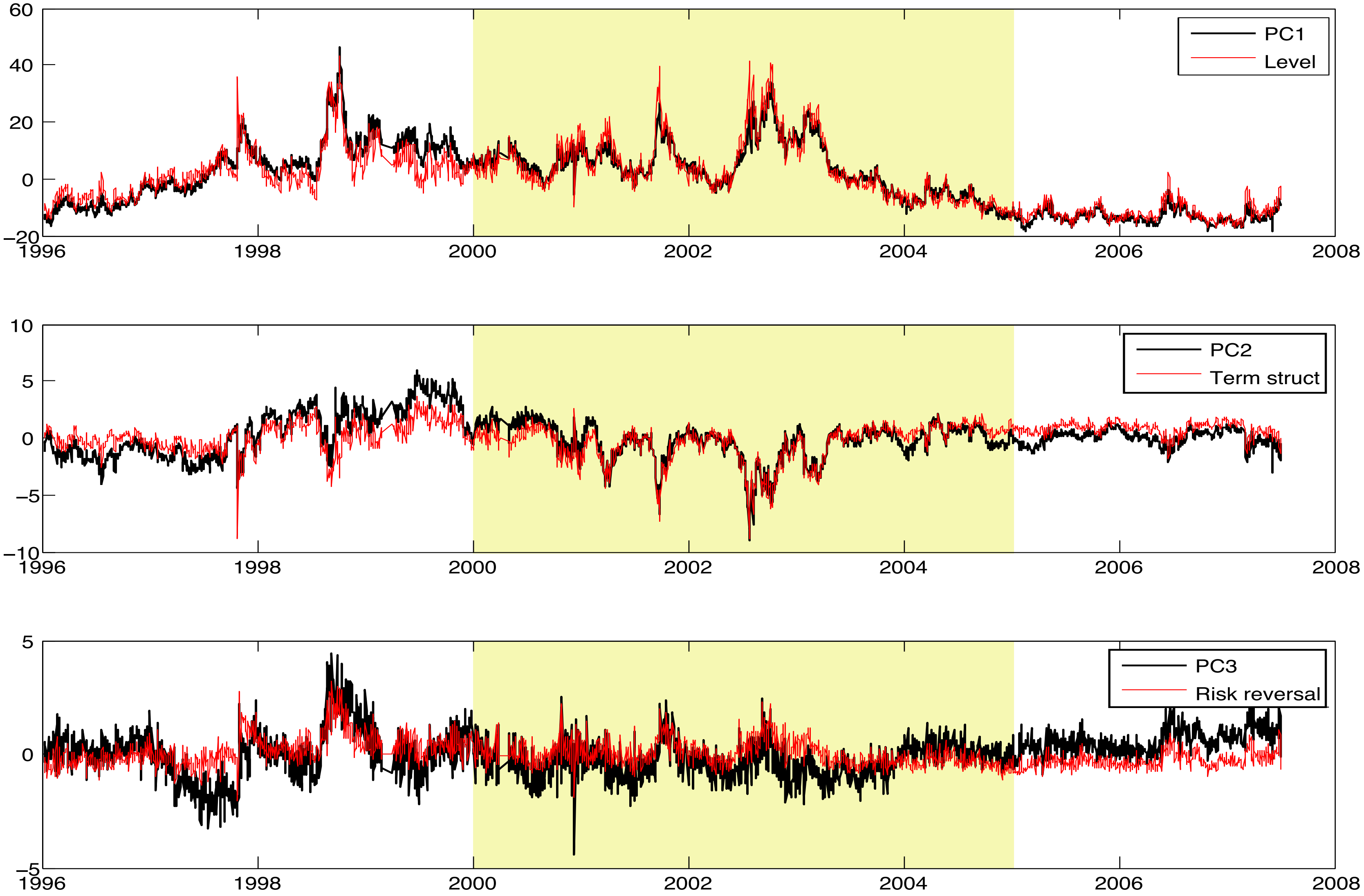
Principal components



IV portfolios

(1) Level	$IV_{\Delta=50,\tau=30}$
(2) Term structure	$IV_{\Delta=50,\tau=30} - IV_{\Delta=50,\tau=365}$
(3) RiskReversal	$IV_{\Delta=75,\tau=30} - IV_{\Delta=25,\tau=30}$
(4) TS hump	$IV_{\Delta=50,\tau=30} + IV_{\Delta=50,\tau=365} - 2IV_{\Delta=50,\tau=91}$
(5) Smile	$IV_{\Delta=75,\tau=30} + IV_{\Delta=25,\tau=30} - 2IV_{\Delta=50,\tau=30}$

PCs and IV portfolios



Model Definition and Identification

Stock return process

$$\frac{dS_t}{S_t} = R(X_t)dt + Tr\left[\sqrt{X_t}dZ_t\right] \quad (1)$$

Matrix-valued factor dynamics $(X_t \in \mathcal{S}_n^+)$

$$dX_t = (\Omega\Omega' + MX_t + X_tM')dt + \sqrt{X_t}dW_tQ + Q'(dW_t)'\sqrt{X_t}$$

Correlated innovations: $Z_t = WR' + B\sqrt{Id - RR'}$

Pure diffusive case: Wishart Process

Bru (1991): $\Omega\Omega' = kQ'Q$ (G.T 2008 $k =$ d.o.f; $k > n-1$)

Focus on 2×2 diffusive case, i.e. a 3-factor model

We follow the definition of identifiability proposed in (C-D G J JF 2008): a model is identifiable if the state vector and the parameter vector can be identified if the prices for a suitable set of spanning securities are given.

- □ P1 If X_t is observable then its drift is observable

$$\mu^{\mathbb{Q}}(t) = \lim_{\Delta \to 0} \frac{E_t^{\mathbb{Q}}[X_{t+\Delta} - X_t]}{\Delta}$$

- □ P2 The quadratic variation of an observable process X_t is observable

$$V_X(t) = \langle X_t, X_t \rangle$$

- □ P3 The quadratic covariation between two observable processes X_t, Y_t is observable

$$V_{XY}(t) = \langle X_t, Y_t \rangle$$

Identifiability in Option Markets

The state vector is rotated to a set of observable factors. We apply the same identification procedure in option markets. El-Karoui Durrleman(QF 2008) relations identify the relation between 3 fundamental contracts and the infinitesimal characteristics of the factor processes in terms of the implied volatility surface $IV_t\left(\tau, K\right)$:

$$IV_t\left(\tau, K\right) = \left|\sigma_t\right| + \tau \mathcal{M}_t + \frac{K - S_t}{S_t} \mathcal{S}_t + O\left\{\left(\frac{K - S_t}{S_t}\right)^2\right\}$$

- □ The Volatility Level $\left|\sigma_t\right|$:

$$\left|\sigma_t\right| = \lim_{\tau \to 0}\left[IV_t\left(\tau, K\right)\right]_{K=S_t} = Tr\left[X_t\right]^{1/2}$$

- □ The Calendar spread $\mathcal{M}_t$:

$$\mathcal{M}_t = \lim_{\tau \to 0}\left[\frac{\partial IV_t\left(\tau, K\right)}{\partial \tau}\right]_{K=S_t} = \frac{Tr\left[kQQ^T\right] - Tr\left[2MX_t\right]}{4Tr[X_t]^{1/2}} - \left|\sigma_t\right|^2 \mathcal{C}_t$$

Stochastic Leverage

- The Risk-Reversal proportional to the skew $\mathcal{S}_t$:

$$\mathbf{S}_t = \rho_t \frac{Tr\left[QQ^T X_t\right]^{1/2}}{2Tr\left[X_t\right]} = \frac{1}{2\left|\sigma_t\right|^3} Tr\left[RQX_t\right]$$

where ρ_t the stochastic correlation leverage is given by:

$$\rho_t = \frac{Tr\left[RQX_t\right]}{Tr\left[X_t\right]^{1/2} Tr\left[QQ^T X_t\right]^{1/2}}$$

- The definition of the factors $|\sigma_t|$ and $\mathcal{M}_t$ depends solely on the trace, which is a spectral invariant.

  The identification of the matrix coordinate system requires a relation between the dynamical evolution of the reference system and the Risk reversal.

Wishart PCA: Eigenvalues

Consider the case $n = 2$; repeating the computation of Bru (1991) it is easy to compute the evolution of the PCA $\Sigma_t = U_t diag\left(\underline{\lambda_t}\right) U_t^T$ for the volatility matrix:

$$
\begin{aligned}
dS_t &= S_t\left(\sqrt{\lambda_1}d\nu_t^1 + \sqrt{\lambda_2}d\nu_t^2\right) \\
d\lambda_i &= \left[k\left(Q^U Q^{TU}\right)_{ii} - 2M_{ii}^U \lambda_{it}\right]dt + \sqrt{\lambda_{it}}2\left(Q^{TU}Q^U\right)_{ii} d\nu^i \\
&\quad - \sum_{j=1}^{2} \frac{(\lambda_{it} + \lambda_{jt})}{(\lambda_{jt} - \lambda_{it})^2}\left[\lambda_{jt}\left(Q^{UT}Q^U\right)_{ii} + \lambda_{it}\left(Q^{UT}Q^U\right)_{jj}\right]dt \\
i &= 1, 2
\end{aligned}
$$

Wishart PCA: Eigenvectors

The eigenvalues of the Wishart behave as two non-crossing square root processes eigenvectors matrix is a 2-dim orthogonal matrix, hence it is parameterized by a single factor:

$$
\begin{aligned}
dA_{12} &= \frac{\left(k\left(Q_t^U Q_t^{TU}\right)_{12} - M_{12t}^U \lambda_{2t} - \lambda_{1t} M_{21t}^U\right)}{\left(\lambda_{2t} - \lambda_{1t}\right)} dt \\
&+ \frac{\sqrt{\left(Q_t^U Q_t^{TU}\right)_{12}\left(\lambda_{1t} + \lambda_{2t}\right)}}{\left(\lambda_{2t} - \lambda_{1t}\right)} d\beta_t
\end{aligned}
$$

where A_t is the stochastic logarithm of U_t:

$$
U_t = \left[\begin{array}{cc} \cos\left(\alpha_t\right) & -\sin\left(\alpha_t\right) \\ \sin\left(\alpha_t\right) & \cos\left(\alpha_t\right) \end{array}\right]
$$

$dU_t = U_t\left(dA_t + d\Gamma_t\right)$, $d\Gamma_t = \frac{1}{2} dA_t dA_t$ and $d\beta, d\nu_1, d\nu_2$ are independent Brownian Motions $d\nu_i d\nu_i^S = R_{ii}^U dt \ i = 1, 2$.

Wishart PCA: Summary

Summarizing, we have a model for returns with

1. Two orthogonal short and long term stochastic volatility factors (two factor Heston models λ_1, λ_2) which cannot cross
2. Factor loadings of the two components evolve stochastically depending on a single factor (the angle α_t)
3. The factor loadings are independent from the eigenvalues if and only if the QQ^T is proportional to the identity

Estimation

Parameter Identification

$\theta = \{M, R, Q, k\}$	$\rightarrow$	13 diffusive parameters
Canonical model	$\rightarrow$	10 parameters to estimate
		Some admissibility constraints (M, R, k)

Two state representations

1. Choleski decomposition: $X_t = S_t' S_t$

2. Eigenvalue representation: $X_t = P_t D_t P_t'$

   - Matrix of eigenvalues $D_t = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

   - Eigenvectors in polar coordinates $P_t = \begin{pmatrix} \cos\alpha_1 & \cos\alpha_2 \\ \sin\alpha_1 & \sin\alpha_2 \end{pmatrix}$

   $\alpha_2 = \alpha_1 + \pi/2 \rightarrow$ symmetry, rotation matrix

Two representations: $\{s_{11}, s_{12}, s_{22}\}$ or $\{\lambda_1, \lambda_2, \alpha\}$

Joint estimation procedure, slightly simplified version of Bates(2000)

- Parameters $\theta = \{M, R, Q, k\}$
- State $\{X_t\}, \quad t = 1, \ldots, 59$
- Unconditional error covariance matrix Q to calculate heteroskedasticity correction $\Omega_t, \quad t = 1, \ldots, 59$ for

$$\max_{\theta, \{X_t\}} \ln L = -\frac{1}{2} \sum_t \ln |\Omega_t| + e_t' \, \Omega_t^{-1} \, e_t$$

Note: dimension of Ω_t changes daily, depending on number of observations.

Step 1: Nested NLS estimation to get starting values for ML

$$\widehat{\theta} = \arg\min_{\theta} \left(\widehat{C}_i(\theta, X_t^*(\theta)) - C_i \right)^2 \tag{7}$$

$$X_t^*(\theta) = \arg\min_{X_t} \left(\widehat{C}_i(\theta, X_t) - C_i \right)^2 \tag{8}$$

Step 2: Heteroskedasticity correction
Bin data in 4 moneyness and 3 maturity bins.
Calc. unconditional within/between bins error cov. matrix Q.
Produce daily unconditional error covariance matrix Ω_t

Step 3: Perform ML parameter estimation.
State still estimated via NLS.
Iterate between step 2 and step 3.

Step 4: Error estimation for θ using BHHH algorithm
Gradients calculated numerically.

$$\widehat{V}_{BHHH} = \left(\frac{1}{N} \sum_i s_i(\hat{\theta}) s_i(\hat{\theta})' \right)^{-1} \tag{9}$$

Semi-out of sample extension

Add daily data within the sample period (59 $\longrightarrow$ 1254 obs)
Use paramters $\hat{\theta}$ from monthly data
Estimate time series of daily state via NLS.
Virtually same performance.

Full of sample extension

Furthermore add daily data form 1996-1999 and 2005-mid2006.
($\longrightarrow$ 2850 obs)
Notably worse performance.

rms \$ errors	monthly data	semi-OOS	ful-OOS
Full Wishart	1.059	1.099	1.547

Results

m_{11}	-0.2906	(0.0218)
m_{21}	-9.3628	(0.0603)
m_{22}	-5.0719	(0.0180)

- Distinctly different mean reversion speeds.
- Long-run factor λ_1, ca. 3yrs
- Short-run factor λ_2, ca. 2 months
- Very fast mean reversion of U_t

q_{11}	0.0340	(0.0049)
q_{22}	0.4789	(0.0079)

- Volatility of short-term shocks is larger.

Parameter estimates (2)

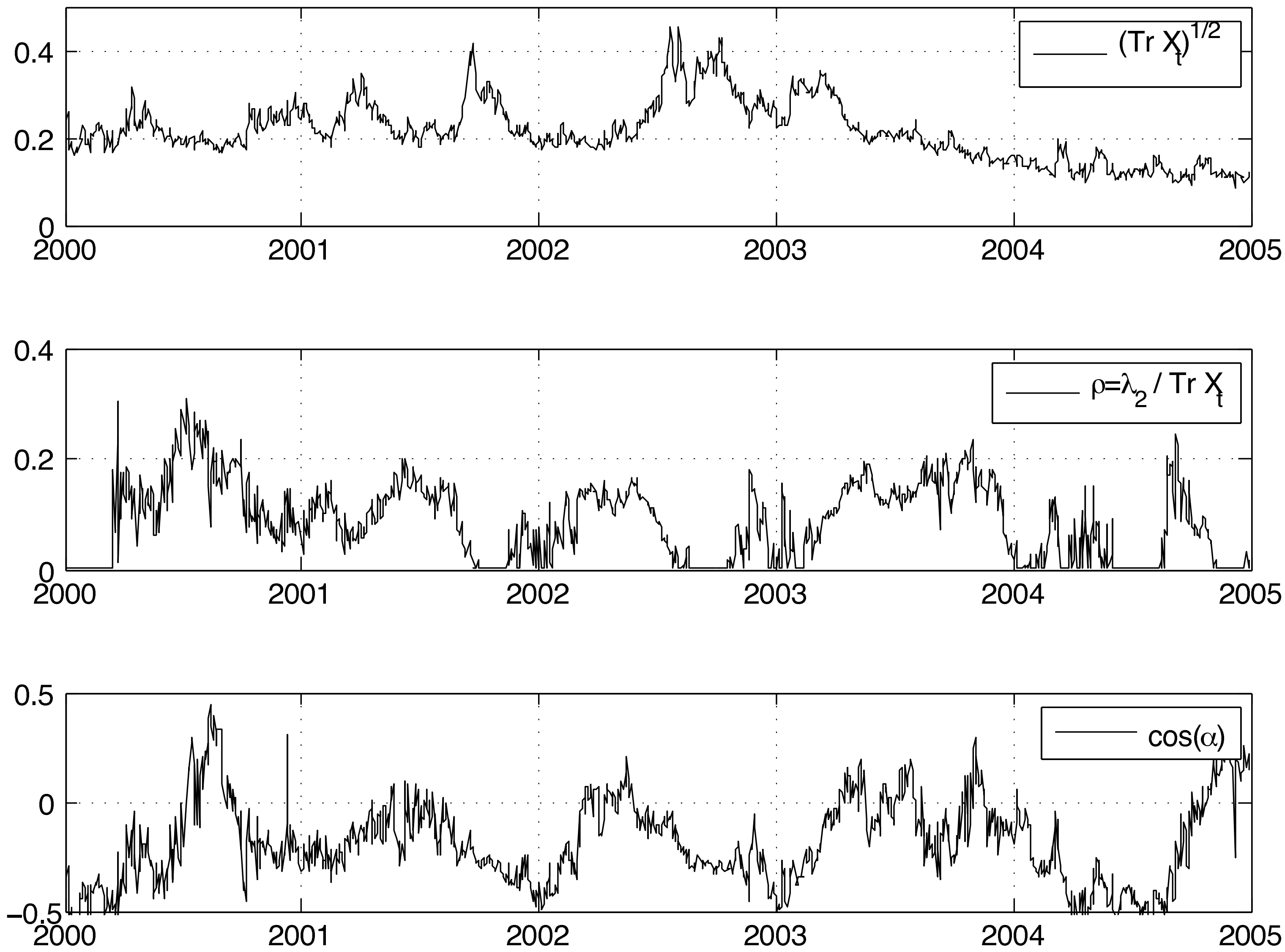
R_{11}	-0.9757	(0.1362)
R_{12}	0.1586	(0.0335)
R_{22}	-0.6905	(0.0186)

- Different contributions of short- and long-run skew.
- Positive R_{12} allows for change of sign in correlation leverage

β	1.0000	(0.0427)

Data: SP500 call options, 59 trading days in 2000-2004 (2nd Wednesday before expiry), overall 9029 observations

Results: State

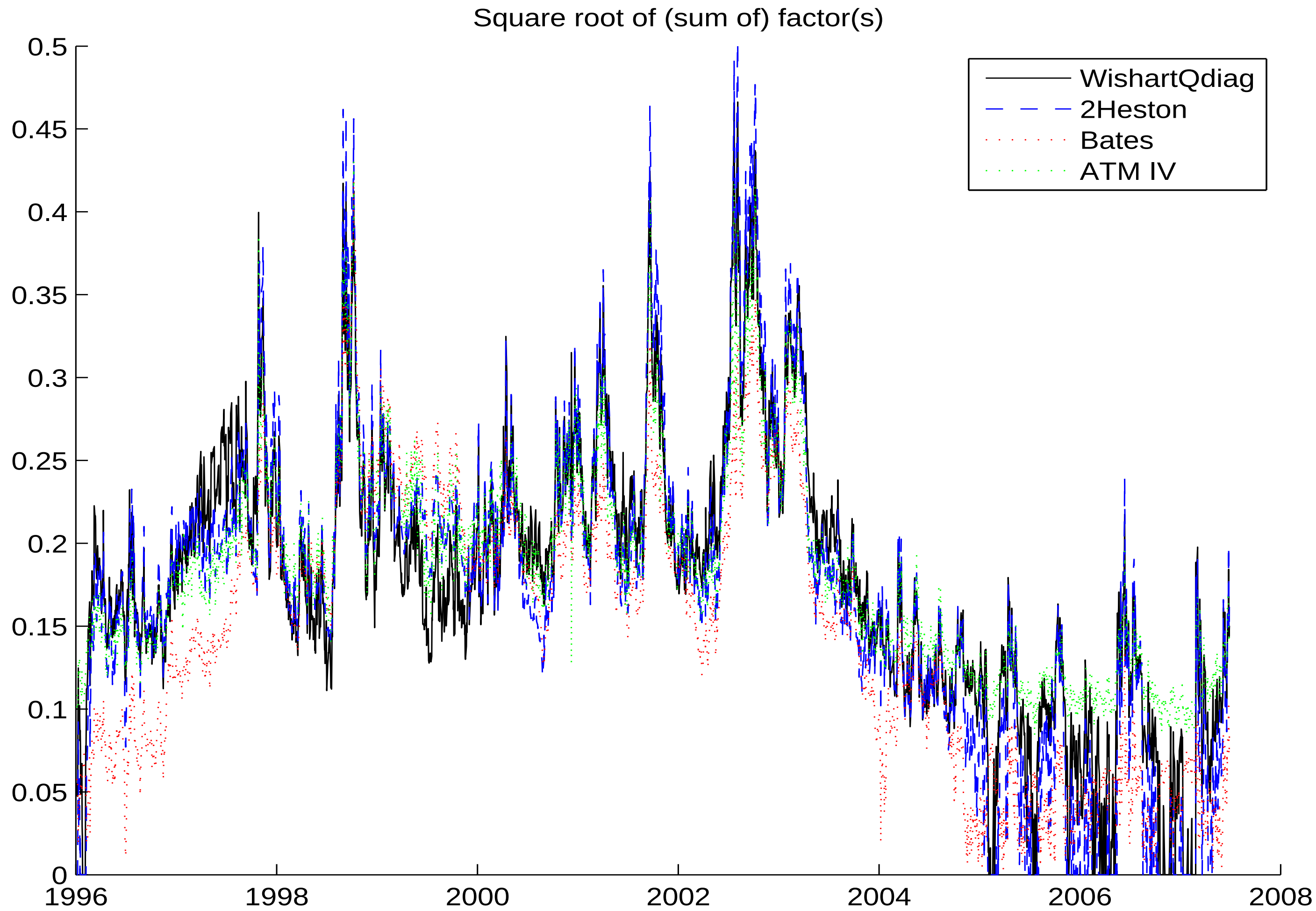


- Short run factor $\ll$ long run factor
- U_t changes sign

Next goal: find an interpretation of the state in our framework of

- Level
- Term structure
- Risk reversal

Level – no magic

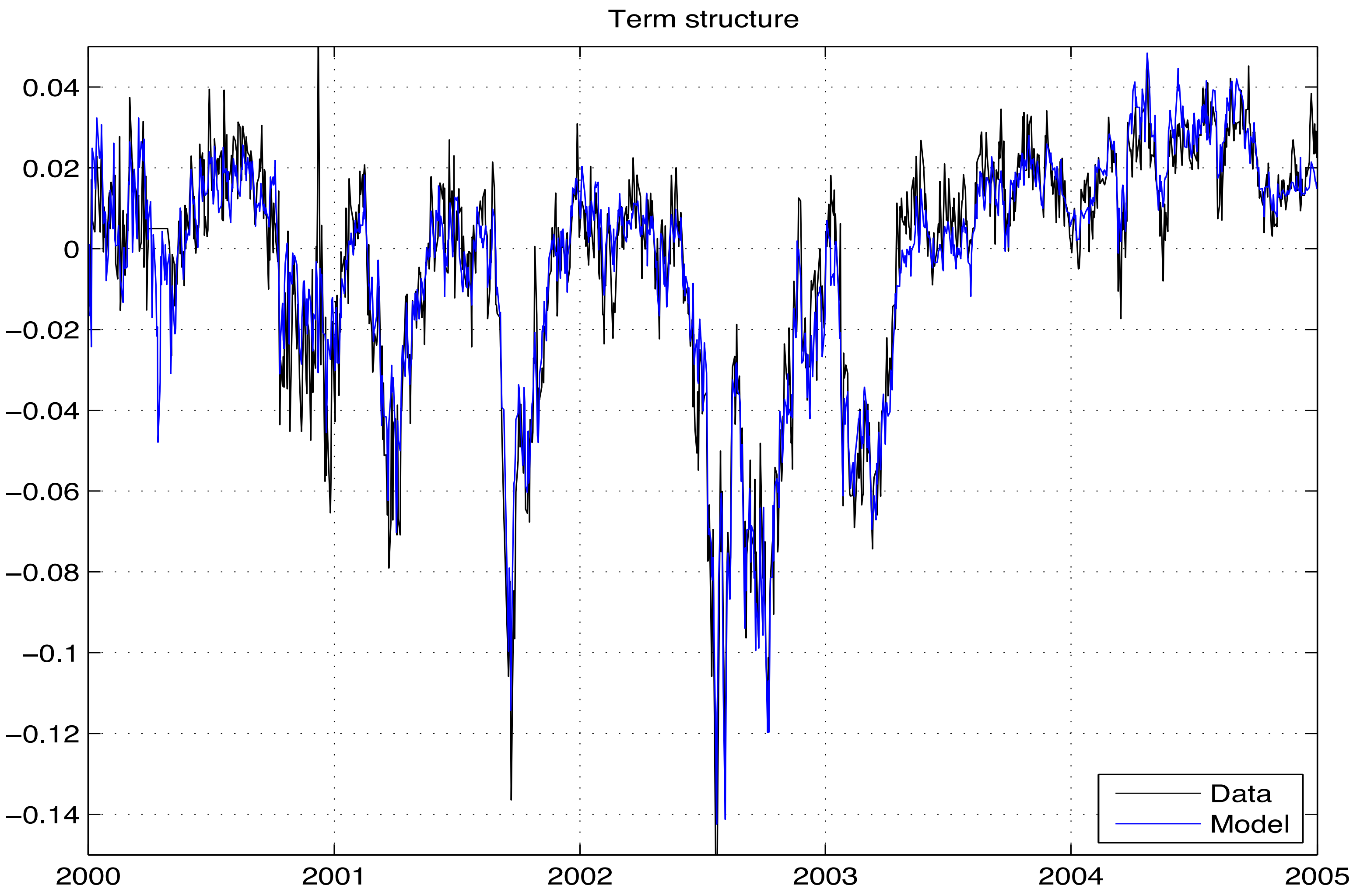


Term structure $\approx \alpha + \beta_1 V_t + \beta_2 \lambda_t^1 / V_t$

Heston-like effect links term structure to level

Mixing of different mean-reversion speeds for λ_t^1 and λ_t^2 allows for deviation

Term structure



$$R^2 = 0.81$$

Risk Reversal

A stylized model from estimation results:

$$Q \cdot Q^T = \begin{bmatrix} 3.7265 \times 10^{-4} & -5.6389 \times 10^{-3} \\ -5.6389 \times 10^{-3} & 0.16941 \end{bmatrix} \simeq \begin{bmatrix} 0 & 0 \\ 0 & 0.16941 \end{bmatrix}$$

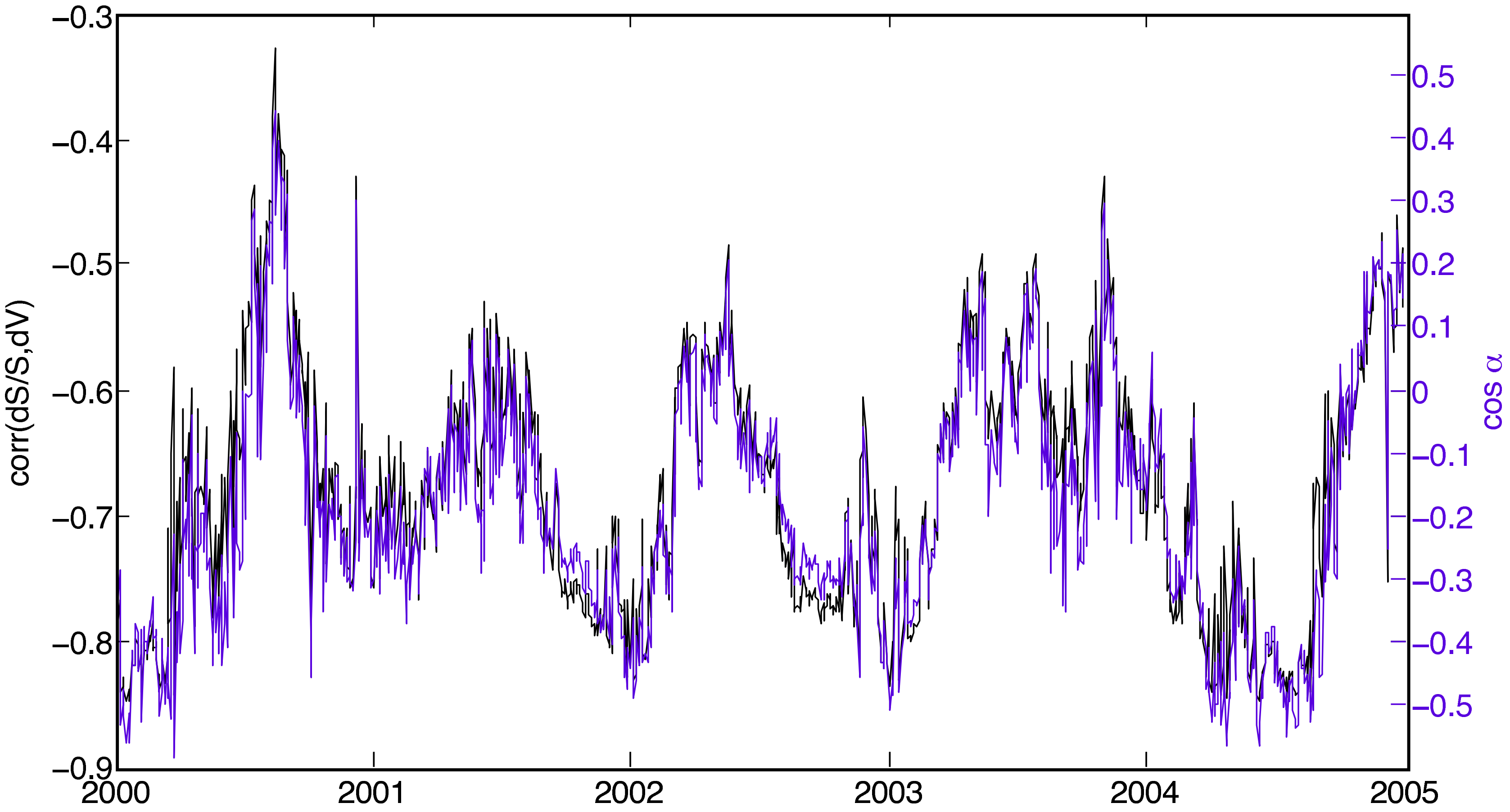
$$RQ = \begin{bmatrix} -6.6341 \times 10^{-3} & -0.23616 \\ 0 & -0.25816 \end{bmatrix} \simeq \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (-0.24716)$$

$$\begin{bmatrix} \sqrt{\lambda_1^\infty} & 0 \\ 0 & \sqrt{\lambda_2^\infty} \end{bmatrix} = \begin{bmatrix} 2.5551 \times 10^{-2} & 0.0 \\ 0.0 & 0.14945 \end{bmatrix}$$

$$\frac{Tr\left[RQ\Sigma_t\right]}{\sqrt{Tr\left[\Sigma_t\right]}\sqrt{Tr\left[Q^TQ\Sigma_t\right]}} \simeq -0.60920\left[1 + \cos\alpha_t\right]$$

The rotation factor is directly proportional to a stochastic skewness component orthogonal to the volatility factors!

Stochastic leverage – $\cos \alpha$



Conclusion

Performance comparison

Data:

- S&P 500, monthly data
- Sample period 2000-2004 (59 observations)
- Time to maturity: $10d \leq \tau \leq 1yr$
- Minimal cuts in moneyness: $\min(C_i, P_i) > \$0.375$

Model	Factors	rms \$ error	remark
Heston	1	1.676	
Bates (1996)	1	1.534	State-ind. jumps
Christoffersen (2008)	2	1.202	Feller cond. violated
Full Wishart	3	1.059	

Conclusion+Outlook

- Identifiability of the Wishart Multifactor Volatility is obtained in terms of observable portfolios
- Good cross-sectional and time-series performance with one parameter set for 5 years of data
- Work in progress: computation of estimation errors, addition of Jumps

Related literature

Empirical facts on the IV surface
Skiadopoulos, Hodges, Clewlow (2000): The dynamics of the S&P 500 implied volatility surface.
Fengler, Härdle, Villa (2001): The dynamics of implied volatilities: A common principle components approach.

Affine processes and Transform Analysis
Carr and Madan (1999): Option Valuation Using the Fast Fourier Transform
Duffie, Pan, Singleton (2000): Transform analysis and asset pricing for affine jump-diffusions
Grasselli and T. (2008): Solvable Affine Term Structure models
Leippold, Trojani (2008): Asset Pricing with Matrix Affine Jump Diffusions

Wishart process + applications
Gourieroux, Suffana (2004): The Wishart Autoregressive Process of Multivariate Risk
Buraschi, Porchia, Trojani (2008): Correlation Risk and Optimal Portfolio Choice

Multifactor option pricing models
da Fonseca et al.(2008): A Multifactor Volatility Heston Model
Christoffersen et al.(2008): The Shape and Term Structure of the Index Option Smirk: Why Multifactor Stochastic Volatility Models Work so Well
Christofersen et al.(2008): Option Valuation with Long-run and Short-run Volatility