# Quantitative Methods for Counterparty Risk:
# II. Implementation Issues and Examples

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**Plan of the presentation**

**1)** Counterparty risk

**2)** Modelling aspects

**3)** FFT based methods in one dimension

**4)** PDE based methods in one dimension

**5)** PDE based methods in two dimensions

**6)** Illustrations

## CDS basics

Credit default swap (CDS) - provides to the buyer a protection against a reference name in return for coupon payments up to contract maturity or the default event

At contract inception, the coupon is set so that the present value of CDS is zero

As the time goes by, the mark-to-market (MtM) value of the CDS contract fluctuates

In particular, when the credit quality of the reference name worsens, the MtM increases

## Counterparty Risk

When the CDS protection is sold by a defaultable counterparty, the protection buyer faces the risk of losing a part of the mark-to-market value of the CDS, if it is positive for the buyer, due to the counterparty default

The loss is profound if the credit quality of both the reference credit and the counterparty worsen simultaneously but the counterparty defaults first

Counterparty risk is crucial at both the firm and economy wide levels ("domino effect")

## Counterparty Charge

Let $\tau_1$ and $\tau_2$ be default times of the reference name and the counterparty, respectively

The counterparty charge, $C(t)$, is the expected maximal potential loss due to counterparty default up to CDS maturity $T$:

$$C(t) = (1 - R_2)\mathbb{E}_t \left[ \int_t^T D(t,t') \max\left\{ \mathbb{E}_{t'} \left[ \tilde{C}(t') \mid E(t') \right], 0 \right\} \mathbf{1}_{\{E(t')\}} dt' \right] \tag{1}$$

$\tilde{C}(t)$ is cash flow of CDS contract (long protection) without counterparty risk discounted to time $t$

$E(t) = \{\tau_1 > t, \tau_2 = t\}$

$R_2$ is recovery rate of counterparty obligations

$D(t,T) = e^{-\int_t^T r(t')dt'}$ is the risk-free discount factor

## Motivation

Model for the counterparty risk evaluation need to:

**1)** describe realistic dynamics of CDS spreads (jump-diffusions)

**2)** create profound correlation effects (correlated jump-diffusions with simultaneous jumps)

Model should match observable market data closely:

**1)** the term structure of CDS spreads

**2)** the term structure of discount factors

**3)** equity and CDS options volatilities

**4)** correlations

Some of model parameters are made time-dependent to fit term structure effects

## Generic 1-d model

$a(t)$ - the value of the firm assets (stochastic)
$l(t)$ - the default barrier (deterministic)
$x(t) = \ln \frac{a(t)}{a(0)}$ - the log of the normalized asset value

Assume that $x(t)$ is an additive process with independent time-dependent increments (Sato (1999)):

$$dx(t) = \mu(t)dt + \sigma(t)dW(t) + jdN(t), \ x(0) = 0 \tag{2}$$

$\mu(t) = -\kappa\lambda(t)$ is the compensator
jumps have probability density function $\varpi(j)$

The default time $\tau$ is defined by:

$$\tau = \min\{t : x(t) \leq b\}, \ b = \ln \frac{l(0)}{a(0)}, \ b < 0$$

The default is triggered either discretely ($t \in \{t_d\}$) or continuously

**Implications for model implementation**

**1)** Ability to handle different pay-off structures in one and two dimensions

**2)** Time dependent model parameters

**3)** Fast calibration through the forward induction

**4)** Pricing through the backward induction consistently with the forward induction

Analytical methods are restrictive given the above considerations

We concentrate on numerical methods to solve the problem efficiently:

**1)** FFT based methods

**2)** PDE based methods

## Generic 1-d Problem. Backward Equation I

$U(t,x)$ is the value function of a generic product with maturity time $T$ specified by:

**1)** terminal pay-off function given survival at $T$, $u(x)$

**2)** coupon function given survival at time $t$, $c(t,x)$

**3)** rebate function given default at time $t$, $q(t,x)$

By Feynman-Kac Theorem:

$$\begin{aligned} &U_t(t,x) + \mathcal{L}U(t,x) = -c(t,x), \\ &U(T,x) = u(x), \\ &U(t,x) = q(t,x),\ \ x \le b \end{aligned} \tag{3}$$

## Generic 1-d Problem. Backward Equation II

$\mathcal{L}$ is the infinitesimal operator corresponding to dynamics (2):

$$\mathcal{L} = \mathcal{D} + \mathcal{J} \tag{4}$$

$\mathcal{D}$ is the diffusion-convection operator:

$$\mathcal{D}U(t,x) \equiv \frac{1}{2}\sigma^2(t)U_{xx} + \mu(t)U_x - (r(t) + \lambda(t))U$$

$\mathcal{J}$ is the integral operator:

$$\mathcal{J}U(t,x) \equiv \lambda(t) \int_{-\infty}^{\infty} U(t, x+j)\varpi(j)dj$$

## Gap Risk and PDE

Formally, the PDE is defined on $(-\infty, \infty)$

For continuous monitoring, we can switch to bounded domain $[b, \infty]$ by augmenting the coupon function by the gap term:

$$Z(t,x) = \lambda(t) \int_{-\infty}^{b-x} q(t, x+j)\varpi(j)dj$$

$Z(t,x)$ measures the gap risk due to jumps (for diffusions $Z(t,x) \equiv 0$)

For a CDS contract under exponential jumps:

$$q(t,x) = (1 - Re^{x-b}),\ \varpi(j) = \nu e^{\nu j}$$

$$Z(t,x) = \lambda(t) \int_{-\infty}^{b-x} \left(1 - Re^{x-b+j}\right) \varpi(j)dj = \lambda(t) \left(1 - R\frac{\nu}{1+\nu}\right) e^{-\nu(x-b)}$$

Although the default triggering boundary is fixed, the actual recovery rate is random

## Computational Challenges

**1) Drift-dominated problem**:

For strong credit names, the asset volatility $\sigma$ (the equity volatility is the asset volatility times the leverage) is small but the mean of the jump amplitude is large, thus compensator $\kappa$ is large

For weak credit names, the asset volatility $\sigma$ is even smaller but the jump frequency is high, thus $\lambda$ is large

Typically, the drift term $-\kappa\lambda$ dominates the diffusion term $\sigma$

**2) Non-local integral part**

Extra complexity to handle the integral term

**Solution with FFT based method**

**1)** Applies in case of discrete default monitoring

**2)** Employs the characteristic function of process $x(t)$

## FFT based method. Characteristic Function

Define the characteristic function of $x(t)$ by:

$$\widehat{G}(t, T, \phi) = \int_{-\infty}^{\infty} e^{i\phi X} G(t, 0; T, X) dX,$$

where $G(t, x; T, X)$ is the TPDF of $x(t)$, $X \equiv x(T)$, $\phi \in \mathbb{R}$ is the transform variable, and $i = \sqrt{-1}$

From the theory of additive processes:

$$\widehat{G}(t, T, \phi) = e^{-\int_t^T \psi(t', \phi) dt'},$$

where $\psi(t, \phi)$ is the characteristic exponent:

$$\psi(t, \phi) = \frac{1}{2}(\sigma(t)\phi)^2 - i\mu(t)\phi - \lambda(t)(\widehat{\varpi}(\phi) - 1), \ \widehat{\varpi}(\phi) = \int_{-\infty}^{\infty} e^{i\phi j} \varpi(j) dj$$

Accordingly, we can compute TPDF of $x(T)$ by:

$$G(t, 0; T, X) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Re \left[ e^{-i\phi X} \widehat{G}(t, T, \phi) \right] d\phi \qquad (5)$$

## FFT based method. Backward Problem

Because the increments in $x(t)$ are independent conditional on the current state values:

$$G(t,x;T,X) \equiv G(t,T,X-x)$$

The value function $U(t,x)$ can be represented as (ignoring coupon and rebate functions):

$$U(t,x) = \int_{-\infty}^{\infty} u(X)G(t,T,X-x)dX$$

Applying the Fourier transformed density function (5) and exchanging the integration order we obtain (Carr-Madan (1999), Lewis (2001), Lipton (2001)):

$$\begin{aligned} U(t,x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} u(X)\int_{-\infty}^{\infty} \Re\left[e^{-i\phi(X-x)}\widehat{G}(t,T,\phi)\right] d\phi dX \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Re\left[e^{i\phi x}\left(\int_{-\infty}^{\infty} e^{-i\phi X}u(X)dX\right)\widehat{G}(t,T,\phi)\right] d\phi \end{aligned} \tag{6}$$

## FFT based method. DFT Algorithm

Observe that Eq.(6) can be computed by applying the two operations of the DFT algorithm:

$$U(t,x) = \mathsf{ifft}\left(\mathsf{fft}(u(x)) \odot \widetilde{G}(t,T,\phi)\right)$$

By discretisation of the state space of $x$ and $\phi$, the relationship $\Delta x \Delta \phi = \frac{2\pi}{N}$ is required for standard DFT algorithm

**FFT based method. Forward Equation for function $U^*(T,X)$:**

$$U^*_T(T,X) + \mathcal{L}^\dagger U^*(T,X) = 0,$$
$$U^*(t,X) = u^*(x)$$

where $\mathcal{L}^\dagger$ is the operator adjoint to $\mathcal{L}$

$U^*(T,X)$ can be represented as the solution to:

$$U^*(T,X) = \int_{-\infty}^{\infty} u^*(x) G(t,T,X-x)dx$$

Applying the Fourier transformed density function (5):

$$\begin{aligned} U^*(T,X) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} u^*(x) \int_{-\infty}^{\infty} \Re\left[e^{-i\phi(X-x)}\widehat{G}(t,T,\phi)\right] d\phi dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Re\left[e^{-i\phi X}\left(\int_{-\infty}^{\infty} e^{i\phi x} u^*(x)dx\right)\widehat{G}(t,T,\phi)\right] d\phi \end{aligned}$$

This can be computed by:

$$U^*(T,x) = \mathrm{fft}\left(\mathrm{ifft}(u^*(x)) \odot \widehat{G}(t,T,\phi)\right) \tag{7}$$

## FFT based method. Time Stepping

In case of discrete monitoring, the value function depends on the state variables observed at discrete times $\{t_m\}_{m=1,\dots,\overline{m}}$

Compute the value function applying the DFT algorithm at each time step:

**1)** Apply the terminal condition by $U(t_{\overline{m}}, x) = u(x)$

**2)** Given $U(t_m, x)$, compute $U(t_{m-1}, x)$ by:

$$U_{m-1}(x) = e^{-\int_{t_{m-1},}^{t_m} r(t')dt'} \text{ifft}\left(\text{fft}(U(t_m, x)) \odot \widehat{G}(t_{m-1}, t_m, \phi)\right)$$

**3)** Set $m \to m-1$ and, if $m > 0$, apply the boundary and coupon conditions and go to **2)**
otherwise, if $m = 0$, the recursion is stopped and the present value is computed

## FFT based method. Implication

Solutions to forward and backward problems are consistent

We use the forward induction to compute:
TPDF $G(0, T, X) \Rightarrow$ survival probability at $T \Rightarrow$ CDS spread at $T$

Given volatility and jump amplitude parameters, we use the forward induction and calibrate the term structure of jump intensity $\lambda(t)$ by bootstrapping

European equity and CDS options can also by computed by the forward induction to calibrate volatility and jump amplitude parameters

We use the backward algorithm defined on the same grid for pricing and counterparty charge evaluation

**FFT based method. Advantages**

**1)** Can be applied for problems with discrete monitoring and piecewise constant model parameters

**2)** Can be applied for jump-diffusions with known characteristic functions (the jump size PDF $\varpi(j)$ can be arbitrary)

**3)** Can be extended to two-dimensional problems with discrete monitoring

**4)** Complexity of the method using the standard DFT in one dimension is $O(2N \log N)$ per time step (the complexity in two dimensions is $O(2N_1 N_2 \log(N_1 N_2)))$

**5)** Relatively fast and easy to implement in one and two dimensions

**FFT based method. Disadvantages**

**1)** DFT assumes that the function is periodic (extend the function)

**2)** Computational domain is required to be uniform (use fractional FFT)

**3)** Convergence is controlled by choosing the grid for transform variable $\phi$ (look at the decay of $\widehat{G}(t_{m-1}, t_m, \phi)$ - can be slow if the volatility parameter is small)

**4)** The scheme is only first order accurate in the space variable because of the discontinuity at the barrier (use Hilbert transform (Feng-Linetsky (2008)))

**5)** Becomes slow if the number of discrete monitoring times is large

**PDE based methods. Considerations**

+ PDE methods are not restricted to uniform grids

+ Convection-dominated problems when drift is large and volatility is small are easier to handle

- Non-local jump term is difficult to handle, especially, in two dimensions

- A direct computation of the integral part by, say, trapezoidal rule leads to $O(N^2)$ complexity

- Using the DFT to compute the convolution part (Andersen-Andreasen (2000)) leads to $O(N \log N)$ complexity but suffers from unpleasant features of the DFT (uniform grids, periodicity, convergence)

However, explicit algorithms of $O(N)$ complexity can be employed if jumps are exponential or discrete (Lipton (2003), Carr-Mayo (2007), Toivanen (2008))

## PDE based method. Discretisation

*For continuous monitoring:*

The computational domain is $[b, x_{\text{max}}]$

The default boundary is enforced continuously

*For discrete monitoring:*

The computational domain is $[x_{\text{min}}, x_{\text{max}}]$

The default boundary is enforced only at default monitoring times (at intermediate time an artificial boundary condition must be enforced)

## PDE based method. Time stepping

To compute the value $U^{l-1}$ at time $t = t_l$ given its value $U^l$ at time $t = t_l$, $l = 1, ..., L$, we use the following splitting:

$$\begin{aligned} \frac{U_n^* - U_n^l}{\delta t_l} &= \mathcal{J}_n^l U^l + c_n^{l-1}, \ n = 2, ..., N-1, \\ \frac{U_n^{l-1} - U_n^*}{\delta t_l} &= \mathcal{D}_n^l U^{l-1}, \ n = 2, ..., N-1 \end{aligned} \tag{8}$$

At the first step, we use the explicit scheme to approximate the integral part and compute the auxiliary function $U^*$ given $U_n^l$

At the second step, we use the implicit scheme to approximate the diffusive step and compute $U_n^{l-1}$ given $U^*$

## PDE based method. Integral part for discrete jumps

Given $\varpi(j) = \delta(j - \nu)$, $\nu > 0$:

$$J_n^l = U(t_l, x_n - \nu), \;\; n = 2, ..., N - 1$$

We approximate $J_n^l$ by linear interpolation with the second order accuracy:

$$J_n^l = \omega_{n_j} U_{n_j - 1}^l + (1 - \omega_{n_j}) U_{n_j}^l$$

where

$$\omega_{n_j} = \frac{y_{n_j} - (y_n - \nu)}{y_{n_j} - y_{n_j - 1}}$$

$$n_j = \min\{j : x_{j-1} \leq x_n - \nu < x_j\}$$

## PDE based method. Integral part for exponential jumps I

Given $\varpi(j) = \nu e^{\nu j}$, $\nu > 0$:

$$J(x) = \nu \int_{-\infty}^{0} e^{\nu j} U(x+j)dj$$

For a small number $h$, $h > 0$:

$$\begin{aligned}
J(x+h) &= \nu \int_{-\infty}^{0} e^{\nu j} U(x+h+j)dj \\
&\overset{z=h+j}{=} \nu e^{-\nu h} \int_{-\infty}^{h} e^{\nu z} U(x+z)dz \\
&= \nu e^{-\nu h} \left( \int_{-\infty}^{0} e^{\nu z} U(x+z)dz + \int_{0}^{h} e^{\nu z} U(x+z)dz \right) \\
&= e^{-\nu h} J(x) + \tilde{J}_1(x)
\end{aligned}$$

## PDE based method. Integral part for exponential jumps II

Expanding $U(x+z)$ in Taylor series around $z=0$ yields:

$$\tilde{J}_1(x) = \nu e^{-\nu h}\int_0^h e^{\nu z} U(x+z)dz = \tilde{a}_0 U(x) + \tilde{a}_1 U' + O(h^3),$$

where

$$\tilde{a}_0 = \nu e^{-\nu h}\int_0^h e^{\nu z} dz = 1 - e^{-\nu h},\ \tilde{a}_1 = \nu e^{-\nu h}\int_0^h z e^{\nu z} dz = \frac{\nu h - \left(1 - e^{-\nu h}\right)}{\nu}$$

Accordingly, with the second order accuracy

$$J(x+h) = e^{-\nu h} J(x) + w_0(\nu, h) U(x) + w_1(\nu, h) U(x+h)$$

where

$$w_0(\nu, h) = \frac{1 - (1 + \nu h)\, e^{-\nu h}}{\nu h},\ w_1(\nu, h) = \frac{\nu h - \left(1 - e^{-\nu h}\right)}{\nu h}$$

## PDE based method. Improving the convergence

Apply the fixed point iterations (d'Halluin *et al*(2005)):

**1)** Set $V^0 = U^l + \delta t_l c_n^{l-1}$;

**2)** For $p = 1, 2, ..., \overline{p}$ apply the scheme (8):

$$\frac{V_n^* - V_n^0}{\delta t_l} = \mathcal{J}_n^l V^{p-1}, \ n = 2, ..., N-1,$$
$$\frac{V_n^p - V_n^*}{\delta t_l} = \mathcal{D}_n^l V^p, \ n = 2, ..., N-1$$

**3)** if norm $||V^p - V^{p-1}||$ in becomes small, stop and set $U^{l-1} = V^p$

Typically, $\overline{p} = 2$ is enough

## PDE based method. Summary

**1)** The scheme has $O(N)$ complexity per each time step

**2)** Although first order in time, the implicit scheme tends to be more stable than the Crank-Nicolson based scheme (especially for forward equation and two-dimensional problems)

**3)** The scheme is second order accurate in the spacial variable if the drift term is not dominant, otherwise it is first order accurate ($\mathcal{D}$ is discretisized appropriately)

**4)** A similar scheme is applied for the forward equation

**5)** As before, using the same grid, the forward scheme is applied for model calibration and the backward scheme is applied for pricing

## Two dimensional dynamics

Introduce stochastic drivers $x_1(t)$ and $x_2(t)$ driven by correlated jump-diffusion process:

$$dx_1(t) = \mu_1(t)dt + \sigma_1(t)dW_1(t) + j_1(dN_{\{1,2\}}(t) + dN_{\{1\}}(t)),\ x_1(0) = 0,$$
$$dx_2(t) = \mu_2(t)dt + \sigma_2(t)dW_2(t) + j_2(dN_{\{1,2\}}(t) + dN_{\{2\}}(t)),\ x_2(0) = 0$$

the jump amplitudes $j_1$ and $j_2$ have the same PDF $\varpi(j_1)$ and $\varpi(j_2)$ as in the marginal dynamics, respectively

jump intensity rates $\lambda_1(t)$ and $\lambda_2(t)$ and volatility parameters $\sigma_1(t)$ and $\sigma_2(t)$ are taken from the marginal marginal dynamics

## Correlation Structure

**1)** Brownian motions $W_1(t)$ and $W_2(t)$ are correlated with parameter $\rho$

**2)** Simultaneous jumps occur according to the Poisson process $N_{\{1,2\}}(t)$ with the intensity rate:

$$\lambda_{\{1,2\}}(t) = \min\{\rho, 0\} \min\{\lambda_1(t), \lambda_2(t)\}$$

**3)** Idiosyncratic jumps occur according to Poisson processes $N_{\{1\}}(t)$ and $N_{\{2\}}(t)$ with jump intensities $\lambda_{\{1\}}(t)$ and $\lambda_{\{2\}}(t)$, respectively:

$$\lambda_{\{1\}}(t) = \lambda_1(t) - \lambda_{\{1,2\}}(t), \; \lambda_{\{2\}}(t) = \lambda_2(t) - \lambda_{\{1,2\}}(t)$$

We need to calibrate the model correlation parameter $\rho$ from either market or historical data

**Jump Size PDF and Instantaneous Correlation**

Consider the instantaneous correlations between $x_1(t)$ and $x_2(t)$ under the assumption of discrete jumps, $\rho_{12}^{\text{dis}}$, and that under exponential jumps, $\rho_{12}^{\text{exp}}$:

$$\begin{aligned}
\rho_{12}^{\text{dis}} &= \frac{\rho\sigma_1\sigma_2 + \lambda_{\{1,2\}}\nu_1\nu_2}{\sqrt{\sigma_1^2 + \lambda_1\nu_1^2}\sqrt{\sigma_2^2 + \lambda_2\nu_2^2}}, \\
\rho_{12}^{\text{exp}} &= \frac{\rho\sigma_1\sigma_2 + \frac{\lambda_{\{1,2\}}}{\nu_1\nu_2}}{\sqrt{\sigma_1^2 + 2\frac{\lambda_1}{\nu_1^2}}\sqrt{\sigma_2^2 + 2\frac{\lambda_2}{\nu_2^2}}}
\end{aligned} \tag{9}$$

If the systematic intensity $\lambda_{\{1,2\}}$ is large, $\rho_{12}^{\text{dis}} \sim 1$ and $\rho_{12}^{\text{exp}} \sim \frac{1}{2}$

From experiments: the maximal implied Gaussian correlation that can be achieved (using $\rho = 0.99$) is about 90% for the model with discrete jumps and about 50% for the model with exponential jumps

The assumption about exponential jumps is not realistic by modelling the joint dynamics of strongly correlated firms belonging to one industry group (such as financial companies)

## Numerical Methods for Two Dimensional Problem

We consider the backward problem for the value function $U(t, x_1, x_2)$:

$$\begin{aligned} U_t + \mathcal{M}U &= -c(t, x_1, x_2) \\ U(T, x_1, x_2) &= u(x_1, x_2) \end{aligned} \tag{10}$$

$$\mathcal{M} = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_{12} + \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_{12}$$

$\mathcal{D}_1$ and $\mathcal{D}_2$ are 1-d diffusion-convection operators in $x_1$ and $x_2$ directions, respectively

$\mathcal{J}_1$ and $\mathcal{J}_2$ are 1-d orthogonal integral operators in $x_1$ and $x_2$ directions, respectively

$\mathcal{D}_{12}$ is the correlation operator, $\mathcal{D}_{12}U(t, x_1, x_2) \equiv \rho\sigma_1(t)\sigma_2(t)U_{x_1 x_2}(t, x_1, x_2)$

$\mathcal{J}_{12}$ is the cross integral operator:

$$\mathcal{J}_{12}U(t, x_1, x_2) \equiv \lambda_{\{1,2\}}(t) \int_{-\infty}^{0} \int_{-\infty}^{0} U(t, x_1+j_1, x_2+j_2)\varpi(j_1)\varpi(j_2)dj_1 dj_2$$

## Counterparty Charge Using Structural Model

Let $x_1(t)$ and $x_2(t)$ be the stochastic drivers for the reference name and the counterparty, respectively

The value of the countrparty charge $U(t, x_1, x_2)$ defined as the solution to (1) solves the following problem:

$$
\begin{aligned}
&U_t + \mathcal{M}U(t, x_1, x_2) = 0, \\
&U(T, x_1, x_2) = 0, \\
&U(t, x_1, x_2) = 0,\ x_1 \le b_1,\ x_2 > b_2\ (\tau_1 < \tau_2); \\
&U(t, x_1, x_2) = (1 - R_2) \max\{C(t, x_1), 0\},\ x_1 > b_1,\ x_2 \le b_2,\ (\tau_1 > \tau_2); \\
&U(t, x_1, x_2) = (1 - R_2)(1 - R_1),\ x_1 \le b_1,\ x_2 \le b_2,\ (\tau_1 = \tau_2); \\
&\lim_{x_1 \to \infty} U(t, x_1, x_2) = 0,\ \lim_{x_2 \to \infty} U(t, x_1, x_2) = 0
\end{aligned}
$$

$C(t, x)$ is the value of CDS contract without counterparty risk

Joint defaults are possible under the discrete monitoring

## Discretisation

We develop a modified Craig-Sneyd (1988) discretization scheme to compute the solution at time $t_{l-1}$, $U^{l-1}_{n,m}$, given the solution at time $t_l$, $U^l_{n,m}$, $l = 1, ..., L$, as follows:

$$(1 - \delta t_l \mathcal{D}_1) U^*_{n,m} = U^l_{n,m} + \delta t_l \left( \left( \mathcal{D}_2 + \mathcal{D}_{12} + \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_{12} \right) U^l_{n,m} + c^{l-1}_{n,m} \right),$$
$$(1 - \delta t_l \mathcal{D}_2) U^{l-1}_{n,m} = U^*_{n,m} - \delta t_l \mathcal{D}_2 U^l_{n,m}$$

In the first line, for each fixed index $m$ we apply the jump operators and diffusion operator in $x_1$ direction, the correlation operator, and coupon payments (if any); and solve the tridiagonal system of equations to get the auxiliary solution $U^*_{\cdot,m}$

In the second line, keeping $n$ fixed, we apply the implicit step in $x_2$ direction and solve the system of tri-diagonal equations to get the solution

## Discretisation of Cross Jump Part

Direct methods are infeasible because of $O(N^2M^2)$ complexity

DFT method (Clift-Forsyth (2008)) has $O(NM \log NM)$ complexity but suffers from problems associated with the DFT

Explicit methods with $O(NM)$ complexity are available for discrete and exponential jumps (Lipton-Sepp (2009))

The simplest case is if jumps are discrete:

$$J_{12}U = U(x_1 - \nu_1, x_2 - \nu_2)$$

This term is approximated by bi-linear interpolation with the second order accuracy leading to the $O(NM)$ complexity

## Discretisation of the Jump Part. Negative exponential jumps

Consider the integral:

$$J(x_1, x_2) = \nu_1 \nu_2 \int_{-\infty}^{0} \int_{-\infty}^{0} e^{\nu_1 j_1 + \nu_2 j_2} U(x_1 + j_1, x_2 + j_2) dj_1 dj_2$$

Take small numbers $h_x$ and $h_y$, $h_x > 0$, $h_y > 0$:

$$J(x_1 + h_1, x_2 + h_2) = \nu_1 \nu_2 \int_{-\infty}^{0} \int_{-\infty}^{0} e^{\nu_1 j_1 + \nu_2 j_2} U(x_1 + h_1 + j_1, x_2 + h_2 + j_2) dj_1$$

$$= \nu_1 \nu_2 e^{-\nu_1 h_1 - \nu_2 h_2} \int_{-\infty}^{h_1} \int_{-\infty}^{h_2} e^{\nu_1 z_1 + \nu_2 z_2} U(x_1 + z_1, x_2 + z_2) dz_1 dz_2$$

$$= \nu_1 \nu_2 e^{-\nu_1 h_1 - \nu_2 h_2} \left( \int_{-\infty}^{0} \int_{-\infty}^{0} + \int_{0}^{h_1} \int_{-\infty}^{0} + \int_{-\infty}^{0} \int_{0}^{h_2} + \int_{0}^{h_1} \int_{0}^{h_2} \left[ e^{\nu_1 z_1 + \nu_2 z_2} U(x_1 \right.\right.$$

$$= e^{-\nu_1 h_1 - \nu_2 h_2} J(x_1, x_2) + e^{-\nu_2 h_2} \widetilde{J}_{10}(x_1, x_2) + e^{-\nu_1 h_1} \widetilde{J}_{01}(x_1, x_2) + \widetilde{J}_{11}(x_1, x_2)$$

Integrals $\widetilde{J}_{10}(x, y)$, $\widetilde{J}_{01}(x, y)$, and $\widetilde{J}_{11}(x, y)$ can be computed by recursion with second order accuracy and $O(NM)$ complexity

**Discretisation of the Jump Part. Improving the convergence**

At each time step, we apply the fixed point iterations as follows ($\bar{p} = 2$ is enough):

**1)** Set $V^0_{n,m} = U^l_{n,m} + \delta t_l \mathcal{C}_{n,m} U^l_{n,m} + \delta t_l c^{l-1}_{n,m}$;

**2)** For $p = 1, 2, ..., \bar{p}$ apply the above scheme:

$$\begin{aligned} V^j_{n,m} &= V^0_{n,m} + \delta t_l (\mathcal{J}_{12} + \mathcal{J}_1 + \mathcal{J}_2) V^{p-1}, \\ (1 - \delta t_l \mathcal{D}_1) V^*_{n,m} &= \delta t_l \mathcal{D}_2 V^{p-1}_{n,m} + V^j_{n,m}, \\ (1 - \delta t_l \mathcal{D}_2) V^p_{n,m} &= V^*_{n,m} - \delta t_l \mathcal{D}_2 V^{p-1}_{n,m} \end{aligned} \tag{11}$$

**3)** if norm $||V^p - V^{p-1}||$ becomes small, stop and set $U^{l-1} = V^p$

## Discretisation. Final Remarks

**1)** The overall complexity of this method per time step is $O(NM)$ operations (using DFT method to compute the convolution leads to $O(NM\,\log(NM))$ complexity)

**2)** The scheme is first order accurate in time

**3)** The scheme is second order accurate in spacial variables (if the drift is not dominant)

**4)** The modified scheme is applied for the forward problem, so that, it needed, the calibration problem in two dimensions can be solved efficiently

## Example. Input data for model calibration

| | JPM | C |
|---|---|---|
| $s(0)$ | 36.49 | 8.47 |
| $L(0)$ | 604.11 | 353.07 |
| $s(0)/L(0)$ | 16.56 | 41.68 |
| $R$ | 40% | 40% |
| $l(0)$ | 241.644 | 141.228 |
| $v(0)$ | 278.134 | 149.698 |
| $b$ | -0.1406 | -0.0582 |
| $\nu(1)$ | 0.1406 | 0.0582 |
| $\nu(2)$ | 0.0703 | 0.0291 |
| $\sigma$ | 0.0262 | 0.0113 |

Use two choices for the jump size:
**1)** $\nu \equiv \nu_1 = -b$ in the model with discrete jumps and $\nu \equiv \frac{1}{\nu_1} = \frac{1}{b}$ in the model with exponential jumps;
**2)** $\nu \equiv \nu_2 = -\frac{1}{2}b$ in the model with discrete jumps and $\nu \equiv \frac{1}{\nu_2} = \frac{1}{2b}$ in the model with exponential jumps

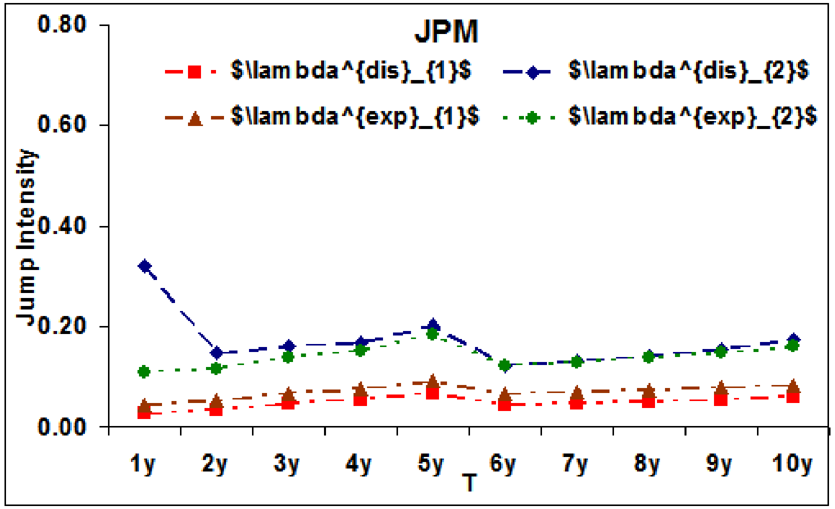
**Example. Input data for model calibration**

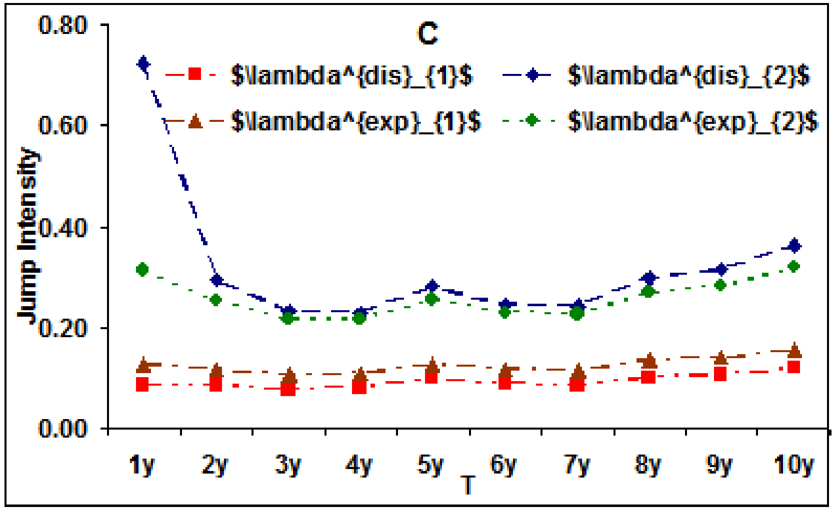
*Spread data for model calibration and the survival probability, default leg, and annuity leg implied using the hazard rate model*

| | CDS Spread | | Survival Prob | | Default Leg | | Annuity Leg | |
|---|---|---|---|---|---|---|---|---|
| $T$ | JPM | C | JPM | C | JPM | C | JPM | C |
| 1y | 0.0105 | 0.0286 | 0.9826 | 0.9535 | 0.0174 | 0.0465 | 0.9913 | 0.9766 |
| 2y | 0.0118 | 0.0271 | 0.9614 | 0.9137 | 0.0386 | 0.0863 | 1.9633 | 1.9099 |
| 3y | 0.0134 | 0.0257 | 0.9348 | 0.8798 | 0.0652 | 0.1202 | 2.9114 | 2.8065 |
| 4y | 0.0147 | 0.0249 | 0.9063 | 0.8475 | 0.0937 | 0.1525 | 3.8320 | 3.6701 |
| 5y | 0.0160 | 0.0248 | 0.8743 | 0.8138 | 0.1257 | 0.1862 | 4.7223 | 4.5007 |
| 6y | 0.0161 | 0.0243 | 0.8498 | 0.7857 | 0.1502 | 0.2143 | 5.5841 | 5.3002 |
| 7y | 0.0162 | 0.0238 | 0.8268 | 0.7590 | 0.1732 | 0.2410 | 6.4223 | 6.0725 |
| 8y | 0.0163 | 0.0236 | 0.8034 | 0.7319 | 0.1966 | 0.2681 | 7.2374 | 6.8179 |
| 9y | 0.0164 | 0.0234 | 0.7804 | 0.7056 | 0.2196 | 0.2944 | 8.0292 | 7.5366 |
| 10y | 0.0165 | 0.0233 | 0.7582 | 0.6801 | 0.2418 | 0.3199 | 8.7985 | 8.2294 |

## Example. Calibrated Intensity Rates

For both choice of jumps size distributions and the jump sizes, the model is calibrated to the term structure of CDS spreads given in Table 2 using the forward induction



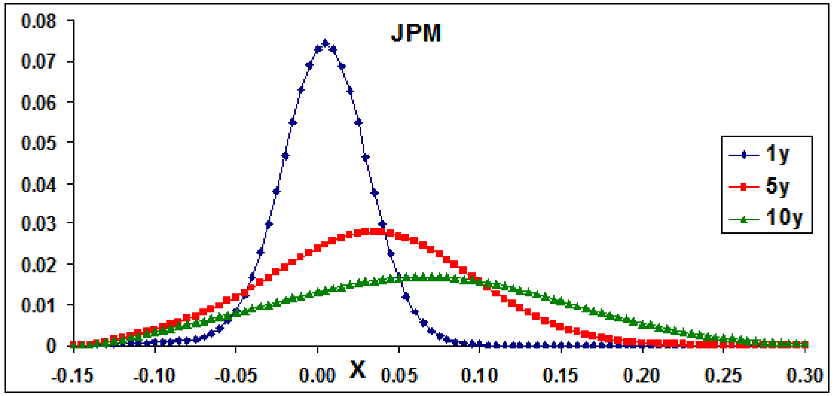



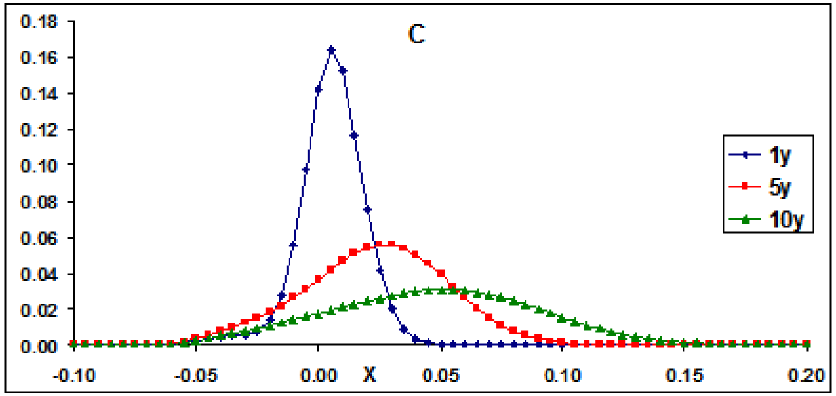

$\lambda_1^{dis}$ (lambda^{dis}_{1}) and $\lambda_2^{dis}$ (lambda^{dis}_{2}) stand for model with discrete jumps with sizes $\nu_1$ and $\nu_2$, respectively

$\lambda_1^{exp}$ (lambda^{exp}_{1}) and $\lambda_2^{exp}$ (lambda^{exp}_{2}) stand for model with exponential jumps with sizes $\frac{1}{\nu_1}$ and $\frac{1}{\nu_2}$, respectively

## Example. Input Data

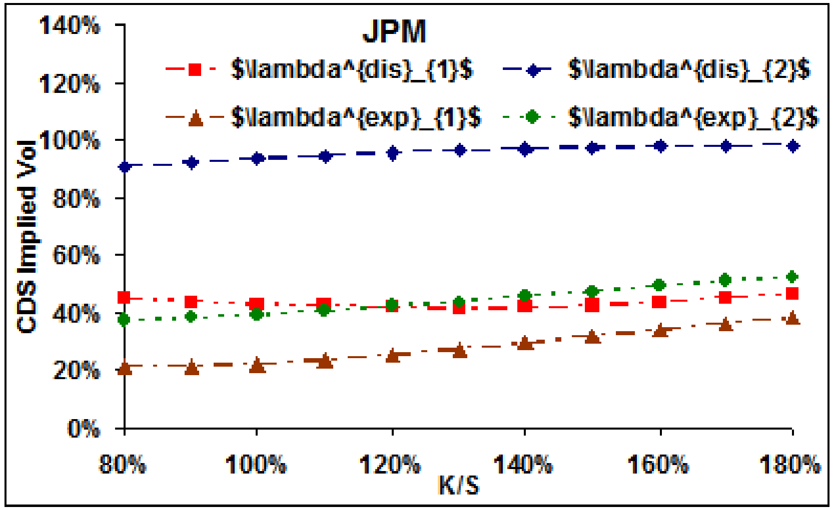
Implied density of the driver $x(t)$ for JMP and C in the model with exponential jump, $\nu = 1/b$, at maturities 1, 5, and 10 years

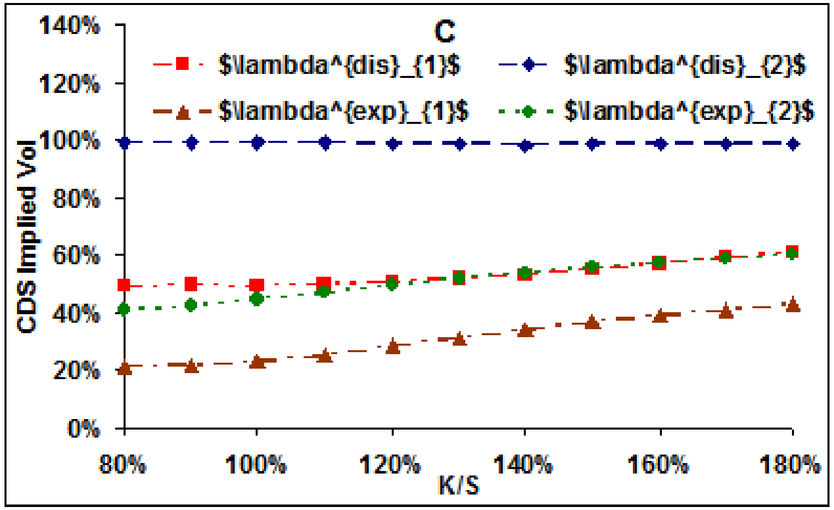


**Example. CDS option volatility**

The log-normal CDS option volatility implied from model values of one year option on five year CDS contract as a function of the moneyness $K_{\alpha,\beta}/S_{\alpha,\beta}$



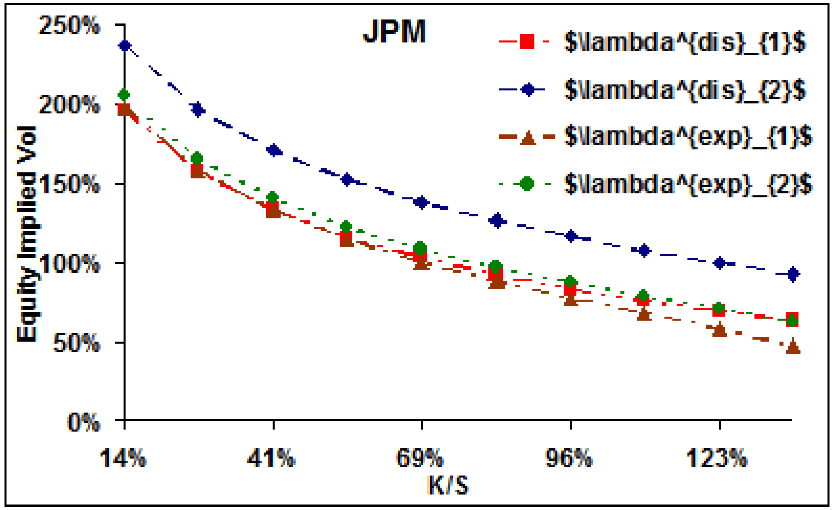



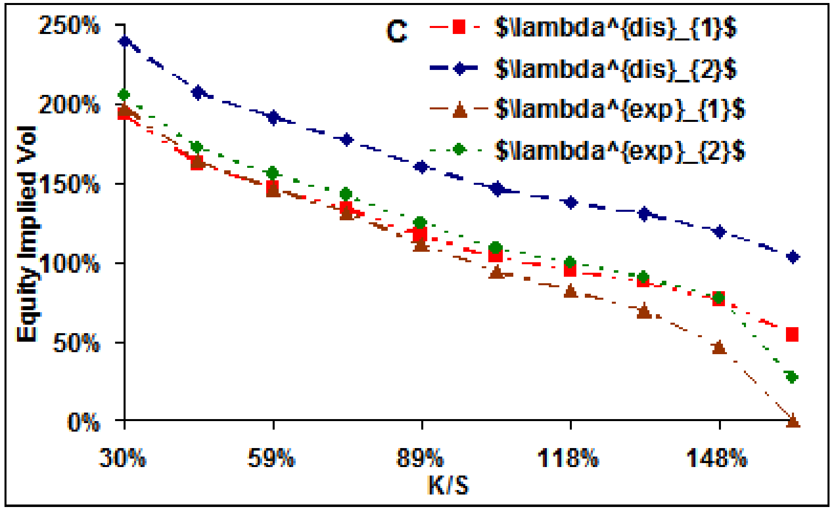

The model implied log-normal volatility $\sigma_{\alpha,\beta}$ exhibits a positive skew

This effect is in line with the market because the CDS spread volatility is expected to increase when the CDS spread increases, so that option sellers charge an extra premium for out-of-the-money CDS options

## Example. Log-normal equity volatility

Log-normal equity volatility implied from model values of put options with maturity 6 months using the Black-Scholes formula



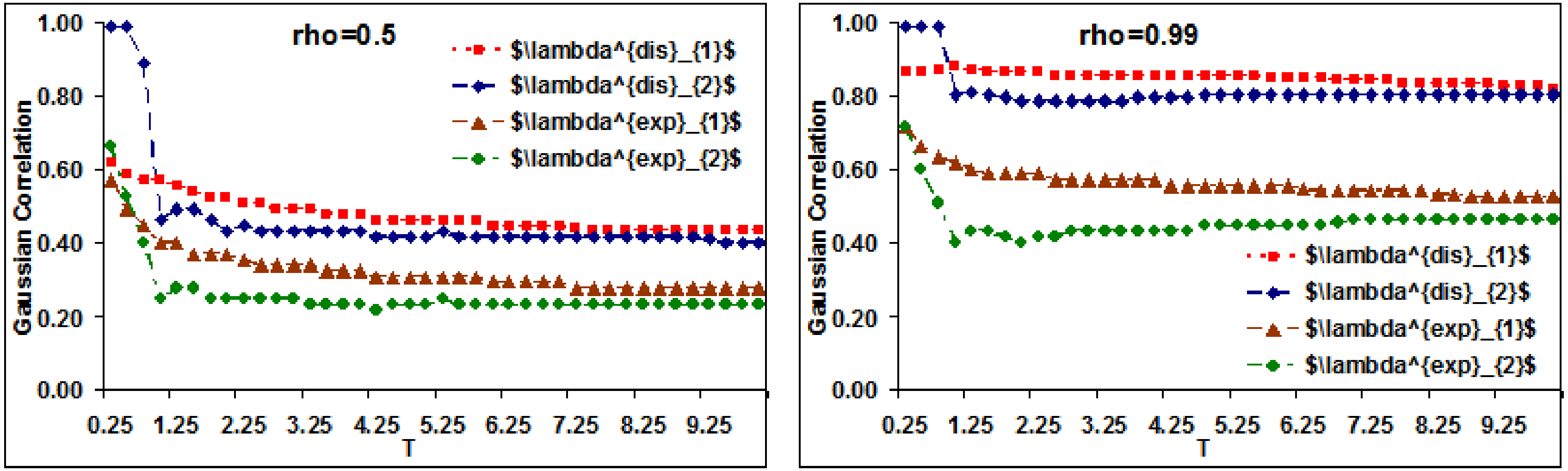

The model implies a remarkable skew in line with that observed in the market

The smaller the jump size the higher is the implied model volatility because the firm value is expected to have more jumps before the barrier crossing so that the realized volatility is expected to be higher

## Example. Implied Gaussian correlation

We compute the model implied Gaussian correlation by equating the fair spread of the first to default swap referencing JPM and C to that computed using the Gaussian copula with implied correlation

We use the two choices for the model correlation parameter: $\rho = 0.50$ and $\rho = 0.99$




The model with exponential jumps produces lower implied correlations

The model with smaller jump amplitudes implies smaller correlations

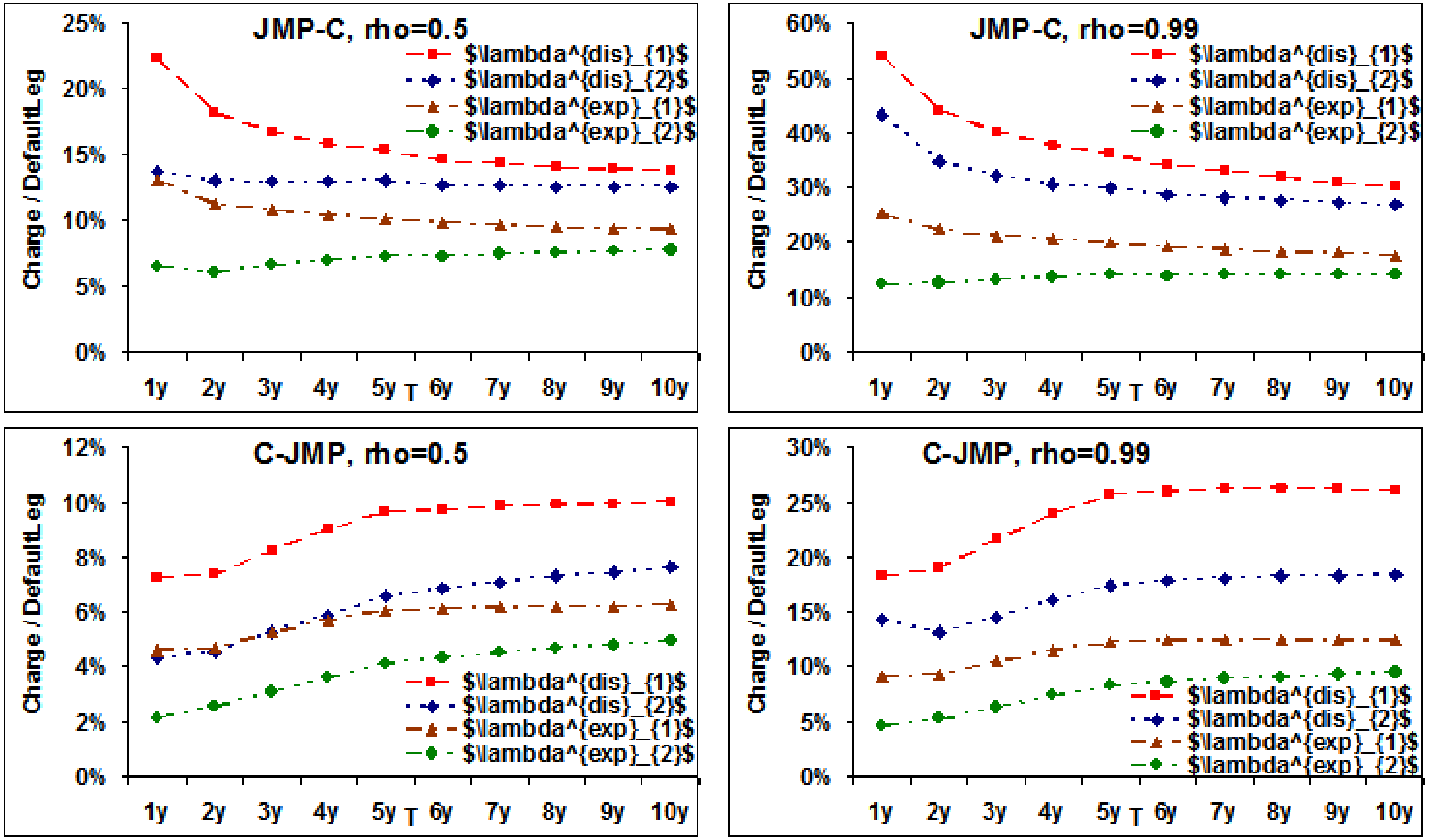
## Illustration. Counterparty charge

We compute the counterparty charge for par CDS on JPM sold by C and that for par CDS on C sold by JPM as functions of CDS maturity using two model correlation parameters: $\rho = 0.5$ and $\rho = 0.99$

We use $R = 0$ for the counterparty recovery and normalize the counterparty charge by the present value of the default leg of CDS on JPM corresponding to CDS maturity

For a moderate correlation assumption with $\rho = 0.50$, the model with discrete large jump implies the countrepaty charge in amount of $10\% - 15\%$ of the present value of the CDS protection leg on the underlying name, while, for a high correlation assumption with $\rho = 0.99$, this proportion grows to $30\% - 40\%$

# Counterparty charge for CDS on JPM sold by C (JMP-C, top) and for CDS on C sold by JPM (C-JPM, bottom)



## Counterparty charge. Conclusions

**1)** The larger is the correlation, the larger is the counterparty charge because, given the counterparty default, the protection lost is larger in case of high correlation

**2)** The larger the jump size, the larger is the counterparty charge because the model with higher jumps implies a larger correlation

**3)** The model with discrete jumps implies a larger counterparty charge than the model with exponential jumps because the former implies larger correlation and CDS spread volatility

**4)** The counterparty charge is not symmetric. It is expected that a more risky counterparty implies a higher counterparty charge

**Conclusions**

**1)** We have proposed an extended structural model capable of fitting arbitrary term structures of CDS spreads

**2)** Applying this model, we have obtained a novel method to analyse the counterparty risk

**3)** We have developed a number of semi-analytical and numerical methods to solve calibration and pricing problems in an efficient way

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