Double-no-touch options and the volatility smile

Aleksandar Mijatović
Department of Mathematics
Imperial College London

Workshop on Spectral and Cubature Methods

University of Leicester

19 June 2009

Based on joint work with **Martijn Pistorius**

Introduction

Double-no-touch (or **range bet**) is a contingent claim that pays one unit of domestic currency at expiry T if neither of the contract defined barrier levels $L < U$ have been breached by a risky security S during the life of the option.

- DNTs are the most liquid exotic options in the financial markets and should be used to extract risk-neutral dynamics of S.
- The arbitrage free price in a model S of a double-no-touch option is given by

$$\begin{aligned} D_{S_0}(T) &= \mathbb{E}_{S_0}\left[\frac{I_{\{\tau_{LU}>T\}}}{B^D_T}\right], \quad \text{where} \\ \tau_{LU} &:= \inf\{t \, : \, S_t \notin (L,U)\}, \end{aligned}$$

B^D denotes the domestic bond and I is the indicator function.

The implied volatility surface

IVol surface is a graph of a function $(K,T) \mapsto \sigma(K,T)$ defined implicitly by the equation

$$\mathrm{BS}(\sigma(K,T)) \quad = \quad C(K,T),$$

where $C(K,T)$ are the market/model specified call option prices and $\mathrm{BS}(\cdot)$ is the Black-Scholes formula.

- $C(K_{ij},T_i)$, $i=1,\ldots,n$, $j=1,2,3$, are the most liquid derivative instruments in the finical markets.
- Knowing σ is equivalent to knowing the one-dimensional marginals in a risk-neutral measure of the underlying process.
- To calibrate to the observed IVol surface the model needs to have stochastic volatility AND jumps.
- If $n=2$ (i.e. two maturities) typically time-dependence of parameters is needed for calibration.

Markov additive model

FX rate $S = (S_t)_{t\geq 0}$, $S_t := \exp(X_t)$, the domestic and foreign bonds $1/B^D = (1/B^D_t)_{t\geq 0}$ and $1/B^F = (1/B^F_t)_{t\geq 0}$ are given by

$$\begin{aligned} B^D_t &:= \exp\left(\int_0^t R_D(Z_s)ds\right), \quad B^F_t := \exp\left(\int_0^t R_F(Z_s)ds\right), \\ X_t &:= x + \int_0^t \mu(Z_s)ds + \int_0^t \sigma(Z_s)dW_s + \sum_{i\in E^0}\int_0^t I_{\{Z_s=i\}}dJ^i_s. \end{aligned}$$

- $Z = \{Z_t; t \geq 0\}$ is a continuous time irreducible Markov chain with finite state space $E^0 = \{1, \ldots, N\}$ and intensity-matrix Q
- $R_D, R_F : E^0 \to \mathbb{R}$ instantaneous domestic and foreign interest rate functions,
- $J_i = \{J_i(t); t \geq 0\}$ are independent compound Poisson processes with double phase-type jumps.

Markov additive model

Phase-type distributions A distribution F on $(0,\infty)$ is said to be of *phase-type*, if it is the distribution of the absorption time of a finite state Markov chain with one state ∂ absorbing and the remaining states transient.
The density of F is given by:

$$f(x) = \alpha' \mathrm{e}^{Tx} t, \qquad x > 0, \tag{1}$$

where

- T is the generator matrix and α the initial distribution,
- $'$ denotes transpose and $t = (-T)\mathbf{1}$, with $\mathbf{1}$ a column vector of ones.

Examples: Hyper-exponential, Erlang
Note: Phase-type distributions are *dense* in the class of all probability distributions on $(0,\infty)$

Markov additive model – European options

Theorem 0.1 *Let $\Lambda(u)$ be a diagonal matrix of size $N \times N$, where the i-th diagonal element equals the characteristic exponent of the process X in regime i*

$$\psi_i(u) := u\mu_i + \sigma_i^2 u^2/2 + \lambda_i p_i \left(\frac{\alpha_i^+}{\alpha_i^+ - u} - 1 \right) + \lambda_i (1-p_i) \left(\frac{\alpha_i^-}{\alpha_i^- + u} - 1 \right).$$

The discounted characteristic function of the Markov process (X, Z) is given by the formula

$$\begin{aligned} \mathbb{E}_{0,i} \left[\frac{\exp(uX_t)}{B_t^D} I_{\{Z_t = j\}} \right] &= e_i' \exp(t(K(u) - \Lambda_D)) e_j, \quad \textit{where} \\ K(u) &:= Q + \Lambda(u), \\ \Lambda_D &:= \operatorname{diag}(R_D). \end{aligned}$$

The drift $\mu : E^0 \to \mathbb{R}$ is given by the formula

$$\Lambda(1) = \Lambda_D - \Lambda_F, \quad \textit{where} \quad \Lambda_F := \operatorname{diag}(R_F).$$

Markov additive model – calibration of stochastic rates

For two maturities $T_1 < T_2$ we have two pairs P^D_{0,T_k}, P^F_{0,T_k}, $k = 1, 2$, of domestic and foreign zero coupon bonds which are in our model equal to $P^F_{0,T_k} = \mathbb{E}_{x,i}[(B^D_{T_k})^{-1} S_{T_k}]/S_0$ and $P^D_{0,T_k} = \mathbb{E}_{x,i}[(B^D_{T_k})^{-1}]$.
To calibrate R_D, R_F we solve the system

$$\begin{aligned} P^D_{0,T_k} &= e'_i \exp\left((Q - \Lambda_D)T_k\right) \mathbf{1}, \\ P^F_{0,T_k} &= e'_i \exp\left((Q - \Lambda_F)T_k\right) \mathbf{1}, \end{aligned}$$

where $k = 1, 2$ and $\Lambda_D = \mathrm{diag}(R_D), \Lambda_F = \mathrm{diag}(R_F)$.
If $N = 2$, this system determines the risk-neutral drift of S, is independent of the calibration to option prices and is very accurate.

Wiener-Hopf factorisation for Brownian motion X

$$\frac{q}{q-u^2/2} \quad = \quad \frac{\rho_+(q)}{u+\rho_+(q)} \cdot \frac{\rho_-(q)}{u+\rho_-(q)}, \quad \text{where} \quad \rho_\pm(q) = \pm\sqrt{2q}$$

are the largest and smallest root of the characteristic equation

$$q - \frac{u^2}{2} = 0.$$

The moment generating function of $\overline{X}_{e_q}$, $\underline{X}_{e_q}$ are given by the factors above, where

$$\overline{X}_t = \max\{X_s \,:\, s \in [0,t]\}, \quad \underline{X}_t = \min\{X_s \,:\, s \in [0,t]\}$$

and e_q is an independent exponential rv with parameter q.

Wiener-Hopf factorisation for Brownian motion X

Therefore $\overline{X}_{e_q}$, $\underline{X}_{e_q}$ are geometric rvs with parameters $\rho_+(q)$, $-\rho_-(q)$ respectively.
Define $\tau_u := \min\{t \geq 0 \,:\, X_t \geq u\}$ and $\tau_l := \min\{t \geq 0 \,:\, X_t \leq l\}$ and note that

$$\{\tau_u > t\} = \{\overline{X}_t < u\}, \quad \{\tau_l > t\} = \{\overline{X}_t > l\}.$$

Hence

$$\mathbb{E}[e^{-q\tau_u}] = e^{-u\rho_+(q)} \quad \text{and} \quad \mathbb{E}[e^{-q\tau_l}] = e^{l\rho_-(q)}.$$

An application of Doob’s optional stopping theorem yields a closed form for the Laplace transform for the two-sided first passage time

$$\tau_{lu} := \inf\{t \,:\, X_t \notin (l,u)\}.$$

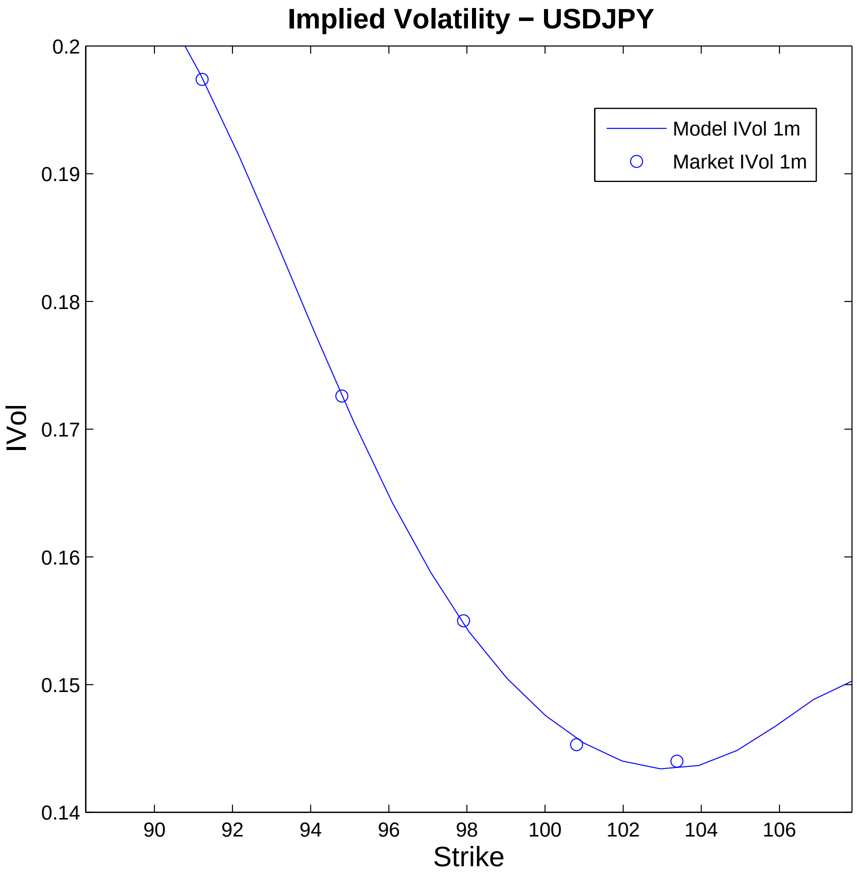
Matrix Wiener-Hopf factorisation

In the general case of the Markov additive process the steps are similar (but the details are very different):

- Fluid-embedding: embed the jumps to get a continuous Markov additive process (phase-type distribution of jumps is used in this step).
- The characteristic equation becomes a quadratic matrix equation.
- The Wiener-Hopf factors can be inverted analytically.
- Closed-form formula for Laplace transform of the one-sided first passage time can be obtained.
- Doob’s optional stopping theorem gives a closed-form formula for the Laplace transform of the two-sided first passage time.

Conclusion: fast pricing (and calibration) for the double-no-touch options is possible in this model.

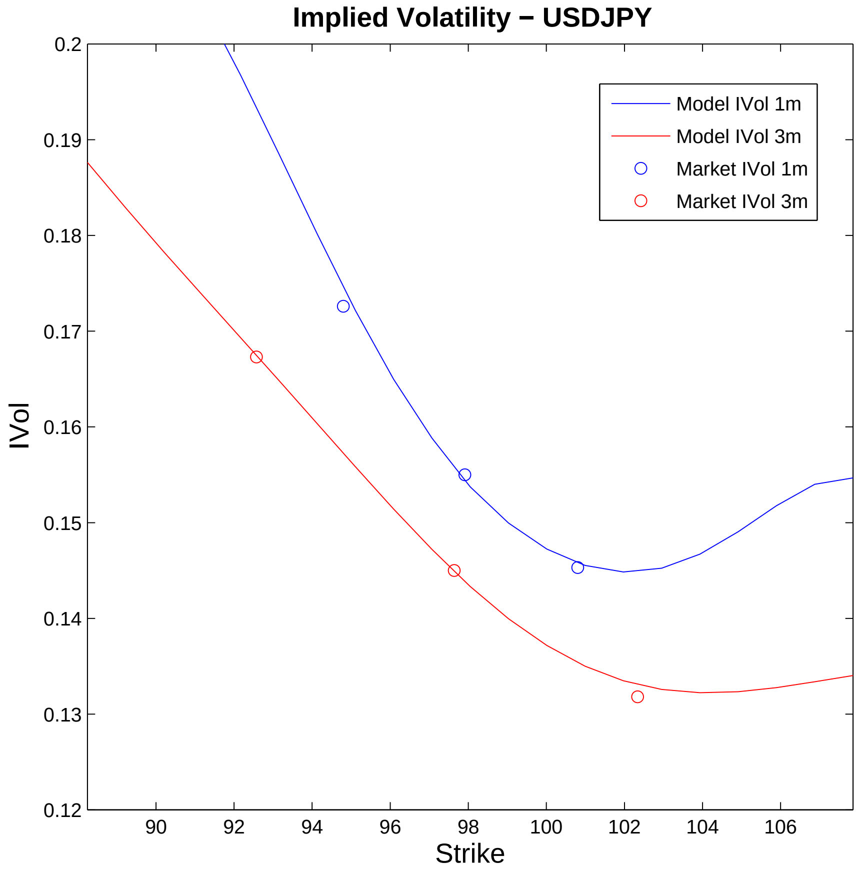
USDJPY – one maturity



Market data: $S_0 = 98.05$, domestic rate $r_d = -0.00036$, foreign rate $r_f = 0.0045$, maturity $T = 1/12$.

Model parameters: $N = 2$, $q_1 = 12, q_2 = 6$, $B_m(1) = \text{diag}(-45, -300)$, $B_p(1) = -100$,$b_m(1) = (0.12, 0.88)$, $\lambda_2 = 0$ (chosen), $\sigma = (0.0423, 0.0628)$, $\lambda_1 = 276.5196$, $p_1 = 0.1610$ (calibrated).

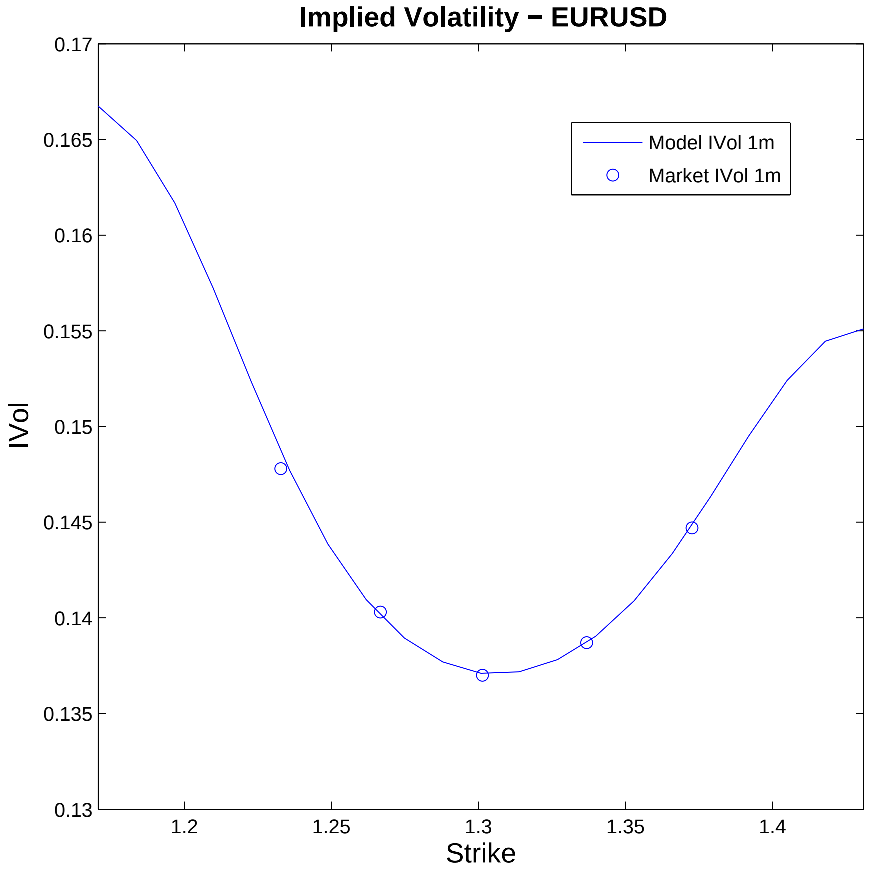
USDJPY – two maturities



Market data: $S_0 = 98.05$, domestic interest rate $r_d = (-0.00036, 0.005)$, foreign interest rate $r_f = (0.0045, 0.0111)$, maturity $T = (1/12, 3/12)$.

Model parameters: $N = 2$, $q_1 = 12$, $q_2 = 6$, $B_m(1) = \mathrm{diag}(-45, -300)$, $b_m(1) = (0.12, 0.88)$, $B_m(2) = -50$, $B_p(1) = -130$, $p_2 = 0$ (chosen), $\sigma = (0.1312, 0)$, $\lambda_1 = 137.4337$, $\lambda_2 = 0.9484$, $p_1 = 0.0386$ (calibrated)

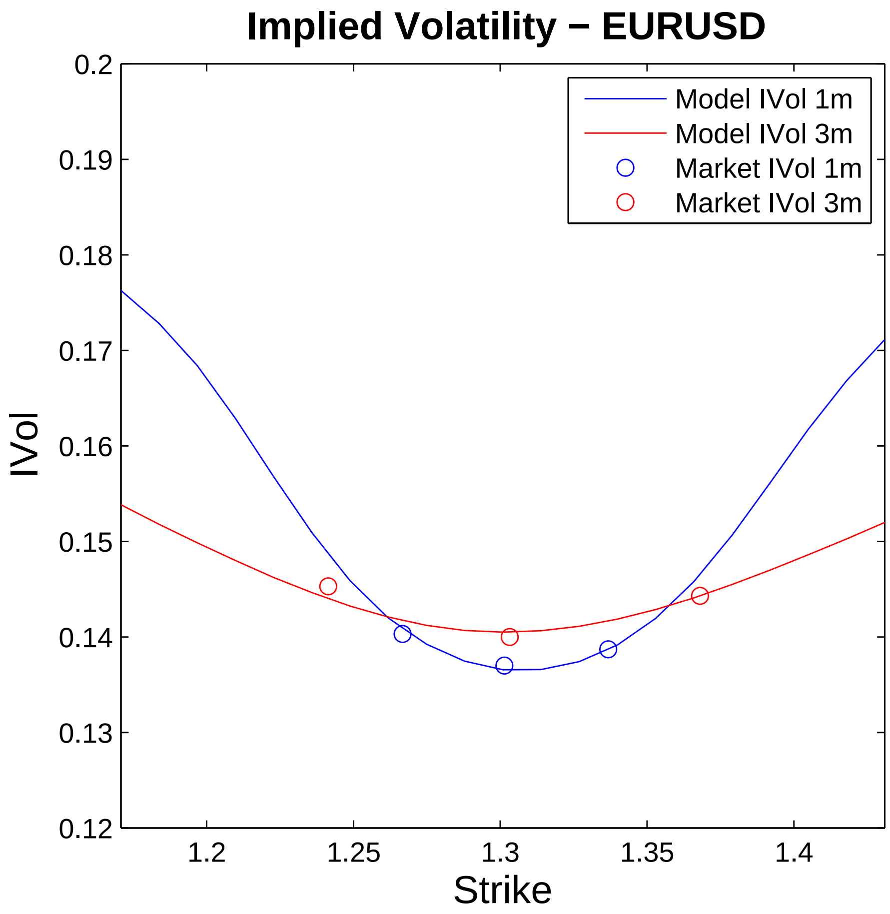
EURUSD – one maturity



Market data: spot $S_0 = 1.3009$, domestic interest rate $r_d = 0.0045$, foreign interest rate $r_f = 0.0084$, maturity $T = 1/12$.

Model parameters: $N = 2$, $q_1 = 12, q_2 = 6$, $B_m(1) = \mathrm{diag}(-45, -300)$,
$b_m(1) = (0.1, 0.9)$, $B_p(1) = -130$, $\lambda_2 = 0$ (chosen)
$\sigma = (0.1352, 0.0490)$, $\lambda_1 = 90.6456$, $p_1 = 0.5231$ (calibrated)

EURUSD – two maturities



Market data: $S_0 = 1.3009$, domestic rate $r_d = (0.0045, 0.0111)$, foreign rate $r_f = (0.0084, 0.0139)$, maturity $T = (1/12, 3/12)$.

Model parameters: $N = 2$, $q_1 = 12, q_2 = 6$, $B_m(1) = -70$, $B_p(1) = -70$, $B_m(2) = -30$, $B_p(2) = -30$, $p_2 = 0.5$ (chosen)

$\sigma = (0.1281, 0.0001)$, $\lambda_1 = 10.7141$, $\lambda_2 = 10.2962$, $p_1 = 0.1084$ (calibrated)