# Solving a BSDE with the cubature method

Konstantinos Manolarakis
Imperial College London

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# Non linear pricing

We consider a market with one risky asset, where different interest rates apply for borowing and lending :

$$\begin{aligned} dX_t &= \mu_t X_t dt + \sigma_t X_t dW_t \\ d\beta_t &= r_t \beta_t dt \\ dB_t &= R_t B_t dt, \qquad R_t > r_t. \end{aligned}$$

i.e. we can borrow money at a rate $R_t$ but invest in the bank account at rate $r_t$.
Let $Y_t$ denote the wealth process of a self financing strategy that allocates $\pi_t$ on $X_t$ and $Y_t - \pi_t$ on the bonds $\beta_t, B_t$. In such an economy we consider the problem of hedging a contigent claim $\xi$ written on $X_t$, $t \in [0, T]$.

# Non linear pricing

The self financing assumption gives :

$$
\begin{aligned}
dY_t &= \pi_t \frac{dX_t}{X_t} + (Y_t - \pi_t)_+ \frac{d\beta_t}{\beta_t} - (Y_t - \pi_t)_- \frac{dB_t}{B_t}, \quad Y_T = \xi \\
&\Rightarrow \\
dY_t &= \pi_t \mu_t dt + \pi_t \sigma_t dW_t + (Y_t - \pi_t)_+ r_t dt - (Y_t - \pi_t)_- R_t dt, \quad Y_T = \xi \\
&\Rightarrow \\
dY_t &= r_t Y_t dt + (\mu_t - r_t)\pi_t dt - (R_t - r_t)(Y_t - \pi_t)_- \, dt + \pi_t \sigma_t dW_t \\
Y_T &= \xi, \; a.s.
\end{aligned}
$$

Goal is to solve it (uniquely) with respect to $(Y_t, \pi_t)$. Then a fair price for $\xi$ at time 0 is $Y_0$.

# Non linear PDE representation

Let $X_s^{t,x}$ be the solution of the forward SDE:

$$dX_s^{t,x} = V_0(X_s^{t,x})dt + \sum_{j=1}^{d} V_j(X_s^{t,x})dW_t^j$$

and let $u(t,x) \in C^{1,2}([0,T] \times \mathbb{R}^d)$ solve the non linear backward Cauchy problem

$$\begin{aligned} u_t(t,x) + V_0(x) \cdot \nabla u(t,x) + \frac{1}{2}\mathrm{Tr}[V(x)V^*(x)D^2u(t,x)] \\ + f(t,x,u(t,x),\nabla u(t,x)V(x)) = 0 \qquad (1) \\ u(T,x) = \Phi(x). \end{aligned}$$

where $V_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $V := (V_1|\dots|V_d) \in \mathbb{R}^{d\times d}$.

# Non linear PDE representation

Apply Ito's lemma to $Y_s^{t,x} := u(s, X_s^{t,x})$:

$$dY_s^{t,x} = \left(u_t + V_0 \cdot \nabla u + \frac{1}{2}\mathrm{Tr}[VV^* D^2 u]\right)\left(s, X_s^{t,x}\right) ds + \nabla u V dW_s$$

Integrating from $t$ to $T$

$$\Phi(X_T^{t,x}) - Y_t^{t,x} = \int_t^T \left(u_t + V_0 \cdot \nabla u + \frac{1}{2}\mathrm{Tr}[VV^* D^2 u]\right) ds + \int_t^T \nabla u V dW_s$$

and since $u$ solves the pde

$$Y_t = \Phi(X_T^{t,x}) + \int_t^T f(s, X_s^{t,x}, \underbrace{u(s, X_s^{t,x})}_{Y_s}, \underbrace{\nabla u V(s, X_s^{t,x})}_{Z_s}) ds - \underbrace{\nabla u V(s, X_s^{t,x})}_{Z_s} dW_s$$

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Our data is as follows :

- A time horizon $[0,T]$, a $d$-dimensional Brownian motion $\{W_t,\, 0 \leq t \leq T\}$ defined on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$.
- Vector fields $\{V_i\}_{i=0}^{d} \in C_b^{\infty}(\mathbb{R}^d;\mathbb{R}^d)$ to define a Stratonovich equation.
- A real valued function $f : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ at least Lipschitz continuous in $x, y, z$ and $\xi \in L^2(\mathcal{F}_T)$.

We will make the usual identification of every vector field with the first order operator :

$$V_i g = \sum_{j=1}^{d} V_i^j \frac{\partial g}{\partial x_j}$$

We then consider the following *decoupled* FBSDE:

$$\begin{aligned} X_s^{t,x} &= \int_t^s V_0(X_u^{t,x})du + \sum_{j=1}^d \int_t^s V_j(X_u^{t,x}) \circ dW_u^j \\ Y_s^{t,x} &= \xi + \int_s^T f(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x})du - \int_s^T Z_u^{t,x} dW_u \end{aligned} \tag{2}$$

Theorem (Pardoux & Peng [1990]): There exists a unique triple of adapted processes $(X, Y, Z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ satisfying (2) in the space of processes

$$\mathbb{E}\left[\sup_{0\leq t\leq T} Y_t^2 + \int_0^T |Z_s|^2 ds\right] < \infty$$

Step 1: Apply the martingale representation theorem to the r.v. $\xi + \int_0^T f(s, X_s, 0)ds$.

$$M_t := \mathbb{E}\left[\int_0^T f(s, X_s, 0)ds + \xi | \mathcal{F}_t\right] = \mathbb{E}\left[\int_0^T f(s, X_s, 0)ds + \xi\right] + \int_0^t Z_s^1 dW_s$$

If $Y_t^1 = M_t - \int_0^t f(s, X_s, 0)ds$ then

$$Y_t^1 = \xi + \int_t^T f(s, X_s, 0)ds - \int_t^T Z_s^1 dW_s$$

Step 2: Construct a sequence $(Y^n, Z^n)$ such that

$$Y_t^n = \xi + \int_t^T f(s, X_s, Y_s^{n-1}, Z_s^{n-1})ds - \int_t^T Z_s^n dW_s$$

Step 3: Show that $(Y, Z) := \lim_{n\to\infty}(Y^n, Z^n)$ satisfies

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$$

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When $\xi$ is simple , i.e. $\xi = \Phi(X_T)$ we have the following non linear Feynman Kac formula. Consider the PDE :

$$\begin{aligned} & u_t(t,x) + Lu + f(t,x,u(t,x), \nabla u(t,x)V(x)) = 0 \\ & u(T,x) = \Phi(x). \end{aligned} \tag{3}$$

where $L := V_0 + 1/2 \sum_{i=1}^{d} V_i^2$.

Theorem (Pardoux & Peng[1992]): If $u$ is the (viscosity) solution of (3), we have

$$Y_s^{t,x} = u(s, X_s^{t,x}), \quad Z_s^{t,x} = \nabla u(s, X_s^{t,x}) V(X_s^{t,x}), \quad a.s.$$

The representation for $Z$ holds, even if $u$ solves (3) in the viscosity sense, so long as $u \in C_b^1(\mathbb{R}^d)$.

Integrating the BSDE between $t$ and $T$ and using the Feynman -Kac formula we have:

$$Y_t^{t,x} = \mathbb{E}\left[\Phi(X_T^{t,x}) + \int_t^T f(s, X_s^{t,x}, u(s, X_s^{t,x}), \nabla u(s, X_s^{t,x}) V(X_s^{t,x}))ds\right]$$

In other words, there exists an implicitly defined functional $\Lambda_t : C_{\mathbb{R}^d}\left[t, T\right] \to \mathbb{R}$ such that

$$Y_t^{t,x} = \mathbb{E}\left[\Lambda_t\left(X^{t,x}\right)\right], \tag{4}$$

Hence, to approximate we have to :

- Replace the functional $\Lambda_t$ with an explicit version $\tilde{\Lambda}_t$
- Integrate $\tilde{\Lambda}_t(\cdot)$ with respect to an approximation of the law of $X_\cdot^{t,x}$.

Descritization along the lines of the algorithms by Zhang(2004), Bouchard & Touzi (2004):

$$Y_{T-\epsilon}^{T-\epsilon,x} = \mathbb{E}\left[\Phi(X_T^{t-\epsilon,x}) + \int_{t-\epsilon}^{T} f(s, X_s^{t-\epsilon,x}, u(s, X_s^{t-\epsilon,x}), \nabla u V(s, X_s^{t-\epsilon,x}) ds\right]$$

$$\begin{aligned} \to Y_{T-\epsilon}^{T-\epsilon,x} \simeq \mathbb{E}&\left[\Phi(X_T^{T-\epsilon,x})\right] \\ &+ \epsilon f(T-\epsilon, x, \underbrace{u(T-\epsilon, x)}_{Y_{T-\epsilon}}, \nabla u(T-\epsilon, x) V(x)) \end{aligned}$$

However, a Stratonovich Taylor expansion tells us

$$\begin{aligned} u(T, X_T^{T-\epsilon,x}) &= u(T-\epsilon, x) + \sum_{i=1}^{d} V_i u(T-\epsilon, x) \int_{T-\epsilon}^{T} \circ dW_s^i \\ &+ \sum_{i,j=1}^{d} V_i V_j u(T-\epsilon, x) \int_{T-\epsilon}^{T} \int_{T-\epsilon}^{s} \circ dW_u^j \circ dW_s^i + O(\epsilon^{3/2}) \end{aligned}$$

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Multiplying by $\Delta W^i$ and integrating, we have

$$\mathbb{E}\left[\Phi(X_T^{T-\epsilon,x})(W_T - W_{T-\epsilon})\right] = \epsilon\nabla u(t-\epsilon,x)V(x) + O(\epsilon^2)$$

Hence,

$$Y_{T-\epsilon}^{T-\epsilon,x} \simeq \mathbb{E}\left[\Phi(X_T^{T-\epsilon,x})\right] + \epsilon f\left(T-\epsilon, x, \underbrace{u(T-\epsilon,x)}_{Y_{T-\epsilon}}, \frac{1}{\epsilon}\mathbb{E}[\underbrace{\Phi(X_T^{T-\epsilon})}_{Y_T}\Delta W_T]\right)$$

Given a partition $\pi := \{0 = t_0 < t_1 < \ldots < t_n = T\}$, we define the one-step operator

$$R_i g(x) = \mathbb{E}\left[g(X_{t_{i+1}}^{t_i,x})\right] + h_{i+1} f\left(t_i, x, R_i g(x), \frac{1}{h_{i+1}}\mathbb{E}\left[g(X_{t_{i+1}}^{t_i,x})\Delta W_{i+1}\right]\right),$$

$$h_i = t_i - t_{i-1}, \quad \Delta W_i = W_{t_i} - W_{t_{i-1}}$$

The approximation of $Y_{t_i}$ is $Y_{t_i}^{\pi} ::= R_i \ldots R_n \Phi(X_{t_i}^{0,x})$

$$\sup_{0\le t\le T} \mathbb{E}\left[|Y_t - Y_{t_i}^{\pi} 1_{[t_i,t_{i+1})}(t)|^2\right] \le C|\pi|$$

Multiplying by $\Delta W^i$ and integrating, we have

$$\mathbb{E}\left[\Phi(X_T^{T-\epsilon,x})(W_T - W_{T-\epsilon})\right] = \epsilon\nabla u(t-\epsilon,x)V(x) + O(\epsilon^2)$$

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$$R_i g(x) = \mathbb{E}\left[g(X_{t_{i+1}}^{t_i,x})\right] + h_{i+1} f\left(t_i, x, R_i g(x), \frac{1}{h_{i+1}}\mathbb{E}\left[g(X_{t_{i+1}}^{t_i,x})\Delta W_{i+1}\right]\right),$$

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$$\sup_{0\le t\le T} \mathbb{E}\left[|Y_t - Y_{t_i}^{\pi} 1_{[t_i,t_{i+1})}(t)|^2\right] \le C|\pi|$$

The involved expectations are not computable in most cases. If $\hat{\mathbb{E}}[\cdot]$ denotes integration with an approximation of the law of $X_{\cdot}^{0,x}$, we set $\{\hat{R}_i\}_i$ :

$$\begin{aligned}\hat{R}_i g(x) &= \hat{\mathbb{E}}\left[g(X_{t_{i+1}}(t_i,x))\right] \\ &\qquad + h_{i+1} f\left(t_i, x, \hat{R}_i g(x), \frac{1}{h_{i+1}} \hat{\mathbb{E}}\left[g(X_{t_{i+1}}(t_i,x))\Delta W_{i+1}\right]\right). \\ \hat{Y}_{t_i} &= \hat{R}_i \ldots \hat{R}_n \Phi(X_{t_i})\end{aligned}$$

Existing methods for $\hat{\mathbb{E}}[\cdot]$:

- Approximation of the density with Voronoi tesselations, Bally& Pages [2001]
- Malliavin representation for regression functions, Bouchard & Touzi [2004]
- Projection on function bases, Gobet,Lemor,Warin [2005]
- The cubature method

Assume that we have a finite measure $\mu$ on $\left(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\right)$. A cubature formula of order $m$ on Euclidean space is a set of points $\{x_i\}_{i=1}^N \subset \mathbb{R}^d$ and a set of non negative numbers $\lambda_i$, $i = 1, \ldots, N$ such that

$$\int_{\mathbb{R}^d} x^k \mu(dx) = \sum_{i=1}^N \lambda_i x_i^k, \quad \forall k \leq m$$

Hence for any smooth function $f : \mathbb{R}^d \to \mathbb{R}$ we have

$$\int_{\mathbb{R}^d} f(x) \mu(dx) \simeq \sum_{i=1}^N \lambda_i f(x_i)$$

The cubature method of Lyons and Victoir realises the same idea on the Wiener space.

Given a nice function $f$, the Sratonovich-Taylor expansion tells us

$$f(X_t^{0,x}) = \sum_{(i_1,\ldots,i_k)\in\mathcal{A}_m} V_{i_1}\ldots V_{i_k} f(x) \int_{0<t_1<\ldots<t_k<t} \circ dW_{t_1}^{i_1}\ldots\circ dW_{t_k}^{i_k} \\ + R_m(t,x,f)$$

with $\|R_m(t,x,f)\|_p = O(t^{\frac{m+1}{2}})$. Let $\mathbb{Q}$ be another measure on $(C([0,t]),\mathcal{B}(C([0,t])))$

If $\mathbb{E}^{\mathbb{Q}}[|R_m(t,x,f)|^p]$ is small and

$$\left(\mathbb{E}^{\mathbb{Q}}-\mathbb{E}\right)\left[\int_{0<t_1<\ldots<t_k<t} \circ dW_{t_1}^{i_1}\ldots\circ dW_{t_k}^{i_k}\right] = 0$$

$$\to \mathbb{E}^{\mathbb{Q}}\left[f(X_t^{0,x})\right] \simeq \mathbb{E}\left[f(X_t^{0,x})\right]$$

### Theorem (Lyons & Victoir (2004), Litterer& Lyons (2008))

*For any* $t > 0$*, there exists paths* $\omega_1, \ldots, \omega_N \in C^0_{0,bv}([0,t];\mathbb{R}^d)$ *and* $\lambda_1, \lambda_2, \ldots, \lambda_N$ *such that*

$$\mathbb{E}\Big(\int_{0<t_1<\ldots<t_k<t} \circ dW^{i_1}_{t_1} \ldots \circ dW^{i_k}_{t_k}\Big) = \sum_{j=1}^{N} \lambda_j \int_{0<t_1<\ldots<t_k<t} d\omega^{i_1}_j(t_1) \ldots d\omega^{i_k}_j(t_k)$$

*If the above is true up to m iterated integrals, we have a cubature formula of order m defined on* $[0,t]$*.*

This new measure, called cubature measure, will be denoted by $\mathbb{Q}^m_t$, i.e $\mathbb{Q}^m_t := \sum_{i=1}^{N} \lambda_i \delta_{\omega_i}$

A bit of notation : Given a multi index $I = (i_1, \ldots, i_k) \in \{0, \ldots, d\}^k$ we write $V_I f := V_{i_1} \ldots V_{i_k} f$ and consider the norms
$|I| = k, \quad \|I\| = k + \#\{1 \leq j \leq k : i_j = 0\}$.

### Theorem (Lyons & Victoir (2004))

*If* $\mathbb{Q}_t^m$ *is a cubature measure of order m defined on* $[0, t]$ *then*

$$\sup_{x \in \mathbb{R}^d} |\mathbb{E}\left[f(X_t^{0,x})\right] - \mathbb{E}_{\mathbb{Q}^m}\left[f(X_t^{0,x})\right]| \leq \sum_{j=m+1}^{m+2} t^{j/2} \sup_{\|I\|=j} \|V_I f\|_\infty$$

Let $\omega \in C_{0,bv}([0,T];\mathbb{R}^d)$. What is $\mathbb{E}^{\delta_\omega}[f(X_t^{0,x})]$?

$$\begin{aligned}\mathbb{E}^{\delta_\omega}[f(X_t^{0,x})] &= f\left(x + \int_0^t V_0(X_s(\omega))ds + \sum_{i=1}^d \int_0^t V_i(X_s(\omega)) \circ dW_s^i(\omega)\right)\\ &= f\left(x + \int_0^t V_0(X_s(\omega))ds + \sum_{i=1}^d \int_0^t V_i(X_s(\omega))d\omega^i(s)\right)\end{aligned}$$

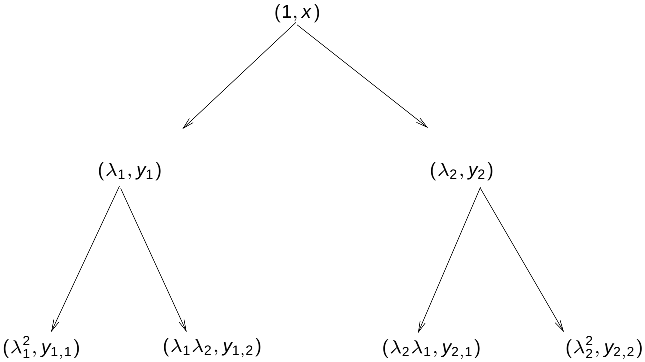
where $X_s(\omega)$ solves the ODE

$$dX_s(\omega) = \sum_{i=0}^d V_i(X_s(\omega))d\omega^i(s).$$

Let us denote by $\Xi_{T,x}(\omega)$ the solution at time $T$ of the ODE $dy_{t,x} = \sum_{i=0}^d V_i(y_{t,x})d\omega^i(t),\ y_0 = x$. Then

$$\mathbb{E}_{\mathbb{Q}^m_{h_{i+1}}}[f(X_{t_{i+1}}^{t_i,x})] = \sum_{j=1}^N \lambda_j \Xi_{h_{i+1},x}(\omega_j).$$

## Cubature with two paths and two time steps



where $y_i = \Xi_{\delta_1,x}(\omega_{\delta_1,i}), \quad y_{i,j} = \Xi_{\delta_2,\Xi_{\delta_1,x}(\omega_{\delta_1,i})}(\omega_{\delta_2,j}), \ i,j = 1,2.$

We focus on $|Y_0 - \hat{Y}_0|$. With a bit of work one can show :

$$|Y_0 - \hat{Y}_0| \simeq \sum_{i=0}^{n-1} |R_i \dots R_n \Phi(X_{t_i}^{0,x}) - \hat{R}_i \dots \hat{R}_n \Phi(X_{t_i}^{0,x})|$$

With a bit more work and involving the Feynman-Kac formula

$$\begin{aligned}|Y_0 - \hat{Y}_0| \simeq O(|\pi|) + \sum_{i=1}^{n} \Big( &|(\mathbb{E} - \mathbb{E}_{\mathbb{Q}^m})[u(t_{i+1}, X_{t_{i+1}}^{t_i,x})]| \\ &+ |(\mathbb{E} - \mathbb{E}_{\mathbb{Q}^m})[u(t_{i+1}, X_{t_{i+1}}^{t_i,x})\Delta W_{i+1}]| \Big)\end{aligned}$$

$$\begin{aligned}|Y_0 - \hat{Y}_0| \simeq O(|\pi|) + \sum_{i=1}^{n} \Bigg( &\sum_{j=3}^{4} h_{i+1}^{(j+1)/2} \sup_{\|I\|=j} \|V_I u(t_i, \cdot)\|_\infty \\ &+ \sum_{j=m+1}^{m+2} h_{i+1}^{j/2} \sup_{\|I\|=j} \|V_I u(t_i, \cdot)\|_\infty \Bigg)\end{aligned}$$

We focus on $|Y_0 - \hat{Y}_0|$. With a bit of work one can show :

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With a bit more work and involving the Feynman-Kac formula

$$\begin{aligned} |Y_0 - \hat{Y}_0| \simeq O(|\pi|) + \sum_{i=1}^{n} \Big( & |(\mathbb{E} - \mathbb{E}_{\mathbb{Q}^m})[u(t_{i+1}, X_{t_{i+1}}^{t_i,x})]| \\ & + |\,(\mathbb{E} - \mathbb{E}_{\mathbb{Q}^m})\,[u(t_{i+1}, X_{t_{i+1}}^{t_i,x})\Delta W_{i+1}]|\Big) \end{aligned}$$

$$\begin{aligned} |Y_0 - \hat{Y}_0| \simeq O(|\pi|) + \sum_{i=1}^{n} \Bigg( & \sum_{j=3}^{4} h_{i+1}^{(j+1)/2} \sup_{\|I\|=j} \|V_I u(t_i, \cdot)\|_\infty \\ & + \sum_{j=m+1}^{m+2} h_{i+1}^{j/2} \sup_{\|I\|=j} \|V_I u(t_i, \cdot)\|_\infty \Bigg) \end{aligned}$$

The bounds depend on the behaviour of the derivatives of $u$. If $\Phi \in C_b^{m+2}$ :

$$\|D^I u\|_\infty < \infty \left(\sim \|D^I \Phi\|_\infty\right) \Rightarrow |Y_0 - \hat{Y}_0| \leq C|\pi|$$

If $\Phi$ only Lipschitz continuous : When $f = 0$ we fall back to the linear PDE case. Then

$$\sup_x |V_I u(t_i, x)| \leq \frac{C}{(T - t_i)^{(\|I\|-1)/2}}$$

Hence

$$|Y_0 - \hat{Y}_0| \simeq \sqrt{h_n} + \sum_{i=1}^{n} \sum_{j=m+1}^{m+2} \frac{h_i^{j/2}}{(T - t_j)^{(j-1)/2}}$$

Aim is to use a partition which gets more dense as we appproach $T$. For example (Kusuoka[2003]) $t_i = T(1 - \left(1 - \frac{i}{k}\right)^\gamma)$.

When $\Phi$ is only Lipschitz and $f \neq 0$ no equivalent characterization of the derivatives of $u$ is available. The trick is to take an $\epsilon$ step back from $T$ projecting $\Phi$.

$$\tilde{\Phi}(x) = \mathbb{E}[\Phi(X_T)|X_{T-\epsilon} = x]$$

and work with the solution of the PDE

$$\tilde{u}_t + L\tilde{u} + f(t, x\tilde{u}, \nabla\tilde{u}V) = 0, \quad \tilde{u}(T-\epsilon, x) = \Phi(x)$$

However in this case $\|D^I u\|_\infty \simeq \|D^I\tilde{\Phi}\|_\infty \simeq h_n^{(1-\|I\|)/2}$. Hence,

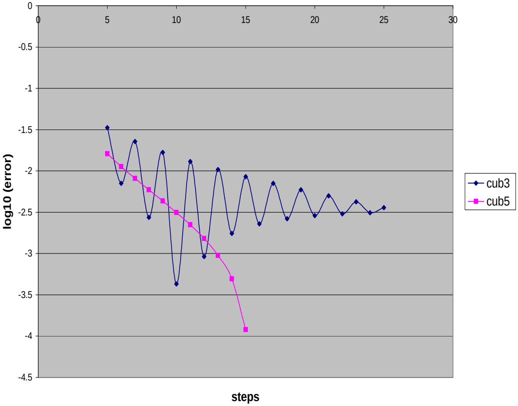
$$|Y_0 - \hat{Y}_0| \simeq \begin{cases} h_n^{1/2} + \sum_{i=1}^{n-1} h_i^2 h_n^{-3/2} & m = 3 \\ h_n^{1/2} + \sum_{i=1}^{n-1} h_i^2 h_n^{-1} & m \geq 5 \end{cases}$$

Need to keep the partition dense enough throughout with $h_i < h_n$.

## Pricing a call option in a Black Scholes economy.

$\Phi(x) = (x - K)_+$, $f(t,x,y,z) = -ry - \theta\sigma z$, $\theta = \frac{\mu - r}{\sigma}$

| | |
|---|---|
| $T$ | 1. |
| $\mu$ | 0.03 |
| $r$ | 0.06 |
| $\sigma$ | 0.2 |
| $X_0$ | 10 |
| $K$ | 10 |



## Pricing a call option with different interest rates:

$$\Phi(x) = (x - K)_+,\ f(t,x,y,z) = -\{ry + \theta z + (R - r)(y - z/\sigma)_-\},\ \theta := \frac{\mu - r}{\sigma}$$

| | |
|---|---|
| $T$ | 1. |
| $\mu$ | 0.05 |
| $r$ | 0.06 |
| $R$ | 0.08 |
| $\sigma$ | 0.2 |
| $X_0$ | 10 |
| $K$ | 10 |