Quantitative Methods for Counterparty Risk

Analytical Formulation and Results

Alexander Lipton and Artur Sepp

Bank of America Merrill Lynch

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Plan

- A brief history of credit modeling.

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- How to model CDSs?

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Acknowledgements

- We am also grateful to other Merrill Lynch colleagues, especially S. Inglis, A. Rennie, D. Shelton, I. Savescu, and members of the Credit Analytics Team and Structured Credit Team for all their help.

References

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- **Counterparty risk:** Blanchet & Patras (2008), Brigo & Chourdakis (2008), Crepey, Jeanblanc & Zargari (2009), Leung & Kwok (2005), Haworth, Reisisnger & Shaw (2006), Hull & White (2000), (2001), Misisrpashayev (2008), Turnbull (2005), Valuzis (2008).

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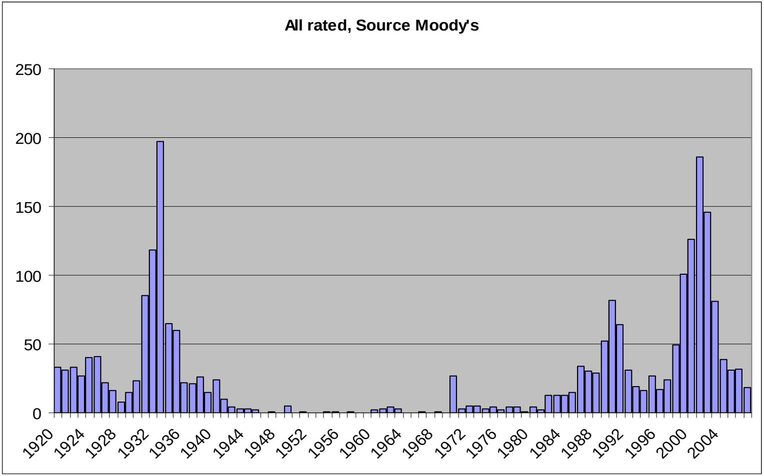
- **Numerical methods:** Andersen & Andreasen (2000), Boyarchenko & Levendorsky (2000), Cont & Tankov (2004), Cont & Voltchkova (2005), Clift & Forsyth (2008), d'Halluin, Forsyth & Vetzal (2005), Feng & Linetsky (2008), Lipton (2003).

Moody's number of defaults figure



Moody's table of recoveries

Year	Sr. Secured Bank Loans	Sr. Secured Bonds	Source: Sr. Unsecured Bonds	Moody's Sr. Subordina Bonds	Subordinated Bonds	Jr. Subord Bonds	All Bonds
1982	NA	$72.50	$35.79	$48.09	$29.99	NA	$35.57
1983	NA	$40.00	$52.72	$43.50	$40.54	NA	$43.64
1984	NA	NA	$49.41	$67.88	$44.26	NA	$45.49
1985	NA	$83.63	$60.16	$30.88	$39.42	$48.50	$43.66
1986	NA	$59.22	$52.60	$50.16	$42.58	NA	$48.38
1987	NA	$71.00	$62.73	$44.81	$46.89	NA	$50.48
1988	NA	$55.40	$45.24	$33.41	$33.77	$36.50	$38.98
1989	NA	$46.54	$43.81	$34.57	$26.36	$16.85	$32.31
1990	$75.25	$33.81	$37.01	$25.64	$19.09	$10.70	$25.50
1991	$74.67	$48.39	$36.66	$41.82	$24.42	$7.79	$35.53
1992	$61.13	$62.05	$49.19	$49.40	$38.04	$13.50	$45.89
1993	$53.40	NA	$37.13	$51.91	$44.15	NA	$43.08
1994	$67.59	$69.25	$53.73	$29.61	$38.23	NA	$45.57
1995	$75.44	$62.02	$47.60	$34.30	$41.54	NA	$43.28
1996	$88.23	$47.58	$62.75	$43.75	$22.60	NA	$41.54
1997	$78.75	$75.50	$56.10	$44.73	$35.96	$30.58	$49.39
1998	$51.40	$48.14	$41.63	$44.99	$18.19	$62.00	$39.65
1999	$75.82	$43.00	$38.04	$28.01	$35.64	NA	$34.33
2000	$68.32	$39.23	$23.81	$20.75	$31.86	$15.50	$25.18
2001	$66.16	$37.98	$21.45	$19.82	$15.94	$47.00	$22.21
2002	$58.80	$48.37	$29.69	$23.21	$24.51	NA	$30.18
2003	$73.43	$63.46	$41.87	$37.27	$12.31	NA	$40.69
2004	$87.74	$73.25	$54.25	$46.54	$94.00	NA	$59.12
2005	$82.07	$71.93	$54.88	$26.06	$51.25	NA	$55.97
2006	$76.02	$74.63	$55.02	$41.41	$56.11	NA	$55.02
2007	$67.74	$80.54	$51.02	$54.47	NA	NA	$53.53

CDS figure

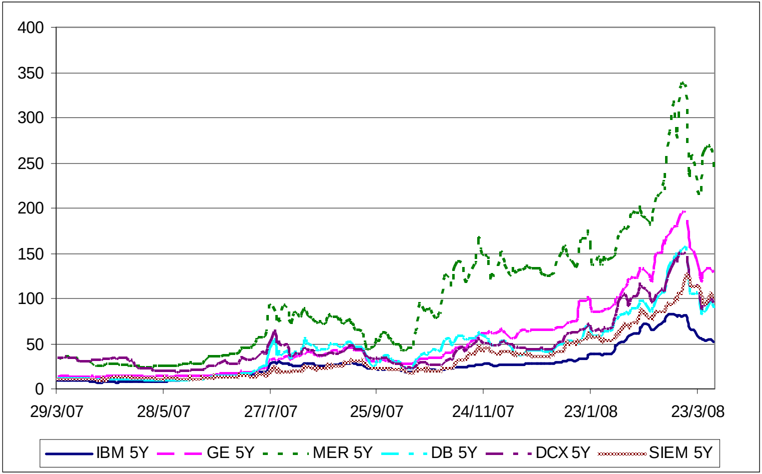


Figure: Time series of 5Y CDS spreads (in bps) for 6 typical companies (IBM, GE, Merrill Lynch, Deutsche Bank, Daimler, and Siemens). Source: Merrill Lynch.

Recent trends

- The volume of CDSs grew by a factor of 100 between 2001 and 2007.

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- Lately, CDS dealers reduced their market by 38% in one year.
- According to the most recent ISDA survery published on April 22nd, the notional value of CDSs outstanding decreased to $38.6 trillion as of Dec 31st 2008 from $54.6 trillion mid-year and $62.2 trillion as of Dec 31st 2007.

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- Then we need to extend our theory to cover indices, tranches, baskets, etc.
- Three complementary approaches to pricing CDSs:
  - reduced form,
  - structural,
  - hybrid
- All of them have pros and cons. Reduced form and structural approaches dominate.

Simple non-risky model

- First, we consider non-risky bonds and introduce the short interest rate $r(t)$ which is driven by the SDE:

$$dr(t) = f_1(t, r(t))\, dt + g_1(t, r(t))\, dW_1(t) \qquad r(0) = r_0$$

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- Let $D(t, T)$ be time t price of the non-risky zero-coupon bond maturing at time T. This price can be written as follows:

$$D(t, T) = V_1(t, r(t)) \qquad D(0, T) = V_1(0, r_0)$$

where $V_1(t, r)$ is the solution of the following backward Kolmogorov problem

$$V_{1,t} + \tfrac{1}{2} g_1^2 V_{1,rr} + f_1 V_{1,r} - r V_1 = 0 \qquad V_1(T, r) = 1$$

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- Functions f_1, g_1 are calibrated to the market to match the yield curve and some swaption volatilities. Alternatively,

$$D(t, T) = \mathbb{E}_t\left\{\exp\left(-\int_t^T r(t')\,dt'\right)\right\} = \exp\left(-\int_t^T \hat{r}(t, t')\,dt'\right)$$

where $\hat{r}$ is the forward rate.

Reduced form model formulation

- The name defaults at the first time a Cox process jumps from 0 to 1. SDE for the default intensity (hazard rate) $X(t)$ is:

$$dX(t) = f(t, X(t))\,dt + g(t, X(t))\,dW(t) + JdN(t), \qquad X(0) = X_0$$

where $W(t)$ is a standard Wiener processes, $N(t)$ is a Poisson process with intensity $\lambda(t)$, and J is a positive jump distribution; W, N, J are mutually independent.

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$$f(t, 0) \geq 0, \quad f(t, \infty) < 0, \quad g(t, 0) = 0$$

- For analytical convenience (rather than for deeper reasons) it is customary to assume that X is governed by the square-root stochastic differential equation (SDE):

$$dX(t) = \kappa(\theta(t) - X(t))\,dt + \sigma\sqrt{X(t)}dW(t) + JdN(t),\ X(0) = X_0$$

  with exponential (or hyper-exponential) jump distribution.

Survival probability

- For practical purposes it is more convenient to consider discrete jump distributions with jump values $J_m > 0$, $1 \leq m \leq M$, occurring with probabilities $\pi_m > 0$; such distributions are more flexible than parametric ones because they allow one to place jumps where they are needed.

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- In this framework, the survival probability of the name from time 0 to time T has the form

$$q(0, T) = \mathbb{E}_0 \left\{ e^{-\int_0^T X(t') dt'} \right\} = \mathbb{E}_0 \left\{ e^{-Y(T)} \right\}$$

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- More generally, the survival probability from time t to time T conditional on no default before time t has the form

$$q(t, T | X(t), Y(t)) = e^{Y(t)} \mathbb{I}_{(\tau > t)} \mathbb{E}_t \left\{ e^{-Y(T)} \middle| X(t), Y(t) \right\}$$

Calculation of expectations

- Expectations of the form $\mathbb{E}_t\left\{ e^{-\xi Y(T)} \middle| X(t), Y(t) \right\}$, can be computed by solving the following augmented partial differential equation (PDE)

$$(\partial_t + \mathcal{L}) V(t, T, X, Y) + X V_Y (t, T, X, Y) = 0$$

$$V(T, T, X, Y) = e^{-\xi Y}$$

where

$$\mathcal{L}V \equiv \kappa(\theta(t) - X) V_X + \frac{1}{2}\sigma^2 X V_{XX} + \lambda \sum_m \pi_m [V(X + J_m) - V(X)]$$

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- Specifically, the following relation holds

$$\mathbb{E}_t\left\{ e^{-\xi Y(T)} \middle| X(t), Y(t) \right\} = V(t, T, X(t), Y(t))$$

Affine ansatz

- The corresponding solution can be written in the so-called affine form:

$$V(t, T, X, Y) = e^{a(t,T,\xi)+b(t,T,\xi)X-\xi Y}$$

where a, b are functions of time governed by the following system of ordinary differential equations (ODEs):

$$\begin{cases} \frac{da(t,T,\xi)}{dt} = -\kappa\theta(t)\, b(t, T, \xi) - \lambda \sum_m \pi_m \left[e^{J_m b(t,T,\xi)} - 1 \right] \\ \frac{db(t,T,\xi)}{dt} = \xi + \kappa b(t, T, \xi) - \frac{1}{2}\sigma^2 b^2(t, T, \xi) \\ a(T, T, \xi) = 0, \qquad b(T, T, \xi) = 0 \end{cases}$$

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$$a(T, T, \xi) = 0, \qquad b(T, T, \xi) = 0$$

- This system cannot be solved analytically, it is very easy to solve it numerically. The survival probability $q(0, T)$ and default probability $p(0, T)$ have the form

$$\begin{aligned} q(0, T) &= e^{a(0,T,1)+b(0,T,1)X_0}, \\ p(0, T) &= 1 - q(0, T) = 1 - e^{a(0,T,1)+b(0,T,1)X_0} \end{aligned}$$

CDS valuation 1

- The value U of a credit default swap (CDS) paying an up-front amount υ and a coupon s in exchange for receiving $1 - R$ (where R is the default recovery) on default as follows:

$$U = -\upsilon + V\left(0, X_0\right)$$

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$$V\left(T, X\right) = 0$$

- Using Duhamel's principle, we obtain the following expression for V:

$$\begin{aligned} V\left(t, X\right) &= -s \int_t^T D\left(t, t'\right) e^{a(t,t',1)+b(t,t',1)X} dt' \\ &\quad - \left(1 - R\right) \int_t^T D\left(t, t'\right) d\left[e^{a(t,t',1)+b(t,t',1)X}\right] \end{aligned}$$

where $D\left(t, t'\right)$ is the discount factor between two times t, t'.

CDS valuation 2

- Accordingly,

$$U = -\upsilon - s \int_0^T D\left(0, t'\right)\left(1 - p\left(0, t'\right)\right) dt' + (1 - R) \int_0^T D\left(0, t'\right) dp\,($$

For a given up-front payment υ, we can represent the corresponding par spread $\hat{s}$ (i.e. the spread which makes the value of the corresponding CDS zero) as follows:

$$\hat{s}\left(T\right) = \frac{-\upsilon + (1 - R) \int_0^T D\left(0, t'\right) dp\left(0, t'\right)}{\int_0^T D\left(0, t'\right)\left(1 - p\left(0, t'\right)\right) dt'}$$

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- Conversely, for a given spread we can represent the par up-front payment in the form

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- In these formulas we implicitly assumed that the corresponding CDS is fully collateralized, so that in the event of default $1 - R$ is readily available.

Two name correlation 1

- It is very tempting to extend the above framework to cover several correlated names. For example, consider two credits, A, B and assume for simplicity that their default intensities coincide,

$$X_A(t) = X_B(t) = X(t)$$

and both names have the same recovery $R_A = R_B = R$.

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- For a given maturity T, the default event correlation ρ is defined as follows

$$\rho(0,T) = \frac{p_{AB}(0,T) - p_A(0,T)\,p_B(0,T)}{\sqrt{p_A(0,T)(1-p_A(0,T))\,p_B(0,T)(1-p_B(0,T))}}$$

where τ_A, τ_B are the default times, and

$$\begin{aligned} p_A(0,T) &= P(\tau_A \le T), \qquad p_B(0,T) = P(\tau_B \le T) \\ p_{AB}(0,T) &= P(\tau_A \le T, \tau_B \le T) \end{aligned}$$

Two name correlation 2

- It is clear that

$$p_A(0,T) = p_B(0,T) = p(0,T) = 1 - e^{a(0,T,1)+b(0,T,1)X_0}$$

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- Simple calculation yields

$$p_{AB}(0,T) = E_0\left\{e^{-\int_0^T (X_A(t')+X_B(t'))dt'}\right\} + p_A(0,T) + p_B(0,T) - 1$$

$$= \mathbb{E}_0\left\{e^{-2\int_0^T X(t')dt'}\right\} + 2p(0,T) - 1$$

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- Thus

$$\rho(0,T) = \frac{e^{a(0,T,2)+b(0,T,2)X_0} - e^{2a(0,T,1)+2b(0,T,1)X_0}}{\left(1 - e^{a(0,T,1)+b(0,T,1)X_0}\right) e^{a(0,T,1)+b(0,T,1)X_0}}$$

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$$p_A(0,T) = p_B(0,T) = p(0,T) = 1 - e^{a(0,T,1)+b(0,T,1)X_0}$$

- Simple calculation yields

$$p_{AB}(0,T) = E_0\left\{e^{-\int_0^T (X_A(t')+X_B(t'))dt'}\right\} + p_A(0,T) + p_B(0,T) - 1$$
$$= \mathbb{E}_0\left\{e^{-2\int_0^T X(t')dt'}\right\} + 2p(0,T) - 1$$

- Thus

$$\rho(0,T) = \frac{e^{a(0,T,2)+b(0,T,2)X_0} - e^{2a(0,T,1)+2b(0,T,1)X_0}}{\left(1 - e^{a(0,T,1)+b(0,T,1)X_0}\right) e^{a(0,T,1)+b(0,T,1)X_0}}$$

- In the absence of jumps, the corresponding event correlation is very low. If large positive jumps are added (while overall survival probability is preserved), then correlation can increase all the way to one. Assuming that $T = 5y$, $\kappa = 0.5$, $\sigma = 7\%$, and $J = 5.0$, we illustrate this observation below.

Event correlation Figure

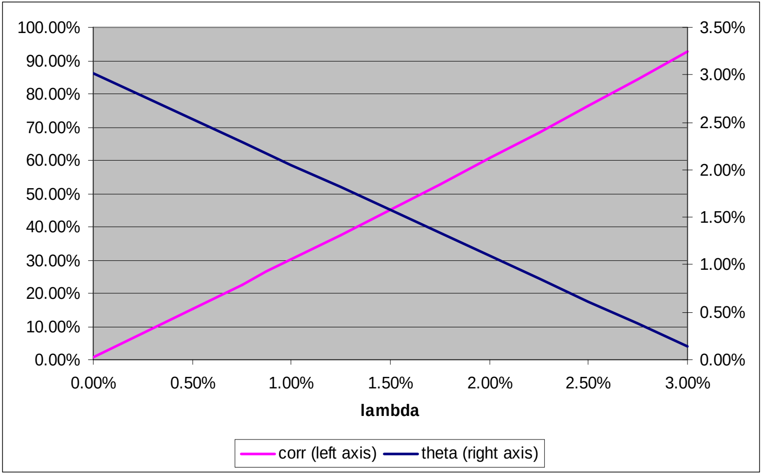

Figure: Correlation ρ and mean-reversion level $\theta = X_0$ as functions of jump intensity λ. Other parameters are as follows: $T = 5y$, $\kappa = 0.5$, $\sigma = 7\%$, and $J = 5.0$.

FTD-STD

- In the two-name portfolio, we can define two types of CDSs which depend on the correlation: (A) the first-to-default swap (FTD); (B) the second-to-default swap (STD). The corresponding par spreads (assuming that there are no up-front payments) are

$$\hat{s}_1(T) = \frac{(1-R)\int_0^T D(0,t')\, d\left[1 - e^{a(0,t',2)+b(0,t',2)X_0}\right]}{\int_0^T D(0,t')\, e^{a(0,t',2)+b(0,t',2)X_0} dt'}$$

$$\hat{s}_2(T) = \frac{(1-R)\int_0^T D(0,t')\, d\left[1 - \left(2e^{a(t',1)+b(t',1)X_0} - e^{a(t',2)+b(t',2)X_{0}}\right.\right.}{\int_0^T D(0,t')\left(2e^{a(t',1)+b(t',1)X_0} - e^{a(t',2)+b(t',2)X_0}\right) dt'}$$

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- It is clear that the relative values of $\hat{s}_1, \hat{s}_2$ very strongly depend on whether or not jumps are present in the model.

FTD-STD Figure

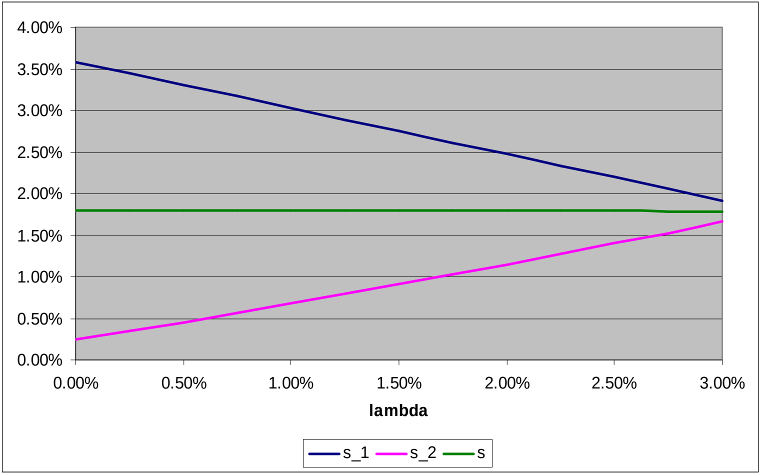

Figure: FTD spread $\hat{s}_1$, STD spread $\hat{s}_2$, and single name CDS spread $\hat{s}$ as functions of jump intensity λ. Other parameters are the same as in Fig.1. It is clear that jumps are necessary to have $\hat{s}_1$ and $\hat{s}_2$ of similar magnitudes.

Counter-party effects

- An important application of the above model is to the evaluation of counter-party effects on fair CDS spreads. Let us assume that name A has written a CDS on reference name B. It is clear that the pricing problem for the value of the uncollateralized CDS $\tilde{V}$ can be written as follows

$$\mathcal{L}\tilde{V}(t,X) - (r+2X)\tilde{V}(t,X)$$
$$= s - (1-R)X - (RV_{+}(t,X) + V_{-}(t,X))X$$

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- It is clear that the discount rate is increased from $r+X$ to $r+2X$, since there are two cases when the uncollateralized CDS can be terminated due to default: when the reference name B defaults; when the issuer A defaults.
- Although this equation is no longer analytically solvable, it can be solved numerically via, say, an appropriate modification of the classical Crank-Nicholson method. It turns out that in the presence of jumps the value of the fair par spread goes down dramatically.

Counter-party Figure

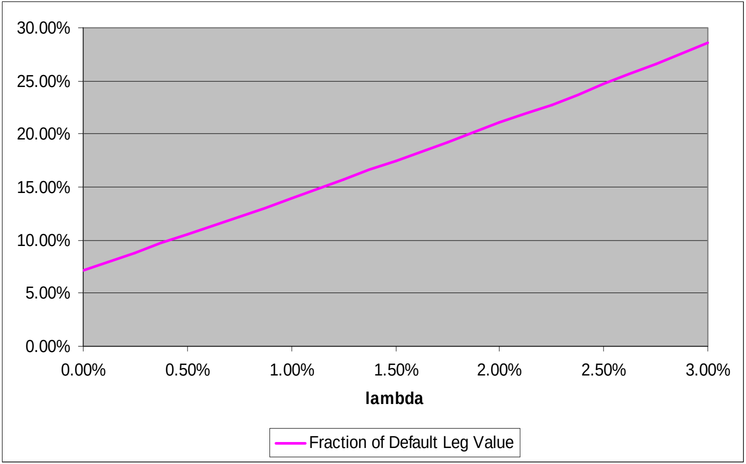

Figure: Reduction in default leg value as function of jump intensity λ. Mean-reversion level $\theta = X_0$ is chosen in order to preserve survival probability. Other parameters are as follows: $T = 5y$, $\kappa = 0.5$, $\sigma = 7\%$, and $J = 5.0$.

Typical structural model without jumps formulation

- A typical structural model without jumps for the evolution of the log-value of the firm has the form:

$$dx = \mu dt + \sigma dW(t), \qquad x(0) = x_0$$

$$x(t) = \ln\left(\frac{v(t)}{B(t)}\right) = \ln\left(\frac{v(t)}{B(0)\exp((r-d)t)}\right)$$

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- This value is governed by a Wiener process. The firm defaults if the value $x(t)$ crosses zero barrier $b(t) = 0$.

Survival probability

- Assuming that all the relevant parameters are constant, the corresponding formula for $Q(t, T)$ is

$$Q(t, T) = I(0, t)\left(\mathbf{N}\left(\frac{x(t) - \frac{1}{2}\sigma^2\tau}{\sqrt{\sigma^2\tau}}\right) - e^{x(t)}\mathbf{N}\left(\frac{-x(t) - \frac{1}{2}\sigma^2\tau}{\sqrt{\sigma^2\tau}}\right)\right)$$

where $\mathbf{N}$ is the cumulative normal distribution. It is easy to verify that

$$Q(0,0) = 1, \ \ Q_T(0,0) = 0$$

The latter fact is rather disconcerting, since it implies that the par short term CDS spread vanishes when $T \to 0$.

Typical structural model with jumps formulation

- A typical structural model with jumps for the evolution of the log-value of the firm has the form:

$$dx = \mu dt + \sigma dW(t) + j_+ dN_+ + j_- dN_-, \qquad x(0) = x_0$$

$$x = \ln\left(\frac{v}{B}\right)$$

$$\mu = -\frac{\sigma^2}{2} - \frac{\lambda_+}{\alpha_+ - 1} + \frac{\lambda_-}{\alpha_- + 1}$$

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- This value is governed by a combination of a Wiener process and a Poisson process with exponentially distributed jumps. The firm defaults if the value $x(t)$ crosses zero barrier $b(t) = 0$. It was realized early on that without jumps (or/and curvilinear or uncertain barriers) it is impossible to explain the short end of the CDS curve within the structural framework. When all the relevant parameters are constant, the problem can be solved analytically via the Laplace transform. However, in general this approach does not work.

Unbounded PDF 1

- When the barrier is absent, Φ can be found via a simple recursion (knowing an analytical solution is useful for benchmarking purposes). Vis-a-vis the Gaussian distribution, Φ has fat tails and a narrow peak. Let

$$\begin{aligned}
\sigma_{\pm} &= \alpha_{\pm}\sigma\sqrt{t} \\
\theta &= \left(x-\xi-\mu t\right)/\sigma\sqrt{t} \\
\Phi_{0,0}\left(x\right) &= \mathbf{n}\left(\theta\right)/\sigma\sqrt{t} \\
\Phi_{1,0}\left(x\right) &= \alpha_{+}\mathbf{P}\left(\theta,-\sigma_{+}\right) \\
\Phi_{0,1}\left(x\right) &= \alpha_{-}\mathbf{P}\left(-\theta,-\sigma_{-}\right)
\end{aligned}$$

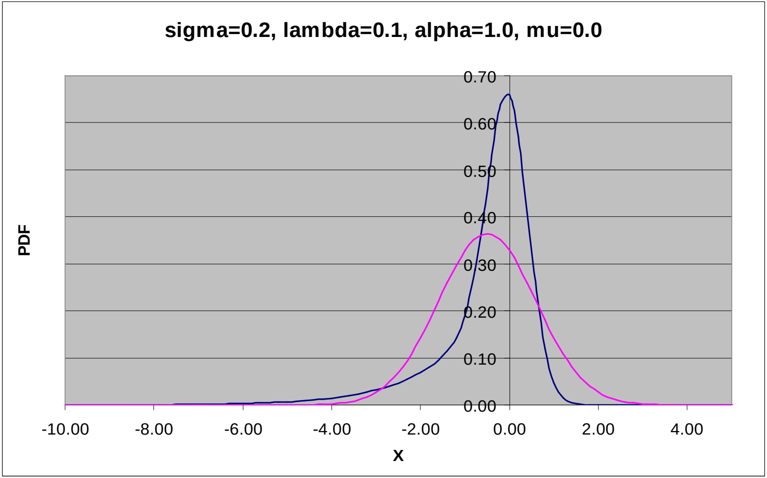
where $\mathbf{P}\left(a,b\right)=\exp\left(ab+b^{2}/2\right)\mathbf{N}\left(a+b\right)$

Unbounded PDF 2

- Then:

$$
\begin{aligned}
\Phi_{k,0}(x) &= \frac{\sigma_+\left(\left(+\theta-\sigma_+\right)\Phi_{k-1,0}+\sigma_+\Phi_{k-2,0}\right)}{k-1} \\
\Phi_{0,l}(x) &= \frac{\sigma_-\left(\left(-\theta-\sigma_-\right)\Phi_{0,l-1}+\sigma_-\Phi_{0,l-2}\right)}{l-1} \\
\Phi_{k,l}(x) &= \frac{\alpha_+\Phi_{k-1,l}+\alpha_-\Phi_{k,l-1}}{\alpha_++\alpha_-} \\
w_{k,l} &= \frac{e^{-(\lambda_++\lambda_-)t}\left(\lambda_+t\right)^k\left(\lambda_-t\right)^l}{k!l!} \\
\Phi(x) &= \sum_{k=0,l=0}^{\infty} w_{k,l}\Phi_{k,l}(x)
\end{aligned}
$$

No barrier PDF figure



Fokker-Planck equation

- We can solve the barrier problem by using the forward Fokker-Planck equation for the t.p.d.f. and putting probabilities below the barrier to zero. This equation has the form:

$$\mathcal{L}^{\dagger}\Phi \equiv \Phi_t - \tfrac{1}{2}\sigma^2\Phi_{xx} + \mu\Phi_x + \lambda\Phi - \lambda\alpha \int_0^{\infty} \Phi\left(t, x+j\right) e^{-\alpha j} dj = 0$$

$$\Phi\left(0, x\right) = \delta\left(x - \xi\right)$$

$$\Phi\left(t, x\right) = 0 \qquad \text{if} \qquad \xi < 0$$

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- When parameters are time-independent, we can find Φ via the Laplace transform.

Laplace transform

- The Laplace transform

$$\frac{1}{2}\sigma^2\hat{\Phi}_{xx} - \mu\hat{\Phi}_x - (\lambda + p)\,\hat{\Phi} + \lambda\alpha\int_0^\infty \hat{\Phi}(p, x + j)\, e^{-\alpha j} dj = -\delta(x - \xi)$$

$$\hat{\Phi}(p, 0) = 0$$

- The Laplace transform

$$\frac{1}{2}\sigma^2\hat{\Phi}_{xx} - \mu\hat{\Phi}_x - (\lambda + p)\,\hat{\Phi} + \lambda\alpha\int_0^\infty \hat{\Phi}\,(p, x + j)\,e^{-\alpha j}dj = -\delta\,(x - \xi)$$

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- Forward characteristic equation:

$$\frac{1}{2}\sigma^2\psi^2 - \mu\psi - (\lambda + p) - \frac{\lambda\alpha}{\psi - \alpha} = 0$$

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- This equation has three roots of which two are positive and one negative. We denote them as $-\psi_i$.

Green's function construction 1

- Hence the overall solution has the form:

$$\hat{\Phi}(p,x) = \begin{cases} C_3 e^{-\psi_3(x-\xi)}; & x \geq \xi \\ D_1 e^{-\psi_1(x-\xi)} + D_2 e^{-\psi_2(x-\xi)} + D_3 e^{-\psi_3(x-\xi)}; & x \leq \xi \end{cases}$$

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$$\hat{\Phi}(p,x) = \left\{ \begin{array}{cc} C_3 e^{-\psi_3(x-\xi)}; & x \geq \xi \\ D_1 e^{-\psi_1(x-\xi)} + D_2 e^{-\psi_2(x-\xi)} + D_3 e^{-\psi_3(x-\xi)}; & x \leq \xi \end{array} \right.$$

- Matching conditions

$$D_1 + D_2 + (D_3 - C_3) = 0$$

$$\psi_1 D_1 + \psi_2 D_2 + \psi_3 (D_3 - C_3) = -\frac{2}{\sigma^2}$$

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- One more condition is obtained from the following observations:

$$\mathcal{L}^{\dagger}\left(e^{-\psi_i(x-\xi)} H(x-\xi)\right) = \frac{\alpha}{\alpha+\psi_i} e^{\alpha(x-\xi)} H(\xi - x)$$

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- Here $\mathcal{L}^{\dagger}$ is the corresponding differential operator and $H(.)$ is the Heaviside function

Green's function construction 2

- The corresponding profile has the form

$$\hat{\Phi}(p,x) = \begin{cases} C_3 e^{-\psi_3(x-\xi)}; & x \geq \xi \\ D_1 e^{\psi_1 \xi}\left[e^{-\psi_1 x} - e^{-\psi_3 x}\right] + D_2 e^{\psi_2 \xi}\left[e^{-\psi_2 x} - e^{-\psi_3 x}\right]; & x \end{cases}$$

$$\begin{aligned} D_1 &= -\frac{2}{\sigma^2}\frac{(\alpha+\psi_1)}{(\psi_1-\psi_2)(\psi_1-\psi_3)} \\ D_2 &= -\frac{2}{\sigma^2}\frac{(\alpha+\psi_2)}{(\psi_2-\psi_1)(\psi_2-\psi_3)} \\ D_3 &= \frac{2}{\sigma^2}\left[\frac{e^{(\psi_1-\psi_3)\xi}(\alpha+\psi_1)}{(\psi_1-\psi_2)(\psi_1-\psi_3)} + \frac{e^{(\psi_2-\psi_3)\xi}(\alpha+\psi_2)}{(\psi_2-\psi_1)(\psi_2-\psi_3)}\right] \end{aligned}$$

$$C_3 = D_1 + D_2 + D_3$$

Stehfest algorithm

- Inverse Laplace transform yields $\Phi(t, x)$. We use Stehfest algorithm to evaluate Q:

$$\Phi(t, x) = p \sum_{k=1}^{N} (-1)^k St_k^N \hat{\Phi}(kp, x)$$

$$p = \frac{\ln 2}{t}$$

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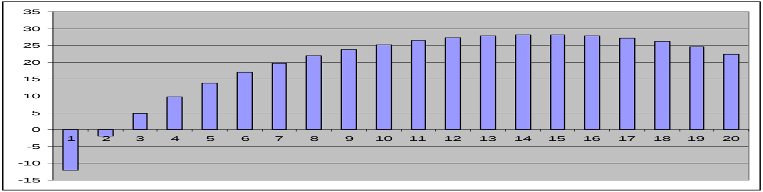
- Coefficients St_k^N are very stiff. We typically choose $N = 20$. For small t inversion can be numerically unstable unless computation is carried with many significant digits.



Hitting time density

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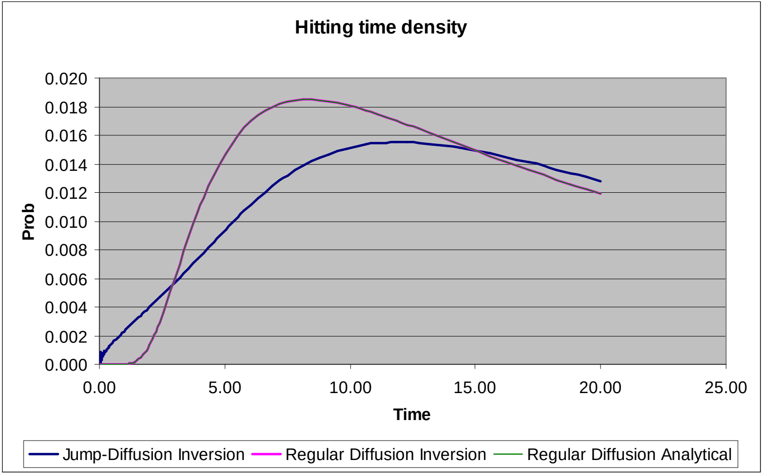
- For the regular diffusion we obtain

$$\hat{f}(p) = e^{\bar{\psi}_2 \xi}$$

$$f(t) = -\frac{\sigma^2}{2}\Phi_x(t, 0) = \frac{\xi e^{-(\xi + \mu t)^2 / 2\sigma^2 t}}{\sqrt{2\pi\sigma^2 t^3}}$$

Hitting time density figure



Green's function expansion

- Perturbative calculation of Green's function in 1D. For brevity we assume that all the parameters are time independent. Time-dependent case can be solved in a similar way. To start with, we make the following transform

$$\Phi(t,x) = \exp\left(-\left(\frac{\mu^2}{2\sigma^2}+\lambda\right)t + \frac{\mu}{\sigma^2}(x-\xi)\right)\tilde{\Phi}(t,x)$$

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$$\Phi(t,x) = \exp\left(-\left(\frac{\mu^2}{2\sigma^2}+\lambda\right)t + \frac{\mu}{\sigma^2}(x-\xi)\right)\tilde{\Phi}(t,x)$$

- The modified Green's function solves the following propagation problem

$$\partial_t\tilde{\Phi}(t,x) - \tfrac{1}{2}\sigma^2\partial^2\tilde{\Phi}(t,x) - \lambda\alpha\int_0^\infty \tilde{\Phi}(t,x+j)\,e^{-\bar{\alpha}j}dj = 0$$

$$\tilde{\Phi}(0,x) = \delta(x-\xi) \qquad \tilde{\Phi}(t,0) = 0, \qquad \tilde{\Phi}(t,\infty) = 0$$

where $\bar{\alpha} = \left(\alpha - \frac{\mu}{\sigma^2}\right)$, $\qquad \mu = -\frac{1}{2}\sigma^2 + \frac{\lambda}{\alpha+1}$

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- We assume that $\lambda \ll 1$ and represent $\tilde{\Phi}$ as follows

$$\tilde{\Phi}(t,x) = \tilde{\Phi}^{(0)}(t,x) + \lambda\tilde{\Phi}^{(1)}(t,x) + \ldots$$

Zero order term

- A simple calculation yields (with $\tau = \sigma^2 t$)

$$\tilde{\Phi}^{(0)}(t,x) = \frac{1}{\sqrt{\tau}}\left[\mathbf{n}\left(\frac{x-\xi}{\sqrt{\tau}}\right) - \mathbf{n}\left(\frac{x+\xi}{\sqrt{\tau}}\right)\right]$$

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- First order rhs

$$H^{(1)}(t,x) = \alpha\mathbf{P}\left(-\frac{x-\xi}{\sqrt{\tau}}, -\bar{\alpha}\sqrt{\tau}\right) - \alpha\mathbf{P}\left(-\frac{x+\xi}{\sqrt{\tau}}, -\bar{\alpha}\sqrt{\tau}\right)$$

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- We use Duhamel's principle and represent $\tilde{\Phi}^{(1)}$ as follows

$$\tilde{\Phi}^{(1)}(t,x) = \int_0^t \int_0^\infty \tilde{\Phi}^{(0)}(t,x;s,\eta_1)\, H^{(1)}(s,\eta_1)\, ds d\eta_1$$

First order term

- A very elaborate calculation yields

$$\tilde{\Phi}^{(1)}(t,x) = \frac{\alpha}{\bar{\alpha}\sigma_1^2}\left[\bar{\alpha}\tau\mathbf{P}\left(-\frac{x-\xi}{\sqrt{\tau}}, -\bar{\alpha}\sqrt{\tau}\right)\right.$$

$$+x\mathbf{P}\left(-\frac{x+\xi}{\sqrt{\tau}}, -\bar{\alpha}\sqrt{\tau}\right) - (x-\bar{\alpha}\tau)\mathbf{P}\left(-\frac{x+\xi}{\sqrt{\tau}}, \bar{\alpha}\sqrt{\tau}\right)$$

$$\left.-(x+\bar{\alpha}\tau)\mathbf{P}\left(-\frac{x}{\sqrt{\tau}}, -\bar{\alpha}\sqrt{\tau}\right) + (x-\bar{\alpha}\tau)\mathbf{P}\left(-\frac{x}{\sqrt{\tau}}, \bar{\alpha}\sqrt{\tau}\right)\right]$$

First order term

- A very elaborate calculation yields

$$\tilde{\Phi}^{(1)}(t,x) = \frac{\alpha}{\bar{\alpha}\sigma_1^2}\left[\bar{\alpha}\tau\mathbf{P}\left(-\frac{x-\xi}{\sqrt{\tau}},-\bar{\alpha}\sqrt{\tau}\right)\right.$$

$$+x\mathbf{P}\left(-\frac{x+\xi}{\sqrt{\tau}},-\bar{\alpha}\sqrt{\tau}\right)-(x-\bar{\alpha}\tau)\mathbf{P}\left(-\frac{x+\xi}{\sqrt{\tau}},\bar{\alpha}\sqrt{\tau}\right)$$

$$\left.-(x+\bar{\alpha}\tau)\mathbf{P}\left(-\frac{x}{\sqrt{\tau}},-\bar{\alpha}\sqrt{\tau}\right)+(x-\bar{\alpha}\tau)\mathbf{P}\left(-\frac{x}{\sqrt{\tau}},\bar{\alpha}\sqrt{\tau}\right)\right]$$

- The formula can be independently checked via the method of heat potentials.

Quality of approximation figure

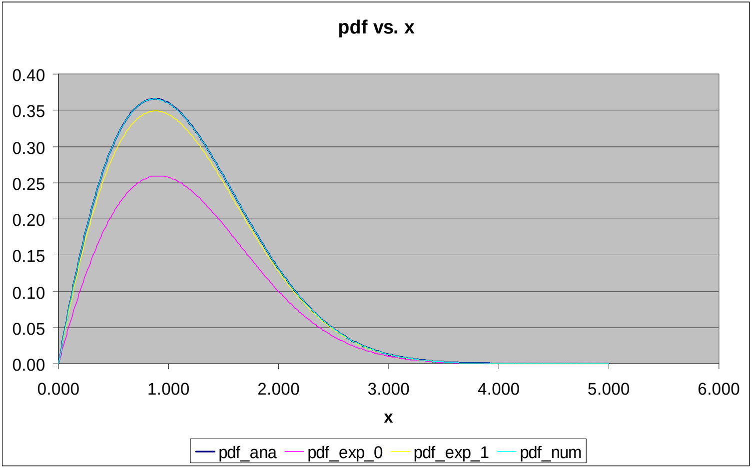
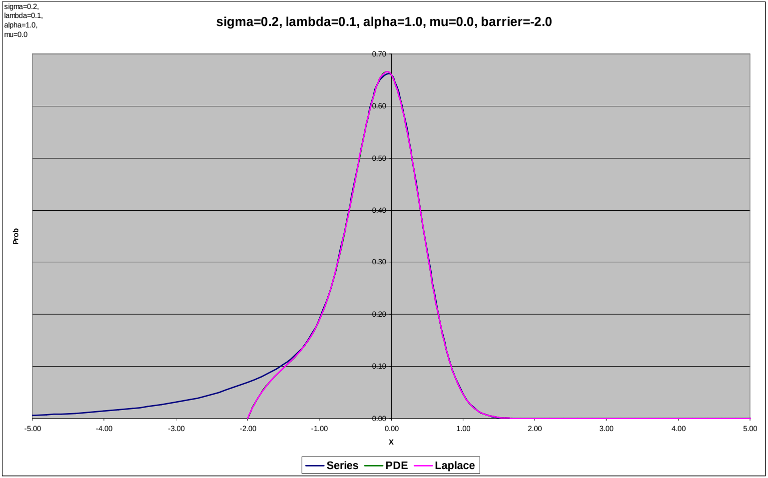


Figure: $G(t,x)$ for $t=20$, $\sigma=20\%$, $\lambda=2\%$, $\alpha=4$, $\xi=1$.

Barrier PDF figure



Hybrid model

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  where $\sigma(t, 0) \neq 0$. From that angle, one can argue that the Bachelier model is actually better than Black-Scholes!!
- Alternatively, one can add a terminal (Samuelson-style) jump to default and model S as follows

$$dS = [r(t) + \lambda(S)] S dt + \sigma(t, S) dW(t) - S dN(t)$$

Model Comparison

- Comparison of numerical and analytical t.p.d.f. as well as the solution of the barrier problem with non-constant barrier is given below.

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Model Comparison

- Comparison of numerical and analytical t.p.d.f. as well as the solution of the barrier problem with non-constant barrier is given below.
- By bootstrapping the barrier, we can reproduce term structure of CDS for most names. In addition, we can price equity derivatives and produce a respectable volatility skew.
- The barrier can assumed to be stochastic as in Duffie and Lando (2001) or CreditGrades (2000). Reduction of the information set makes reduced form and structural modeling almost identical, Jarrow, Protter, Yildirim (2004).

Survival probability figure

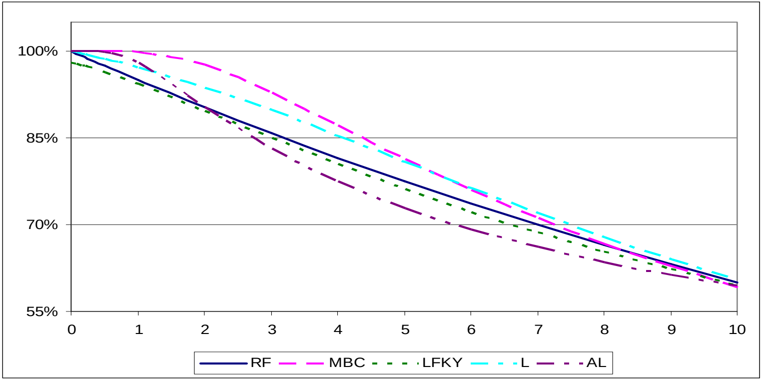

Figure: Typical survival probabilities $Q(0, T)$ as functions of T (in years) for the reduced-form, Merton-Black-Cox, Lardy *et al.*, Lipton, and Atlan-Lelanc models. Notice that $Q_{MBC}(0, T)$ and $Q_{AL}(0, T)$ are too "flat" when $T \to 0$, while $Q_{LFHY}(0, 0) < 1$. $Q_{RF}(0, T)$ and $Q_{L}(0, T)$ behave "properly".

Green's function without jumps

Without jumps, we need to find Green's function for the correlated heat equation in the quarter-plane, i.e., to solve the following problem

$$\Phi_t - \frac{1}{2}\left(\sigma_1^2\Phi_{x_1x_1} + 2\rho\sigma_1\sigma_2\Phi_{x_1x_2} + \sigma_2^2\Phi_{x_2x_2}\right) + \mu_1\Phi_{x_1} + \mu_2\Phi_{x_2} = 0$$

$$\Phi\left(t, x_1, 0\right) = 0, \qquad \Phi\left(t, 0, x_2\right) = 0$$

$$\Phi\left(t, x_1, x_2\right) \to \delta\left(x_1 - \xi_1\right)\delta\left(x_2 - \xi_2\right)$$

Standard transform

$$\Phi = \exp\left(\alpha t + \beta_1\left(x_1 - \xi_1\right) + \beta_2\left(x_2 - \xi_2\right)\right)\tilde{\Phi}$$

removes drift terms provided that

$$\Sigma\beta = \mu, \qquad \alpha = -\frac{1}{2}\Sigma^{-1}\mu\cdot\mu$$

Transforms

A sequence of linear transforms allows us to rewrite the pricing problem as follows

$$\tilde{\Phi}_t^{(2)} - \frac{1}{2}\left(\tilde{\Phi}^{(2)}_{x_1^{(2)}x_1^{(2)}} + \tilde{\Phi}^{(2)}_{x_2^{(2)}x_2^{(2)}}\right) = 0$$

$$w\left(t, x_1^{(2)}, 0\right) = 0, \qquad w\left(t, -\rho x_2^{(2)}/\bar{\rho}, x_2^{(2)}\right) = 0$$

$$w\left(t, x_1^{(2)}, x_2^{(2)}\right) \to \delta\left(x_1^{(2)} - \xi_1^{(2)}\right)\delta\left(x_2^{(2)} - \xi_2^{(2)}\right)$$

where $\bar{\rho} = \sqrt{1-\rho^2}$. We now have the standard heat equation in an angle and can use our previous results. This angle is formed by the horizontal axis $\xi_2^{(2)} = 0$ and a sloping line $\xi_1^{(2)} = -\rho\xi_2^{(2)}/\bar{\rho}$. It is acute when $\rho < 0$ and blunt otherwise. The size of this angle is $\alpha =$atan$(-\bar{\rho}/\rho)$.

Eigenfunction expansion

The solution of the above problem via the method of images was independently introduced in finance by He *et al.*, Zhou and Lipton. We want to find the Green's function for the following parabolic equation

$$\tilde{\Phi}_t^{(2)} - \frac{1}{2}\left(\tilde{\Phi}_{rr}^{(2)} + \frac{1}{r}\tilde{\Phi}_r^{(2)} + \frac{1}{r^2}\tilde{\Phi}_{\phi\phi}^{(2)}\right)$$

supplied with the boundary conditions of the form

$$\tilde{\Phi}^{(2)}(t, r, \phi) \underset{r\to 0}{\longrightarrow} C < \infty, \quad \tilde{\Phi}^{(2)}(t, r, \phi) \underset{r\to\infty}{\longrightarrow} 0, \quad \tilde{\Phi}^{(2)}(t, r, 0) = 0, \quad \tilde{\Phi}^{(2)}$$

and the initial condition

$$\tilde{\Phi}^{(2)}(t, r, \phi) \underset{t\to 0}{\longrightarrow} \frac{\delta(r - r')\,\delta(\phi - \phi')}{r'}$$

The fundamental solution has the form

$$\Phi_\alpha(t, r, \phi | 0, r', \phi') = \frac{2e^{-\left(r^2 + r'^2\right)/2t}}{\alpha t} \sum_{n=1}^{\infty} I_{n\pi/\alpha}\left(\frac{rr'}{t}\right) \sin\left(\frac{n\pi\phi'}{\alpha}\right) \sin\left(\frac{n\pi\phi}{\alpha}\right.$$

Non-periodic Green's function

It turns out that we can find the fundamental solution via the method of images too. This approach seems to be new. To start with, we find non-periodic solution of the heat equation written in polar co-ordinates in the positive half-plane $0 < r < \infty, \quad -\infty < \phi < \infty$:

$$\Phi\left(t, r, \phi | 0, r', \phi'\right) = \Phi_1\left(t, r, \phi | 0, r', \phi'\right) - \Phi_2\left(t, r, \phi | 0, r', \phi'\right)$$

where

$$\Phi_1\left(t, r, \phi | 0, r', \phi'\right) = \frac{e^{-\left(r^2 + r'^2\right)/2t}}{2\pi t} \frac{\left(s_+ + s_-\right)}{2} e^{(rr'/t)\cos(\phi - \phi')}$$

$$\Phi_2\left(t, r, \phi | 0, r', \phi'\right) = \frac{e^{-\left(r^2 + r'^2\right)/2t}}{2\pi^2 t} \int_0^\infty \frac{\left[\begin{array}{c} s_+ e^{-(rr'/t)\cosh((\pi + (\phi - \phi'))\zeta)} \\ + s_- e^{-(rr'/t)\cosh((\pi - (\phi - \phi'))\zeta)} \end{array}\right]}{\zeta^2 + 1} d\zeta$$

$$s_\pm = \operatorname{sign}\left(\pi \pm \left(\phi - \phi'\right)\right)$$

Graph of Green's function

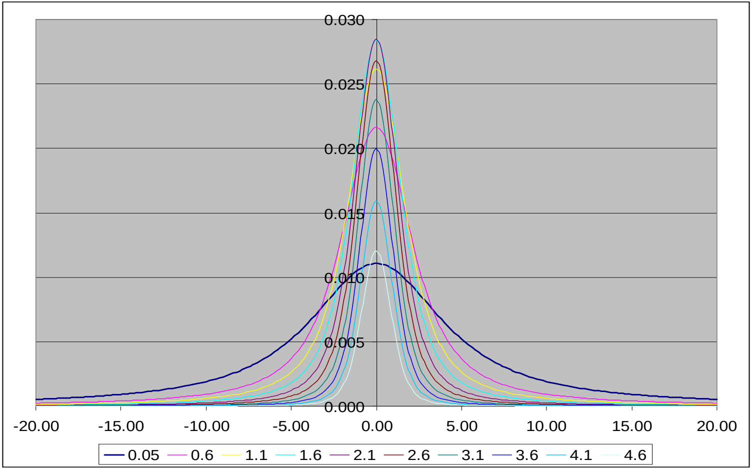

Figure: Cross-section of non-periodic Green's function. Parameters are as follows $t = 5, r' = 1.5, \phi' = 0$.

Method of images

Simple balancing of terms shows that

$$\Phi_2\left(t, r, \phi | 0, r', \phi'\right) = O\left(\left|\phi - \phi'\right|^{-2}\right)$$

Next, we represent the fundamental solution in the form

$$\Phi_\alpha\left(t, r, \phi | 0, r', \phi'\right) = \sum_{n=-\infty}^{\infty} \left[\begin{array}{c} \Phi\left(t, r, \phi | 0, r', \phi' + 2n\alpha\right) \\ -\Phi\left(t, r, \phi | 0, r', -\phi' + 2n\alpha\right) \end{array} \right]$$

Indeed, it is clear that the sum converges, every term solves the parabolic equation, only one term has a pole inside the angle, and $\Phi_\alpha(0) = \Phi_\alpha(\alpha) = 0$.

Correlated PDF figure

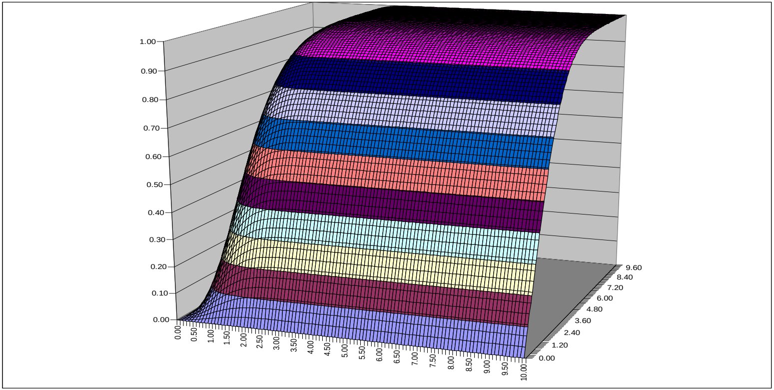

Figure: PDF in the positive quarter-plane

Green's function with jumps

With jumps, we need to find Green's function for the correlated jump-diffusion equation in the quarter-plane. In the simplest case we have to solve the following problem

$$\Phi_t - \frac{1}{2}\left(\sigma_1^2 \Phi_{x_1 x_1} + 2\rho\sigma_1\sigma_2 \Phi_{x_1 x_2} + \sigma_2^2 \Phi_{x_2 x_2}\right) + \mu_1 \Phi_{x_1} + \mu_2 \Phi_{x_2}$$

$$-\lambda\alpha_1\alpha_2 \int_0^\infty \int_0^\infty \Phi\left(x_1 + j_1, x_2 + j_2\right) e^{-\alpha_1 j_1 - \alpha_2 j_2} dj_1 dj_2 + \lambda\Phi = 0$$

$$\Phi\left(t, x_1, 0\right) = 0, \qquad \Phi\left(t, 0, x_2\right) = 0$$

$$\Phi\left(t, x_1, x_2\right) \to \delta\left(x_1 - \xi_1\right)\delta\left(x_2 - \xi_2\right)$$

This problem has been recently solved (numerically) by Lipton and Sepp. Its analytical solution is not known (??).

Multi-name case

- Consider N companies and assume that their asset values are driven by the following SDEs

$$\frac{dV_i(t)}{V_i(t)} = (r - d_i - \kappa_i \lambda_i(t))\,dt + \sigma_i(t)\,dW_i(t) + \left(e^{J_i} - 1\right) dN_i(t)$$

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- In log coordinates

$$x_i(t) = \ln\left(\frac{V_i(t)}{B_i(t)}\right)$$

$$dx_i(t) = \beta_i(t)\, dt + \sigma_i(t)\, dW_i(t) + J_i dN_i(t) \qquad x_i(0) = \ln\left(\frac{v_i}{b_i}\right) = \xi_i$$

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- Thus, we assume that there are both common and idiosyncratic jump sources.

Multi-name pricing problem

- We now formulate a typical pricing equation in the positive cone $R_+^{(N)}$. We have

$$\partial_t U(t,\vec{x}) + \mathcal{L}^{(N)} U(t,\vec{x}) = \chi(t,\vec{x})$$

$$U(t,\vec{x}_{0,k}) = \phi_{0,k}(t,\vec{y}), \qquad U(t,\vec{x}_{\infty,k}) = \phi_{\infty,k}(t,\vec{y})$$

$$U(T,\vec{x}) = \psi(\vec{x})$$

where $\vec{x}_{0,k}$, $\vec{x}_{\infty,k}$, $\vec{y}_k$ are N and $N-1$ dimensional vectors, respectively,

$$\vec{x}_{0,k} = \left(x_1, ..., \underset{k}{0}, ...x_N\right), \quad \vec{x}_{\infty,k} = \left(x_1, ..., \underset{k}{\infty}, ...x_N\right), \quad \vec{y}_k = (x_1, ..., x_N$$

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- The integro-differential operator $\mathcal{L}^{(N)}$ can be written as

$$\begin{aligned} \mathcal{L}^{(N)} f(\vec{x}) &= \tfrac{1}{2} \sum_i \sigma_i^2 \partial_i^2 f(\vec{x}) + \sum_{i,j,j>i} a_{ij} \partial_i \partial_j f(\vec{x}) + \sum_i \beta_i \partial_i f(\vec{x}) \\ &\quad - \tilde{\gamma} f(\vec{x}) + \sum_{\pi \in \Pi^{(N)}} \lambda_\pi \prod_{i \in \pi} \mathcal{J}_i f(\vec{x}) \end{aligned}$$

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Jump terms

- In the case of negative exponential jumps,

$$J_i f\left(\vec{x}\right) = \int_{-x_i}^{0} f\left(x_1, ..., x_i + j, ...x_N\right) \varpi_i\left(j\right) dj$$

while in the case of discrete negative jumps

$$J_i f\left(\vec{x}\right) = \mathbf{H}\left(x_i + J_i\right) f\left(x_1, ..., x_i + J_i, ...x_N\right)$$

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- We naturally split the operator $\mathcal{L}^{(N)}$ into the local (differential) and non-local (integral) parts:

$$\mathcal{L}^{(N)} f(\vec{x}) = \mathcal{D}^{(N)} f(\vec{x}) + \mathcal{I}^{(N)} f(\vec{x})$$

Adjoint operator

- The corresponding adjoint operator is

$$\mathcal{L}^{(N)\dagger} g(\vec{x}) = \frac{1}{2}\sum_i \sigma_i^2 \partial_i^2 g(\vec{x}) + \sum_{i,j,j>i} a_{ij}\partial_i\partial_j g(\vec{x}) - \sum_i \beta_i \partial_i g(\vec{x}) - \tilde{\gamma} g(\vec{x}) + \sum_{\pi\in\Pi^{(N)}} \lambda^{\pi} \prod_{i\in\pi} \mathcal{J}_i^{\dagger} g(\vec{x})$$

where

$$J_i^{\dagger} g(\vec{x}) = \int_0^{\infty} g(x_1, ..., x_i - j, ...x_N)\, \varpi_i(j)\, dj$$

$$J_i^{\dagger} g(\vec{x}) = g(x_1, ..., x_i - J_i, ...x_N)$$

Green's formula

- We introduce Green's function $\Phi(t,\vec{x})$, or, more explicitly, $\Phi\left(t',\vec{x};t,\vec{\xi}\right)$, such that

$$\partial_{t'}\Phi(t',\vec{x}) - \mathcal{L}^{(N)\dagger}\Phi(t',\vec{x}) = 0$$

$$\Phi(t',\vec{x}_{0k}) = 0, \qquad \Phi(t',\vec{x}_{\infty k}) = 0, \qquad \Phi(t,\vec{x}) = \delta\left(\vec{x}-\vec{\xi}\right)$$

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- It can be shown that

$$\begin{aligned}U\left(t,\vec{\xi}\right) &= \int_{R_+^{(N)}}\psi\left(\vec{x}\right)\Phi\left(T,\vec{x};t,\vec{\xi}\right)d\vec{x}\\ &+\sum_k\int_t^T\int_{R_+^{(N-1)}}\phi_k\left(t',\vec{y}\right)\Phi_k\left(t',\vec{y};t,\vec{\xi}\right)dt'd\vec{y}\\ &-\int_t^T\int_{R_+^{(N)}}\chi\left(t',\vec{x}\right)\Phi\left(t',\vec{x};t,\vec{\xi}\right)dt'd\vec{x}\end{aligned}$$

In other words, instead of solving the backward pricing problem with nonhomogeneous rhs and boundary conditions, we can solve the forward propagation problem for Green's function with homogeneous

Single-name case

- We now are in a position to describe the relevant operators in more detail for $N = 1, 2$.

Single-name case

- We now are in a position to describe the relevant operators in more detail for $N = 1, 2$.
- When $N = 1$ we deal with the reference name alone. Since $\Pi^{(1)}$ consists of just one element,

$$\Pi^{(1)} = \{\{1\}\}$$

$$\mathcal{L}^{(1)} f = \frac{1}{2}\sigma_1^2 \partial_1^2 f + \beta_1 \partial_1 f - \tilde{\gamma} f + \lambda_1 \mathcal{J}_1 f = \mathcal{D}^{(1)} f + \mathcal{I}^{(1)} f$$

$$\mathcal{L}^{(1)\dagger} g = \frac{1}{2}\sigma_1^2 \partial_1^2 g - \beta_1 \partial_1 g - \tilde{\gamma} g + \lambda_1 \mathcal{J}_1^{\dagger} g = \mathcal{D}^{(1)\dagger} g + \mathcal{I}^{(1)\dagger} g$$

$$\tilde{\gamma} = \gamma + \lambda_1$$

$$\mathcal{I}^{(1)} f(x_1) = \lambda_1 \int_{-x_1}^{0} f(x_1 + j_1) e^{\alpha_1 j_1} d(\alpha_1 j_1) = \lambda_1 \int_0^{x_1} f(y_1) e^{-\alpha_1 (x_1 - y_1)}$$

$$\mathcal{I}^{(1)\dagger} g(x_1) = \lambda_1 \int_{-\infty}^{0} g(x_1 - j_1) e^{\alpha_1 j_1} d(\alpha_1 j_1) = \lambda_1 \int_{x_1}^{\infty} g(y_1) e^{\alpha_1 (x_1 - y_1)}$$

$$\mathcal{I}^{(1)} f(x_1) = \lambda_1 \mathbf{H}(x_1 + J_1) f(x_1 + J_1)$$

$$\mathcal{I}^{(1)\dagger} f(x_1) = \lambda_1 f(x_1 - J_1)$$

Two-name case 1

- When $N = 2$ we deal with the reference name 1 and protection seller 2, say. Since $\Pi^{(2)}$ consists of three elements,

$$\Pi^{(2)} = \{\{1\}, \{2\}, \{1,2\}\}$$

we have

$$\begin{aligned}
\mathcal{L}^{(2)}f &= \tfrac{1}{2}\sigma_1^2\partial_1^2 f + \tfrac{1}{2}\sigma_2^2\partial_2^2 f + \sigma_1\sigma_2\rho_{12}\partial_1\partial_2 f + \beta_1\partial_1 f + \beta_2\partial_1 f - \tilde{\gamma}f \\
&\quad +\lambda_1\mathcal{J}_1 f + \lambda_2\mathcal{J}_2 f + \lambda_{1,2}\mathcal{J}_1\mathcal{J}_2 f \\
&= \mathcal{D}^{(2)}f + \mathcal{I}^{(2)}f
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}^{(2)\dagger}g &= \tfrac{1}{2}\sigma_1^2\partial_1^2 g + \tfrac{1}{2}\sigma_2^2\partial_2^2 g + \sigma_1\sigma_2\rho_{12}\partial_1\partial_2 g - \beta_1\partial_1 g - \beta_2\partial_1 g - \tilde{\gamma}g \\
&\quad +\lambda_1\mathcal{J}_1^\dagger g + \lambda_2\mathcal{J}_2^\dagger g + \lambda_{1,2}\mathcal{J}_1^\dagger\mathcal{J}_2^\dagger g \\
&= \mathcal{D}^{(2)\dagger}g + \mathcal{I}^{(2)\dagger}g
\end{aligned}$$

$$\tilde{\gamma} = \gamma + \lambda_1 + \lambda_2 + \lambda_{1,2}$$

Two-name case 2

- Specifically,

$$
\begin{aligned}
\mathcal{I}^{(2)} f(x_1, x_2) &= \lambda_1 \int_0^{x_1} f(y_1, x_2)\, e^{-\alpha_1 (x_1 - y_1)} d(\alpha_1 y_1) \\
&+ \lambda_2 \int_0^{x_2} f(x_1, y_2)\, e^{-\alpha_2 (x_2 - y_2)} d(\alpha_2 y_2) \\
&+ \lambda_{1,2} \int_0^{x_1} \int_0^{x_2} f(y_1, y_2)\, e^{-\alpha_1 (x_1 - y_1) - \alpha_2 (x_2 - y_2)} d(\alpha_1 y_1
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}^{(2)\dagger} g(x_1, x_2) &= \lambda_1 \int_{x_1}^{\infty} g(y_1, x_2)\, e^{-\alpha_1 (x_1 - y_1)} d(\alpha_1 y_1) \\
&+ \lambda_2 \int_{x_2}^{\infty} g(x_1, y_2)\, e^{-\alpha_2 (x_2 - y_2)} d(\alpha_2 y_2) \\
&+ \lambda_{1,2} \int_{x_1}^{\infty} \int_{x_2}^{\infty} g(y_1, y_2)\, e^{-\alpha_1 (x_1 - y_1) - \alpha_2 (x_2 - y_2)} d(\alpha_1 y_1
\end{aligned}
$$

Non-risky CDS pricing problem

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Non-risky CDS pricing problem

- We are now in a position to formulate the pricing problems of interest.
- We start with a CDS on a reference credit 1 which is sold by a nonrisky seller to a nonrisky buyer. Assuming for simplicity that the coupon is paid continuously at the rate s, and that the default boundary is continuous, we write the pricing equation as follows

$$\partial_t U^{(1)}(t, x_1) + \mathcal{L}^{(1)} U^{(1)}(t, x_1) = \chi(t, x_1)$$

$$U^{(1)}(t, 0) = \phi_0(t), \qquad U^{(1)}(t, \infty) = \phi_\infty(t)$$

$$U^{(1)}(T, x_1) = 0$$

Here

$$\chi(t, x_1) = s - \lambda_1 \left(1 - \hat{R}_1\right) e^{-\alpha_1 x_1} \qquad \hat{R}_1 = \frac{\alpha_1 R_1}{\alpha_1 + 1}$$

$$\phi_0(t) = (1 - R_1), \qquad \phi_\infty(t) = -s \int_t^T D(t, t') \, dt'$$

Risky seller CDS pricing problem 1

- For a CDS on a reference credit 1 which is sold by a risky seller 2 to a nonrisky buyer, we write the pricing equation as follows

$$\partial_t U^{(2)}(t, x_1, x_2) + \mathcal{L}^{(2)} U^{(2)}(t, x_1, x_2) = \chi(t, x_1, x_2)$$

$$U^{(2)}(t, 0, x_2) = \phi_{0,1}(t, x_2), \qquad U^{(2)}(t, \infty, x_2) = \phi_{\infty,1}(t, x_2)$$

$$U^{(2)}(t, x_1, 0) = \phi_{0,2}(t, x_1), \qquad U^{(2)}(t, x_1, \infty) = \phi_{\infty,2}(t, x_1)$$

$$U^{(2)}(T, x_1, x_2) = 0$$

Risky seller CDS pricing problem 2

- Here

$$\chi(t, x_1, x_2) = s - \lambda_1 v_1(t, x_1, x_2) - \lambda_2 v_2(t, x_1, x_2) - \lambda_{1,2} v_{1,2}(t, x_1, x_2)$$

$$v_1(t, x_1, x_2) = \left(1 - \hat{R}_1\right) e^{-\alpha_1 x_1}$$

$$v_2(t, x_1, x_2) = \tilde{U}^{(1)}(t, x_1) e^{-\alpha_2 x_2}$$

$$v_{1,2}(t, x_1, x_2) = \left(1 - \hat{R}_1\right) e^{-\alpha_1 x_1} \left(1 - e^{-\alpha_2 x_2}\right)$$

$$+ \int_{-x_1}^{0} \tilde{U}^{(1)}(t, x_1 + j_1) e^{\alpha_1 j_1} d(\alpha_1 j_1) e^{-\alpha_2 x_2} + \left(1 - \hat{R}_1\right) \hat{R}_2 e^{-\alpha_1 x_1 - \alpha_2 x_2}$$

$$\tilde{U}^{(1)}(t, x_1) = \hat{R}_2 U_+^{(1)}(t, x_1) + U_-^{(1)}(t, x_1)$$

$$\phi_{0,1}(t, x_2) = 1 - R_1, \qquad \phi_{\infty,1}(t, x_2) = -s \int_t^T D(t, t')\, dt'$$

$$\phi_{0,2}(t, x_1) = R_2 U_+^{(1)}(t, x_1) + U_-^{(1)}(t, x_1), \qquad \phi_{\infty,2}(t, x_1) = U^{(1)}(t, x_1)$$

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- We showed how to build meaningful structural default models and connected them to reduced form models.
- We showed that even the simplest elementary building blocks of credit universe are difficult to describe properly.
- Next talk will address computational aspects of the model.
- The opinions expressed in this talk are those of the speaker and do not necessarily reflect the views or opinions of Bank of America Merrill Lynch.