THE HESTON MODEL WITH STOCHASTIC INTEREST RATES UNDER FOURIER BASED PRICING ALGORITHMS

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The Objectives of the Research

The main objectives of our research is to build an Equity-Interest Rate Hybrid model which:

- ⇒ generates smiles on the equity side;
- ⇒ the interest rate component is well fitted to (at least) ATM swaptions, caplets, etc. and preserves an implied volatility hump;
- ⇒ hybrid handles non-zero correlations between the underlying processes (preferably full matrix of correlations);
- ⇒ can be efficiently and accurately evaluated by means of the Fourier inverse based algorithms (the calibration purposes);

Hybrid Model Construction

- In general the equity-interest hybrid is given by:

$$\begin{cases} \mathrm{d}\mathbf{X}_t = & a(\mathbf{X}_t,\mathbf{R}_t)\mathrm{d}t + b(\mathbf{X}_t)\mathrm{d}\mathbf{W}_t^{\mathbf{X}}, & \text{Equity} \\ \mathrm{d}\mathbf{R}_t = & c(\mathbf{R}_t)\mathrm{d}t + d(\mathbf{R}_t)\mathrm{d}\mathbf{W}_t^{\mathbf{R}}, & \text{Interest Rate} \\ \mathbf{Z}_t\mathbf{Z}_t^{\mathsf{T}} = & \mathbf{C}^{\mathbf{H}}\mathrm{d}t, & \text{Correlation} \end{cases}$$

with $\mathbf{H_t} = [\mathbf{X}_t, \mathbf{R}_t]$, and $\mathbf{Z}_t = [\mathrm{d}\mathbf{W}_t^{X}, \mathrm{d}\mathbf{W}_t^{R}]^{\mathsf{T}}$,

- where:
  - $\mathbf{C}^{\mathbf{H}}$ is a $(n+m)\times(n+m)$ matrix which represents the instantaneous correlation matrix between the Brownian motions;
  - the correlation within the asset classes is allowed, i.e.: $\mathbf{C}^{\mathbf{X}} = (\mathrm{d}\mathbf{W}_t^{\mathbf{X}})(\mathrm{d}\mathbf{W}_t^{\mathbf{X}})^{\mathsf{T}}$, $\mathbf{C}^{\mathbf{Y}} = (\mathrm{d}\mathbf{W}_t^{\mathbf{R}})(\mathrm{d}\mathbf{W}_t^{\mathbf{R}})^{\mathsf{T}}$.

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Hybrid Model Construction (Equity Part)

- In particular, the Heston model for the state vector $\mathbf{X}_t = [x_t = \log S_t, \sigma_t]^{\mathrm{T}}$ is described by the following system of SDEs:

$$\begin{cases} \mathrm{d}x_t = & \left(r_t - \frac{1}{2}\sigma_t\right)\mathrm{d}t + & \sqrt{\sigma_t}\mathrm{d}W_t^x, & x_0 > 0, \\ \mathrm{d}\sigma_t = & \epsilon\left(\bar{\sigma} - \sigma_t\right)\mathrm{d}t + & \omega\sqrt{\sigma_t}\mathrm{d}W_t^\sigma, & \sigma_0 > 0, \\ \mathbf{C}_{1,2}^{\mathbf{X}} = & \rho_{x,\sigma}\mathrm{d}t, & & \end{cases}$$

with the speed of mean reversion $\epsilon > 0$, long-term mean $\bar{\sigma} > 0$, and correlation $|\rho_{x,\sigma}| < 1$.

Hybrid Model Construction (Interest Rate)

- For the interest rate process we take the state vector $\mathbf{R}_t = [r_t, v_t]^\mathrm{T}$:

$$\begin{cases} \mathrm{d}r_t = & \kappa(\theta_t + p v_t - r_t)\mathrm{d}t + & \eta\sqrt{v_t}^{(1-p)}\mathrm{d}W_t^r, \quad r_0 > 0, \\ \mathrm{d}v_t = & \lambda(\bar{v}(1-p) - v_t)\mathrm{d}t + & \gamma\sqrt{v_t}^{(1-p)}\mathrm{d}W_t^v, \\ \mathbf{C}^\mathbf{R}_{1,2} = & \rho_{r,v}\mathrm{d}t, & \end{cases}$$

with

$$\begin{cases} v_0 > 0 \quad \text{for } p = 0, \\ v_0 = 0 \quad \text{for } p = 1, \end{cases}$$

where $p = \{0, 1\}$,

- depending on p we have two different models for the short-rate:
  - $p = 0$: stochastic volatility short rate process (H2++) [Heidari *et al.*-2007]
  - $p = 1$: Hull-White two factor model (G2++) [Hull-2006, Brigo and Mercurio-2006]

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Hybrid Model Construction: Correlation Structure

- A hybrid model $\mathbf{H}_t^p = [\mathbf{X}_t, \mathbf{Y}_t]^\mathrm{T} = [x_t, \sigma_t, r_t, v_t]^\mathrm{T}$ has the following instantaneous correlation structure:

$$\mathbf{C}^\mathbf{H} = \left(\begin{array}{cc|cc} 1 & \rho_{x,\sigma} & \rho_{x,r} & \rho_{x,v} \\ * & 1 & \rho_{\sigma,r} & \rho_{\sigma,v} \\ \hline * & * & 1 & \rho_{r,v} \\ * & * & * & 1 \end{array}\right)_{4\times 4}.$$

- the equity and the interest rate asset classes are linked by correlations in the right-upper and left-lower diagonal blocks of matrix $\mathbf{C}^\mathbf{H}$;
- the instantaneous covariance matrix $\mathbf{S} := \Sigma_\mathbf{H}\Sigma_\mathbf{H}^\mathrm{T}$:

$$\mathbf{S} = \left(\begin{array}{cc|cc} \sigma_t & \rho_{x,\sigma}\omega\sigma_t & \rho_{x,r}\eta\sqrt{\sigma_t}\sqrt{v_t}^{(1-p)} & \rho_{x,v}\gamma\sqrt{\sigma_t}\sqrt{v_t}^{(1-p)} \\ * & \omega^2\sigma_t & \rho_{\sigma,r}\eta\omega\sqrt{\sigma_t}\sqrt{v_t}^{(1-p)} & \rho_{\sigma,v}\omega\gamma\sqrt{\sigma_t}\sqrt{v_t}^{(1-p)} \\ \hline * & * & \eta^2 v_t^{1-p} & \rho_{r,v}\eta\gamma v_t^{1-p} \\ * & * & * & \gamma^2 v_t^{1-p} \end{array}\right)_{4\times 4}.$$

The model for $\rho_{x,r} \neq 0$,$\rho_{x,v} \neq 0$, $\rho_{\sigma,r} \neq 0$ and $\rho_{\sigma,v} \neq 0$ is not affine!

Square Root Process

- Distribution of σ_t [Broadie,Kaya-2006] is given by:

$$\sigma_t = c(t)\chi^2\left(d, \lambda(t)\right), \quad t > 0,$$

with

$$c(t) = \frac{1}{4\epsilon}\omega^2(1 - \mathrm{e}^{-\epsilon t}), \quad d = \frac{4\epsilon\bar{\sigma}}{\omega^2}, \quad \lambda(t) = \frac{4\epsilon \mathrm{e}^{-\epsilon t}\sigma_0}{\omega^2(1 - \mathrm{e}^{-\epsilon t})}.$$

- with

$$\begin{aligned} \mathbb{E}(\sigma_t|\sigma_0) &= \quad c(t)(d + \lambda(t)), \\ \mathbb{V}\mathrm{ar}(\sigma_t|\sigma_0) &= \quad c^2(t)(2d + 4\lambda(t)). \end{aligned}$$

What can be said about $\sqrt{\sigma_t}$ and its moments?

Square Root Process

- How to find $\mathbb{E}(\sqrt{\sigma_t})$?
  - Delta method based approximations [Swishchuk-2004],
  - Integral exact representation [Gatheral-2006],

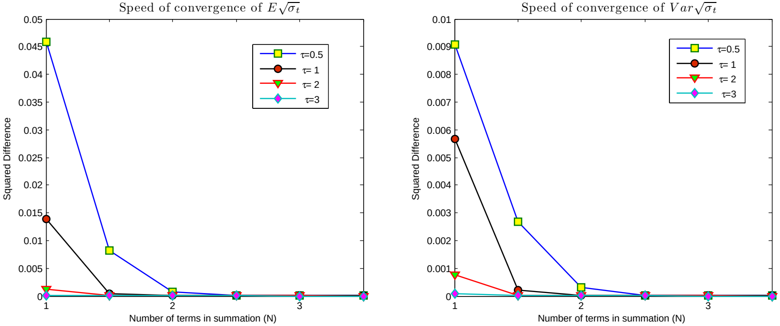
Lemma (Expectation and variance for $\sqrt{\sigma_t}$)

For a given time $t > 0$ the expectation and variance of $\sqrt{\sigma_t}$ are given by:

$$\mathbb{E}(\sqrt{\sigma_t}|\sigma_0) = \sqrt{2c(t)}\mathrm{e}^{-\lambda(t)/2} \sum_{k=0}^{\infty} \frac{1}{k!} (\lambda(t)/2)^k \frac{\Gamma\left(\frac{1+d}{2}+k\right)}{\Gamma(\frac{d}{2}+k)},$$

$$\mathbb{V}\mathrm{ar}\left(\sqrt{\sigma_t}|\sigma_0\right) = \mathbb{E}(\sigma_t|\sigma_0) - 2c(t)\mathrm{e}^{-\lambda(t)} \left(\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\lambda(t)}{2}\right)^k \frac{\Gamma\left(\frac{1+d}{2}+k\right)}{\Gamma\left(\frac{d}{2}+k\right)} \right)^2.$$

Square Root Process



Speed of convergence of $E\sqrt{\sigma_t}$

Speed of convergence of $Var\sqrt{\sigma_t}$

Figure: Speed of convergence of expectation $\mathbb{E}(\sqrt{\sigma_t})$ (LEFT) and variance $\mathbb{V}\mathrm{ar}(\sqrt{\sigma_t})$ (RIGHT) for different t. Other parameters are chosen: $\epsilon = 1.2$, $\omega = 0.4$, $\bar{\sigma} = 0.2$ and $\sigma_0 = 0.05$.

Normal approximation for dynamics of $\sqrt{\sigma_t}$

- For $t \to \infty$, noncentrality parameter $\lambda(t) \to 0$ so by [Fisher,1922]

$$\lim_{t\to\infty} \sqrt{\sigma_t} \sim \mathcal{N}\left(\sqrt{\bar{\sigma} - \frac{\omega^2}{8\epsilon}}, \frac{\omega^2}{8\epsilon}\right).$$

- For $t > 0$ we have found an approximation:

$$\sqrt{\sigma_t} \sim \mathcal{N}\left(\sqrt{c(t)(\lambda(t)-1) + c(t)d + \frac{c(t)d}{2(d+\lambda(t))}}, c(t) - \frac{c(t)d}{2(d+\lambda(t))}\right),$$

with

$$c(t) = \frac{1}{4\epsilon}\omega^2(1-\mathrm{e}^{-\epsilon t}), \quad d = \frac{4\epsilon\bar{\sigma}}{\omega^2}, \quad \lambda(t) = \frac{4\epsilon\mathrm{e}^{-\epsilon t}\sigma_0}{\omega^2(1-\mathrm{e}^{-\epsilon t})}.$$

Normal approximation for dynamics of $\sqrt{\sigma_t}$

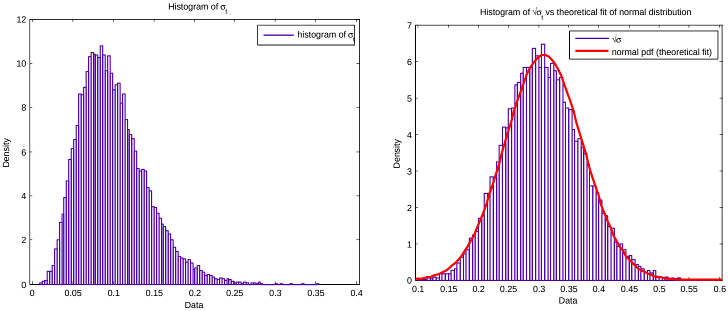


Figure: For maturity $T = 5$; LEFT: a histogram for σ_t, RIGHT: a histogram for $\sqrt{\sigma_t}$ and the theoretical fit of normal distribution. The Monte Carlo simulation was performed with 20.000 paths with 500 steps for $\epsilon = 1.2$, $\omega = 0.2$, $\bar{\sigma} = 0.1$, $\sigma_0 = 0.05$.

Normal approximation for dynamics of $\sqrt{\sigma_t}$

- For finite t normal distribution provides highly-satisfactory approximation;
- The moments of $\sqrt{\sigma_t}$ are functions of time;

$$\sqrt{\sigma_t} \sim \mathcal{N}\left(\mathbb{E}(\sqrt{\sigma_t}|\sigma_0), \mathbb{V}\mathrm{ar}(\sqrt{\sigma_t}|\sigma_0)\right).$$

- By Itô's lemma:

$$\mathrm{d}\sqrt{\sigma_t} = \left(-\frac{\omega^2}{8\sqrt{\sigma_t}} + \frac{\epsilon}{2}\left(\frac{\bar{\sigma}}{\sqrt{\sigma_t}} - \sqrt{\sigma_t}\right)\right)\mathrm{d}t + \frac{\omega}{2}\mathrm{d}W_t^{\sigma}, \quad \sqrt{\sigma_0} > 0.$$

- Define an approximation: $\sqrt{\sigma_t} \approx u_t$

$$\mathrm{d}u_t = \mu_t^u \mathrm{d}t + \psi_t^u \mathrm{d}W_t^{\sigma}, \qquad u_0 = \sqrt{\sigma_0} > 0,$$

for certain time dependent: μ_t^u and ψ_t^u chosen such that first two moments of u_t and $\sqrt{\sigma_t}$ match.

Construction of the Affine Hybrid Model

- The instantaneous covariance matrix:

$$\mathbf{S}=\left(\begin{array}{cc|cc}\sigma_t & \rho_{x,\sigma}\omega\sigma_t & \rho_{x,r}\eta\sqrt{\sigma_t}\sqrt{v_t}^{(1-p)} & \rho_{x,v}\gamma\sqrt{\sigma_t}\sqrt{v_t}^{(1-p)}\\ * & \omega^2\sigma_t & \rho_{\sigma,r}\eta\omega\sqrt{\sigma_t}\sqrt{v_t}^{(1-p)} & \rho_{\sigma,v}\omega\gamma\sqrt{\sigma_t}\sqrt{v_t}^{(1-p)}\\ \hline * & * & \eta^2 v_t^{1-p} & \rho_{r,v}\eta\gamma v_t^{1-p}\\ * & * & * & \gamma^2 v_t^{1-p}\end{array}\right)_{\mathbf{4\times4}}.$$

- Element $(1,3)$ (for example) from matrix $\mathbf{S}$ is given by:

$$\mathbf{S}_{(1,3)}=\rho_{x,r}\eta\sqrt{\sigma_t}\sqrt{v_t}^{(1-p)},$$

- *In our previous research for the Heston-Hull-White hybrid model ($p=1$) we have used the following approximation of $\mathbf{S}_{(1,3)}$:*

$$\widetilde{\mathbf{S}}_{(1,3)}=\rho_{x,r}\mathbb{E}(\sqrt{\sigma_t})\approx\rho_{x,r}\left(a+b\mathrm{e}^{-ct}\right),$$

for certain constants a, b and c which resulted in closed form solution of corresponding characteristic function.

Construction of the Affine Hybrid Model

- Define an approximation:

$$\widetilde{\mathbf{S}}_{(1,3)} = \rho_{x,r}\eta u_t^{\sigma} \left(u_t^{v}\right)^{(1-p)},$$

with:

$$\begin{cases} \mathrm{d}u_t^{\sigma} = & \mu_t^{\sigma}\mathrm{d}t + & \psi_t^{\sigma}\mathrm{d}W_t^{\sigma}, & u_0^{\sigma} = \sqrt{\sigma_0} > 0, \\ \mathrm{d}u_t^{v} = & \mu_t^{v}\mathrm{d}t + & \psi_t^{v}\mathrm{d}W_t^{v}, & u_0^{v} = \sqrt{v_0} > 0, \\ \rho_{\sigma,v}\mathrm{d}t = & \mathrm{d}W_t^{\sigma}\mathrm{d}W_t^{v}, & & \end{cases}$$

- Depending on the values of parameter p:

$$\begin{cases} p = 0: & \widetilde{\mathbf{S}}_{(1,3)} = & \rho_{x,r}\eta u_t^{\sigma} u_t^{v}, & \text{Not Affine!} \\ p = 1: & \widetilde{\mathbf{S}}_{(1,3)} = & \rho_{x,r}\eta u_t^{\sigma}, & \text{Affine.} \end{cases}$$

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Construction of the Affine Hybrid Model

- In order to *repair* this for the case of $p=0$ take

  $$z_t := u_t^\sigma u_t^v,$$

  with:

  $$\mathrm{d}z_t = \Delta_t \mathrm{d}t + u_t^v \psi_t^\sigma \mathrm{d}W_t^\sigma + u_t^\sigma \psi_t^v \mathrm{d}W_t^v,$$

  where: $\Delta_t = \mu_t^\sigma u_t^v + \mu_t^v u_t^\sigma + \psi_t^\sigma \psi_t^v \rho_{\sigma,v}$.
- The process above is not affine, but since $u_t^\sigma \approx \sqrt{\sigma_t}$ and $u_t^v \approx \sqrt{v_t}$ the following holds:

  $$\mathrm{d}z_t = \quad \Delta_t \mathrm{d}t + \quad \sqrt{v_t}\psi_t^\sigma \mathrm{d}W_t^\sigma + \quad \sqrt{\sigma_t}\psi_t^v \mathrm{d}W_t^v.$$

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Construction of the Affine Hybrid Model

- By combining the equations the linearized hybrid is given by:

$$\text{Hybrid Model}\begin{cases} \text{Equity} & \begin{cases} \mathrm{d}x_t = \left(r_t - \frac{1}{2}\sigma_t\right)\mathrm{d}t + \sqrt{\sigma_t}\mathrm{d}W_t^x, \\ \mathrm{d}\sigma_t = \epsilon\left(\bar{\sigma} - \sigma_t\right)\mathrm{d}t + \omega\sqrt{\sigma_t}\mathrm{d}W_t^\sigma, \end{cases} \\ \text{Short Rate} & \begin{cases} \mathrm{d}r_t = \kappa(\theta_t + pv_t - r_t)\mathrm{d}t + \eta\sqrt{v_t}^{(1-p)}\mathrm{d}W_t^r, \\ \mathrm{d}v_t = \lambda(\bar{v}(1-p) - v_t)\mathrm{d}t + \gamma\sqrt{v_t}^{(1-p)}\mathrm{d}W_t^v, \end{cases} \\ \text{Affinity correction} & \begin{cases} \mathrm{d}u_t^\sigma = \mu_t^\sigma\mathrm{d}t + \psi_t^\sigma\mathrm{d}W_t^\sigma, \\ \mathrm{d}u_t^v = \mu_t^v\mathrm{d}t + \psi_t^v\mathrm{d}W_t^v, \\ \mathrm{d}z_t = \Delta_t\mathrm{d}t + \sqrt{v_t}\psi_t^\sigma\mathrm{d}W_t^\sigma + \sqrt{\sigma_t}\psi_t^v\mathrm{d}W_t^v, \end{cases} \end{cases}$$

with

$$\begin{cases} u_t^\sigma := \sqrt{\sigma_t}, & u_0^\sigma = \sqrt{\sigma_0} > 0, \\ u_t^v := \sqrt{v_t}, & u_0^v = \sqrt{v_0} > 0, \\ z_t := u_t^\sigma u_t^v \overset{def}{:=} \sqrt{\sigma_t}\sqrt{v_t}, & z_0 = \sqrt{\sigma_0}\sqrt{v_0} > 0. \end{cases}$$

Is the model affine???

Construction of the Affine Hybrid Model

- Because of the model linearization the dimension of the original $4D$ state vector, $\mathbf{X}_t = [x_t, \sigma_t, r_t, v_t]^\mathrm{T}$ has increased;
- $5D$ vector $\widetilde{\mathbf{H}}_t^1 = [x_t, \sigma_t, r_t, v_t, u_t^\sigma]^\mathrm{T}$ for $p = 1$, H-G2++

$$\mathbf{\Sigma}_{\widetilde{\mathbf{H}}_t^1}\mathbf{\Sigma}^\mathrm{T}_{\widetilde{\mathbf{H}}_t^1} = \left(\begin{array}{cc|cc|c} \sigma_t & \rho_{x,\sigma}\omega\sigma_t & \rho_{x,r}\eta u_t^\sigma & \rho_{x,v}\gamma u_t^\sigma & \rho_{x,\sigma}\psi_t^\sigma u_t^\sigma \\ * & \omega^2\sigma_t & \omega\rho_{\sigma,r}\eta u_t^\sigma & \rho_{\sigma,v}\omega\gamma u_t^\sigma & \omega\psi_t^\sigma u_t^\sigma \\ \hline * & * & \eta^2 & \rho_{r,v}\eta\gamma & \rho_{r,\sigma}\eta\psi_t^\sigma \\ * & * & * & \gamma^2 & \rho_{\sigma,v}\gamma\psi_t^\sigma \\ * & * & * & * & (\psi_t^\sigma)^2 \end{array}\right)_{5\times 5}.$$

- $7D$ vector $\widetilde{\mathbf{H}}_t^0 = [x_t, \sigma_t, r_t, v_t, u_t^\sigma, u_t^v, z_t]^\mathrm{T}$, for $p = 0$, H-H2++ (similar to $5D$ case).

Now, the extended models are of affine form.

Characteristic Function of the Heston Hybrid Model

- For a given affine state-vector $\mathbf{H}_t$ the discounted characteristic function [Duffie *et al.*-2000] is given by:

  $$\phi(\mathbf{u},\mathbf{X}_t,t,T)=\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds+i\mathbf{u}^T\mathbf{X}_T}|\mathcal{F}_t\right)=e^{A(\mathbf{u},\tau)+\mathbf{B}^{\mathbf{T}}(\mathbf{u},\tau)\mathbf{X}_t},$$

  where the expectation is taken under the risk-neutral measure, $\mathbb{Q}$. For a time lag, $\tau:=T-t$.

- The coefficients $A(\mathbf{u},\tau)$ and $\mathbf{B}^T(\mathbf{u},\tau)$ have to satisfy the following complex-valued ordinary differential equations (ODEs):

  $$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}(\mathbf{u},\tau)= & -r_1+a_1^T\mathbf{B}+ & \frac{1}{2}\mathbf{B}^Tc_1\mathbf{B}, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}A(\mathbf{u},\tau)= & -r_0+\mathbf{B}^Ta_0+ & \frac{1}{2}\mathbf{B}^Tc_0\mathbf{B}, \end{cases}$$

  with certain matrices $a_i,c_i,r_i,\ i=0,1$.

Characteristic Function of the Heston Hybrid Model

- H-G2++ with $\widetilde{\mathbf{H}}_t^1 = [x_t, \sigma_t, r_t, v_t, u_t^\sigma]^\mathsf{T}$, the CF under risk neutral measure indicated by $\mathbb{Q}$, to be of the following form:

$$\begin{cases} \phi_{\text{H-G2++}}(\mathbf{u}, \widetilde{\mathbf{H}}_\tau^1, \tau) = & \mathrm{e}^{A(\mathbf{u},\tau)+\mathbf{B}^\mathsf{T}(\mathbf{u},\tau)\widetilde{\mathbf{H}}_0^1}, \\ \phi_{\text{H-G2++}}(\mathbf{u}, \widetilde{\mathbf{H}}_0^1, 0) = & \mathrm{e}^{i\mathbf{u}^\mathsf{T}\widetilde{\mathbf{H}}_0^1}. \end{cases}$$

$\Rightarrow$ Partially analytical solution for CF is available.

- for H-H2++ with $\widetilde{\mathbf{H}}_t^0 = [x_t, \sigma_t, r_t, v_t, u_t^\sigma, u_t^v, z_t]^\mathsf{T}$,

$$\begin{cases} \phi_{\text{H-H2++}}(\mathbf{u}, \widetilde{\mathbf{H}}_t^0, \tau) = & \mathrm{e}^{A(\mathbf{u},\tau)+\mathbf{B}^\mathsf{T}(\mathbf{u},\tau)\widetilde{\mathbf{H}}_0^0}, \\ \phi_{\text{H-H2++}}(\mathbf{u}, \widetilde{\mathbf{H}}_0^0, 0) = & \mathrm{e}^{i\mathbf{u}^\mathsf{T}\widetilde{\mathbf{H}}_0^0}. \end{cases}$$

$\Rightarrow$ Requires numerics for solving ODEs.

Characteristic Function of the Heston Hybrid Model

- Crucial in evaluating CF is fast calculation of the coefficients $A(u,\tau)$ and $B_i(u,\tau)$ for any $i \in \{1,\ldots,n\}$.

$$\frac{\mathrm{d}}{\mathrm{d}\tau}B_i(u,\tau) = f(\tau, B_1(u,\tau),\ldots,B_n(u,\tau)),$$

- Solve ODEs with explicit Runge-Kutta method;
  - ⇒ the method allows evaluating CF for a whole strip of u's.
- In order to gain speed use a pricing engine which requires as few points as possible:
  - ⇒ the Fourier Cosine expansion method by [Fang,Oosterlee-2008].

Series Coefficients of the Density and the CF

- Fourier-Cosine expansion of density function on interval $[a,b]$:

$$f(x) = {\sum_{n=0}^{\infty}}' F_n \cos\left(n\pi \frac{x-a}{b-a}\right),$$

with $x \in [a,b] \subset \mathbb{R}$ and the coefficients defined as

$$F_n := \frac{2}{b-a}\int_a^b f(x)\cos\left(n\pi\frac{x-a}{b-a}\right)dx.$$

- F_n has direct relation to ch.f., $\phi(\omega) := \int_{\mathbb{R}} f(x)e^{i\omega x}dx$ ($\int_{\mathbb{R}\setminus[a,b]} f(x) \approx 0$),

$$\begin{aligned} F_n \approx A_n &:= \frac{2}{b-a}\int_{\mathbb{R}} f(x)\cos\left(n\pi\frac{x-a}{b-a}\right)dx \\ &= \frac{2}{b-a}\Re\phi\left(\frac{n\pi}{b-a}\right)\exp\left(-i\frac{na\pi}{b-a}\right). \end{aligned}$$

Recovering Densities with Fourier Expansion Technique

- Replace F_n by A_n, and truncate the summation:

$$f(x) \approx \frac{2}{b-a}{\sum_{n=0}^{N-1}}' \Re\phi\left(\frac{n\pi}{b-a};x\right)\exp\left(in\pi\frac{-a}{b-a}\right)\cos\left(n\pi\frac{x-a}{b-a}\right),$$

- Example: $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$, $[a,b] = [-10,10]$ and $x = \{-5,-4,\cdots,4,5\}$.

N	4	8	16	32	64
error	0.2538	0.1075	0.0072	4.04e-07	3.33e-16
cpu time (sec.)	0.0025	0.0028	0.0025	0.0031	0.0032

Exponential error convergence in N.

Pricing European Options

- Start from the risk-neutral valuation formula:

$$v(x,t_0)=e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[v(y,T)|x\right]=e^{-r\Delta t}\int_{\mathbb{R}}v(y,T)f(y|x)dy.$$

- Truncate the integration range:

$$v(x,t_0)=e^{-r\Delta t}\int_{[a,b]}v(y,T)f(y|x)dy+\varepsilon.$$

- Replace the density by the COS approximation, and interchange summation and integration:

$$\hat{v}(x,t_0)=e^{-r\Delta t}{\sum_{n=0}^{N-1}}{}' \Re\phi\left(\frac{n\pi}{b-a};x\right)e^{-in\pi\frac{a}{b-a}}V_n,$$

where the series coefficients of the payoff, V_n, are analytic.

Pricing European Options

- Log-asset prices: $x := \ln(S_0/K) \quad \text{and} \quad y := \ln(S_T/K)$,
- The payoff for European options reads

$$v(y,T) \equiv [\alpha \cdot K(e^y - 1)]^+.$$

- For a call option:

$$\begin{aligned} V_k^{call} &= \frac{2}{b-a}\int_0^b K(e^y-1)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \frac{2}{b-a}K\left(\chi_k(0,b)-\psi_k(0,b)\right), \end{aligned}$$

- For a vanilla put:

$$V_k^{put} = \frac{2}{b-a}K\left(-\chi_k(a,0)+\psi_k(a,0)\right).$$

Numerical Example

Pricing for 21 strikes $K = 50, 55, 60, \cdots, 150$ under Heston's model. Other parameters: $S_0 = 100, r_t = 0$, $T = 1, \epsilon = 1.5768, \omega = 0.5751, \bar{\sigma} = 0.0398, \sigma_0 = 0.0175, \rho_{x,\sigma} = -0.5711$.

COS	no. of expansion terms N	96	128	160
	(msec.)	2.039	2.641	3.220
	max. abs. err.	4.52e-04	2.61e-05	$4.40e-06$

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Pricing European Options with the Hybrid Models

Table: The models were evaluated with the following parameters: $\epsilon = 1.2$, $\bar{\sigma} = 0.1$, $\omega = 0.05$, $\kappa = 1.5$, $\theta = 0.05$, $\eta = 0.1$, $\lambda = 1.5$, $\gamma = 0.1$, $\rho_{x,\sigma} = -0.4$, $\rho_{x,r} = 0.4$, $\rho_{x,v} = -0.6$, $\rho_{\sigma,r} = 0.1$,$\rho_{\sigma,v} = 0.2$, $\rho_{r,v} = 0.3$, and initials: $S_0 = 1$, $\sigma_0 = 0.05$, $r_0 = 0.03$, for H-G2++ $v_0 = 0$ and for H-H2++ $v_0 = 0.05$.

characteristic			number of expansion terms (N)		
model	**mat.**	**des.**	50	100	200
H-G2++	$\tau = 1y$	SSE	1.0661	5.756E-11	1.011E-16
		time [s]	0.073s	0.084s	0.093s
	$\tau = 10y$	SSE	2.352E-5	1.357E-11	1.263E-11
		time [s]	0.095s	0.105s	0.115s
H-H2++	$\tau = 1$	SSE	16.671	2.746E-8	1.045E-17
		time [s]	0.450s	0.512s	0.612s
	$\tau = 10y$	SSE	2.231E-3	8.479E-16	1.453E-16
		time [s]	0.817s	0.932s	1.023s

The prices were calculated for a whole strip of strikes i.e.,$K = \{0.1, 0.2, \ldots, 5\}$.

The Accuracy of Hybrid Approximation

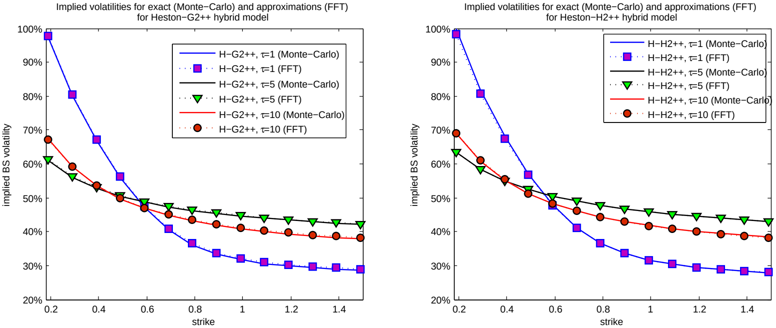

Figure: Implied BS volatilities for the Equity. LEFT: H-G2++, RIGHT: H-H2++ For both models the parameters were chosen to: $\epsilon = 1.2$, $\bar{\sigma} = 0.1$, $\omega = 0.05$, $\kappa = 1.5$, $\theta = 0.05$, $\eta = 0.1$, $\lambda = 1.5$, $\gamma = 0.1$, $\rho_{x,\sigma} = -0.4$, $\rho_{x,r} = 0.4$, $\rho_{x,v} = -0.6$, $\rho_{\sigma,r} = 0.1$, $\rho_{\sigma,v} = 0.2$ and $\rho_{r,v} = 0.3$. For the H-G2++ $v_0 = 0$, in H-H2++ $v_0 = 0.05$.

The Correlation Impact on Implied Volatilities

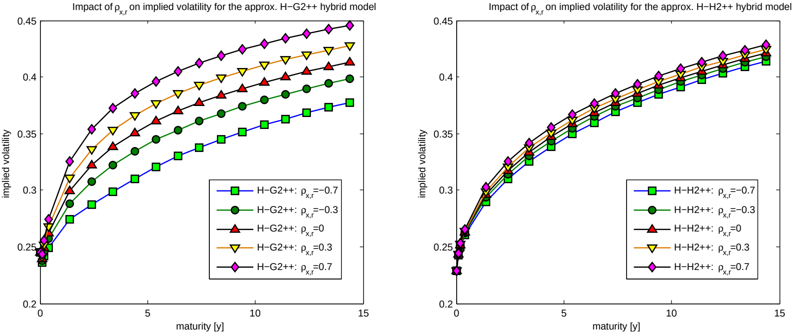

Figure: Effect of the hybrid correlation, $\rho_{x,r}$, between the stock and interest rate on Equity implied volatility. LEFT: H-G2++, Right: H-H2++.

Conclusions

Conclusions

- ⇒ We have build an approximation of the CF of the Heston-Hull-White two-factor and Heston-Stochastic volatility IR Hybrid Models;
- ⇒ With help of Fourier Cosine expansion method and Explicit Runge-Kutta method the pricing of European Options can be done in milliseconds;
- ⇒ The idea of extending the space-vector can be generalized to any Hybrid models involving correlated square-root processes.
- ⇒ Analytical CF can be obtained by setting some correlations to zero.

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