Asymptotics for local volatility and Sabr models

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Outline

1. Collaborators in work presented today
2. Outline of Work to be discussed
   - Overview
3. Background
   - Models
     - Local Volatility Models
     - Stochastic Volatility models
   - Methodology to be used
   - Curvature
4. Our Approach and Results
   - Local volatility Models revisited

Collaborators in work presented today

- Jim Gatheral, Merrill Lynch and Courant Institute
- Elton Hsu, Northwestern University
- Cheng Ouyang, Northwestern University
- Tai-Ho Wang, Baruch College, CUNY

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Outline of Results

Two lines:
Contributions of

- **Theoretical nature**
  Provide rigorous proofs of short time to maturity expansion formulas for i) call prices and ii) implied volatility in local volatility setting.
- **Practical Nature**
  New expansion formulas for call prices and implied volatility. I.e. expansion up to second order with optimal (in a certain sense) coefficients. Already order 1 more accurate for several models tested than earlier expansions tested.

$$\sigma_{BS}(t,T) = \sigma^{0}_{BS}(t) + \sigma^{(1)}_{BS}(t)(T-t) + \underbrace{\sigma^{(2)}_{BS}(t)(T-t)^2}_{\text{second order coefft}}$$

$$+o(T-t)^2$$

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Local volatility

The Local volatility model

$$dS_t = b(t)S_t dt + a(S_t, t)dW_t$$

where

- $\{S_t\}_{t\geq 0}$ is price process for the stock
- $\{W_t\}_{t\geq 0}$ is a Brownian motion. Pioneered by Bruno Dupire. Still popular model, some say, in certain (eg. French) banks.

Sabr type models

- Sabr Model in it's original form (Hagan and Woodward, Hagan, Kumar, Lesniewski and Woodward, Andreasen-Andersen)

$$\begin{aligned} dF_t &= F_t^{\beta} y_t dW_{1t} \\ dy_t &= \alpha y_t dW_{2t} \\ < dW_{1t}, dW_{2t} &>= \rho dt \end{aligned}$$

  Calibrates well to smile, but for only *one* maturity.

- **"Dynamic Sabr Model"**

$$\begin{aligned} dF_t &= \boxed{\gamma(t)} C(F_t) y_t dW_{1t} \\ dy_t &= \boxed{\nu(t)} y_t dW_{2t} \\ < dW_{1t}, dW_{2t} &>= \rho(t) dt, \end{aligned}$$

  time dependent parameters. Can be calibrated to implied volatility surface for several maturities.

Heston

Heston Model

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_t \\ dV_t &= \kappa(\theta - V_t) dt + \sigma \sqrt{V_t} dZ_t \\ dW_t dZ_t &= \rho dt \end{aligned}$$

where

- $\{S_t\}_{t \geq 0}$ and $\{V_t\}_{t \geq 0}$ are price and volatility processes
- $\{W_t\}_{t \geq 0}$ and $\{Z_t\}_{t \geq 0}$ are Wiener processes with correlation ρ
- θ is long-run mean, κ is the rate of reversion and σ is volatility of volatility

Heston + local vol

The Heston Model with local vol

$$dS_t = \mu S_t dt + \sqrt{V_t}\,\sigma(S_t, t)\,dW_t$$

$$dV_t = \kappa(\theta - V_t)dt + \bar{\sigma}\sqrt{V_t}dZ_t$$

$$< dW_t, dZ_t >= \rho dt$$

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- θ is long-run mean, κ is the rate of reversion and $\bar{\sigma}$ is volatility of volatility.

Andreasen and others.

Lipton-Andersen Quadratic SV Model

Lipton-Andersen Model

$$dS(t)$$
$$= \lambda(t)\sqrt{z(t)}\left(b(t)S(t) + (1-b(t))S_0 + \frac{1}{2}\frac{c(t)}{S_0}(S(t)-S_0)^2\right)dW_t$$
$$dz(t) = \kappa(1-z(t))dt + \eta(t)\sqrt{z(t)}dZ(t)$$
$$z(0) = 1$$

where

$$< dW(t), dZ(t) >= \rho dt$$

Needs adjustment at the wings, since local martingale but not a martingale in general.
Can be seen as special case of Heston-local vol model.

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Methods

Passage from stochastic volatility model to local vol model:

- Gyongy-Dupire-Derman and Britten-Jones and Neuberger method for reducing the computation of call prices in stochastic volatility model to computation of a effective local volatility in a local volatility model.
  Combine with:

Methods

Passage from stochastic volatility model to local vol model:

- Gyongy-Dupire-Derman and Britten-Jones and Neuberger method for reducing the computation of call prices in stochastic volatility model to computation of a effective local volatility in a local volatility model.
  Combine with:
- Heat kernel method for the determination of transition probability density in the local and stochastic volatility models.
  This reduction requires knowledge of the corresponding Riemannian distance function and/or geodesics for the SV model. These are known in Sabr models.

Gyongy-Dupire: From stochastic volatility to local volatility

Stochastic volatility models:

$$dF_t = \alpha_t b(F_t) dW_{1t}$$
$$d\alpha_t = g(\alpha_t) dW_{2t}$$
$$F_0 = F, \alpha_0 = \alpha \quad \text{initial conditions}$$
$$< dW_{1t}, dW_{2t} >= \rho dt$$

Obtaining a local volatility model with same F marginals:
"Equivalent" local volatility function is given by:

$$\sigma^2_{loc}(K, T) = b^2(K) E[\alpha_T^2 \mid F_T = K]$$

Gyongy-Dupire: Effective parameters

More general result, giving rise to the concept of "mimicking":

SV model

$$
\begin{aligned}
dS_t &= c(S_t, \nu_t, t)dt + b(S_t, t)g(\nu(t), t)dW_{1t} \\
d\nu_t &= \zeta(\nu_t)dt + \beta(\nu_t)dW_{2t}dt \\
&< dW_{1t}, dW_{2t} >= \rho dt \\
S(0) &= S, \quad \nu(0) = \nu,
\end{aligned}
$$

yields the same marginal distributions with respect to the S variable as the following sde:

$$
\begin{aligned}
dS_t &= \gamma(S, t)dt + \sigma(S_t, t)d\bar{W}_t, \\
S(0) &= S
\end{aligned}
$$

where, effective parameters are $\sigma^2(K, T) = b^2(K, T)E\left[g^2 \mid S_T = K\right]$ and $\gamma(K, T) = E\left[c \mid S_T = K\right]$

Laplace asymptotics

Local volatility

Representation

$$\sigma^2(k,t) = \frac{\int_0^\infty y^2 p(t,(s_0,y_0),(k,y))dy}{\int_0^\infty p(t,(s_0,y_0),(k,y)dy}$$

Now use $p(t,(s_0,y_0),(k,y)) = \frac{1}{2\pi t} e^{-\frac{d_R^2((s_0,y_0),(K,y))}{2t}} f(K,y)$, where d_R is the natural Riemannian distance, and f comes from heat kernel expansion.

Apply Laplace asymptotics to express (for small t) in terms of

min

$$y_{min} = \text{argmin}_y \; d_R^2((s_0,y_0),(K,y))$$

Heat kernel Series solution for fundamental solution

Seek solution of *backward heat equation in* y, τ in the form:

Heat Kernel Series: Time homogeneous case

$$F(y,x,\tau) = \frac{\sqrt{g(x)}}{(2\pi\tau)^{n/2}}\sqrt{\Delta(x,y)}\mathcal{P}(x,y)e^{-\frac{d^2(x,y)}{2\tau}}\sum_{n=1}^{+\infty}U_n(x,y)\tau^n, \quad \tau \to 0$$

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where,

- $d(x,y)$ is the geodesic distance between x and y, i.e., minimizer of the functional

$$\int_0^1 g_{ij} \frac{d\bar{x}^i}{dt} \frac{d\bar{x}^j}{dt} dt$$
$$\bar{x}(0) = x \quad \bar{x}(1) = y,$$

where $g(x) = \det(g_{ij})$ and where
$g = a^{-1}$, here $a = \{a_{ij}\}$ is principal part of elliptic operator $a_{ij} \frac{\partial^2}{\partial x^i \partial x^j}$

Heat kernel ct'd

- $$f_\tau - \textcircled{a_{ij}} \frac{\partial^2}{\partial x_i \partial x_j} f - b_i \frac{\partial}{\partial x_i} f = 0$$
  Solution in the form :
  $$\frac{\sqrt{g(x)}}{(4\pi\tau)^{n/2}} \sqrt{\Delta(x,y)} \mathcal{P}(x,y) e^{-\frac{d^2(x,y)}{4\tau}} \sum_{n=1}^{+\infty} a_n(x,y)\tau^n, \quad \tau \to 0$$
- $$\Delta(x,y) = |g(x)|^{-1/2} \det \left(\frac{\partial \frac{d^2}{2}}{\partial x \partial y} \right) |g(y)|^{-1/2}$$ Van-Vleck-DeWitt determinant
- $\mathcal{P} =$ exponential of work done by field $\mathcal{A}$, $e^{\int_{C(x,y)} < \mathcal{A}, \mathbf{dl} >_{\mathcal{R}}}$
- $\mathcal{A}$ is constructed from PDE, using two ingredients: diffusion matrix and from the drift b, i.e.
  $$\mathcal{A}^i = b^i - \det(g)^{-1/2} \frac{\partial}{\partial x^j} \left(\det(g)^{1/2} g^{ij} \right)$$

Time inhomogeneous case

- Suppose the coefficients of the diffusion and/or drift *depend explicitly on time*. How does the heat expansion change?

Time inhomogeneous case

$$F(y,x,t,T) = \frac{\sqrt{g(x,T)}}{(4\pi(T-t))^{n/2}}\sqrt{\Delta(x,y,t)}\mathcal{P}(x,y,t)e^{-\frac{d^2(x,y,t)}{4(T-t)}} \times$$

$$\{\sum_{n=1}^{+\infty} U_n(x,y,t)(T-t)^n\},$$

$$\text{as } T-t \to 0$$

-

satisfies the backward Kolmogorov equation in the variables (y,t).

Finding the coefficients in the heat kernel expansion

- Zero-th order coefficient can be solved for in closed form only when we know the distance function in closed form. This is why Sabr model succeeds since in Sabr model Riemannian distance is diffeomorphic image of distance in the hyperbolic plane (In formula below $f^{\beta-1}$ is local vol, i.e.$df_t = f^{\beta} y_t dW_t$).

distance in diffeormorphic image of hyperbolic plane

$$d(X,Y) = \operatorname{arccosh}\left[1 + \frac{\left(\int_X^x \frac{1}{f^{\beta}} du^2\right)^2 - 2\rho(y-Y)\int_X^x \frac{1}{f^{\beta}} du + (y-Y)^2}{2(1-\rho^2)yY}\right]$$

- Coefficients in the heat equation satisfy the so-called transport equations, i.e. ordinary equations along the geodesics connecting points y and x. Cannot usually solve these in closed form but can Taylor expand for y close to x.

Heat kernel coefficients one or higher

One way to get around the inability to solve the transport equations explicitly proposed by Henry-Labordère:

- Use *on diagonal* (say first order) heat kernel coefficients: $U_1(x,x)$
- Approximate *off diagonal* heat kernel coefficient $U_1(x,y)$ by

$$U_1(x,y) = U_1\left(\frac{x+y}{2}, \frac{x+y}{2}\right)$$

This method works quite well when we are near the diagonal, i.e., y close to x.

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Influence of curvature

G. Ben Arous, P.L., Tai-Ho Wang

Theorem

- *Consider the SV model*

$$dx_t = b(x_t) y_t dW_{1t} + \mu_x dt$$
$$dy_t = \gamma y_t^{q+1} dW_{2t} + \mu_y dt$$
$$< dW_{1t}, dW_{2t} >= \rho dt$$

  where ρ and γ are constants. Then

- *The (Gaussian) curvature of the Riemannian metric naturally associated to the problem is independent of the factor $b(x)$ and independent of the correlation and of the drift.*
- *The curvature is equal to*

$$(q-1)y^{2q}$$

  Thus
- *The curvature is identically zero if and only if $q = 1$, ie. in the quadratic case, and is negative when $q < 1$.*

influence of curvature II: $(q-1)y^{2q}$

- When $q = 0$, the curvature is constant.
  This is the original lognormal Sabr model.
- When $q = -1$ i.e. Heston model, the curvature is negative and it blows up at $y = 0$. In fact the curvature blows up at $y = 0$ as soon as $q < 0$.
- Note: The sign and size of the curvature is important in the heat kernel asymptotic approach to the heat kernel.
  Here is why: On Riemannian manifolds of negative Riemannian curvature, the cut locus is empty.

Hyperbolic Space

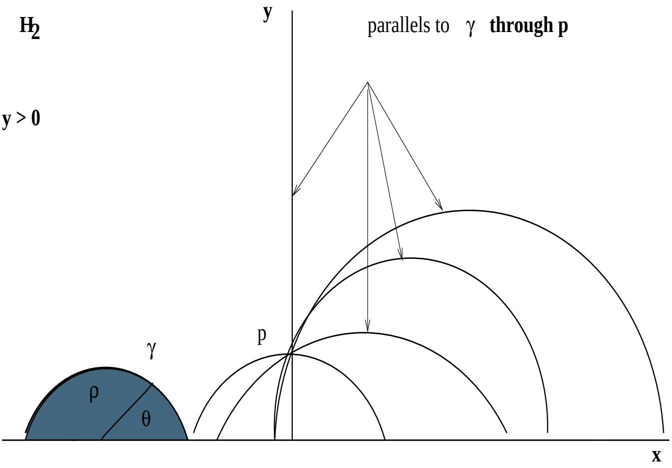
$$\mathcal{H} : ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$$

Space of constant negative Gaussian curvature Gc equal to -1:

$$Gc = \frac{1}{2H}\left\{\frac{\partial}{\partial u}\left[\frac{F}{EH}\frac{\partial E}{\partial v} - \frac{1}{H}\frac{\partial G}{\partial u}\right] + \frac{\partial}{\partial v}\left[\frac{2}{H}\frac{\partial F}{\partial u} - \frac{1}{H}\frac{\partial E}{\partial v} - \frac{F}{EH}\frac{\partial E}{\partial u}\right]\right\}$$

where $ds^2 = Edx^2 + 2Fdxdy + Gdy^2, \& H = \sqrt{EG - F^2}$ and where, in the case of hyperbolic plane : $E = G = \frac{1}{y^2}, \quad F = 0$

geodesics



Geodesics in the hyperbolic plane

PDE

If $0 < C_1 < \sigma < C_2$:

Berestycki, Busca Florent (2002, 2004) The implied volatility lies in $W^{1,2,p}$ for all $1 < p < \infty$ and satisfies the equation

$$2\tau\phi\phi_\tau + \phi^2 - \sigma^2(x,\tau)(1 - x\frac{\phi_x}{\phi})^2$$

$$-\sigma^2(x,\tau)\tau\phi\phi_{xx} + \frac{1}{4}\sigma^2(x,\tau)\tau^2\phi^2\phi_x^2 = 0$$

where $x = \log(\frac{Se^{r\tau}}{K})$. Also, short time limit

$$\lim_{\tau\to 0}\phi(x,\tau) = \frac{1}{\int_0^1 \frac{ds}{\sigma(sx,0)})}$$

Note, however, that BBF require the diffusions be non-degenerate,i.e. $\sigma(x,\tau) > C > 0$.

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Local Volatility Models revisited: Motivations

Highly accurate approximations for transition density in local volatility models of interest because

- Local volatility model of independent interest.
- Asymptotics for local volatility models when combined with Gyongy projection technique, provide highly accurate asymptotics for stochastic volatility models (two factor).

Heat Kernel coefficients time inhomogeneous case

- Heat kernel coefficients in time inhomogeneous case satisfy transport equations, i.e., first order, inhomogeneous ordinary differential equations along the geodesics associated to the natural Riemannian metric. In one-D can be integrated exactly. For example:

Heat Kernel coefficients time inhomogeneous case

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Coefficients

$$u_0(s,K,t) = \exp\left[-\int_K^s \frac{1}{d(K,\eta,t)}\left(-\frac{1}{2} + \frac{\frac{a^2}{2}(d^2)_{\eta\eta} + b(d^2)_\eta}{2} + \frac{(d^2)_t}{2}\right)\frac{d\eta}{a(\eta,t)}\right],$$

- $\mathcal{L} = \frac{1}{2}a(s,t)\frac{\partial^2}{\partial s^2} + b(s,t)\frac{\partial}{\partial S} + c(s,t)$.

$$d(s,K) = \int_s^K \frac{1}{a(u,t)}du$$

Example:Driftless one dimensional case

- $$Lu = \frac{1}{2} a^2(y,t) u_{yy}$$
- Heat kernel coefficients in closed form:

Coefficients: 1 D case

$$u_0 = \exp(\frac{1}{2} \log \frac{a(y,t)}{a(x,t)})$$

$$\times \exp\left(\int_x^y \left(\frac{1}{a(\tilde{y},t)} \int_x^{\tilde{y}} \frac{a_t}{a^2(u,t)} du\right) d\tilde{y}\right)$$

$$= \frac{\sqrt{a(y,t)}}{\sqrt{a(x,t)}} \exp\left(-\int_x^y \frac{b(y,t)}{a(y,t)^2}\right) \exp\left(\int_x^y \frac{1}{a(\tilde{y},t)}) \int_x^{\tilde{y}} \frac{a_t}{a^2(u,t)} du d\tilde{y}\right)$$

Heat Kernel coefficients 2

$b = c = 0$ in PDE & a independent of time, obtain the following integral:

off diagonal: time homegeneous case

$$u_1(x,y)$$
$$\frac{1}{4}U_0 \frac{1}{\int_x^y \frac{1}{a(u)}du} \int_x^y (a_{\bar{y}\bar{y}} - \frac{1}{2}\frac{a_{\bar{y}}^2}{a})d\bar{y}$$
$$= \frac{1}{4} \underbrace{\frac{1}{\int_x^y \frac{1}{a(u)}}}_{\text{harmonic mean of volatility}} \left\{ \frac{\sqrt{a(y)}}{\sqrt{a(x)}} \left(a_y(y) - a_y(x) - \frac{1}{2}\int_x^y \frac{a_{\bar{y}}^2}{a} \right) d\bar{y} \right\} \quad (1)$$

Solving for implied volatility

Use Dupire-Derman-Kani to express Call Prices in the form:

Call price asymptotics

$$Call(y,x,t,T) = (y-x)^{+} + \frac{1}{2}E\left[\int_t^T \left[a^2 \mathbf{1}_{y_t=x}\right]\right]$$

$$= (y-x)^{+} + \frac{1}{2}\int_t^T a^2(x,u)\left(\frac{1}{a((x,T)}\frac{1}{(4\pi(u-t))^{1/2}}e^{-\frac{d^2}{4(u-t)}}\left[u_0(x,y,t)\right.\right.$$

$$\left.\left. +(u-t)u_1(x,y,t)+(u-t)^2u_2(x,y,t)+\ldots)\right]\right)du$$

Call Prices: Illustration in time homogeneous case

Grouping powers of $T - t = \bar{T}$, this leads to expressions for call prices of the form

Call Prices

$$C(y,x,\bar{T}) = (y-x)^+ + \frac{1}{2\sqrt{2\pi}}(U_0(x,y)u_0(x,y,\bar{T}) + \dots)$$
$$= (y-x)^+ + \frac{1}{2\sqrt{2\pi}}\sum_{i=0}^{n} \underbrace{u_i(x,y)}_{\text{heat kernel coefficients}} \quad U_i(x,y,\bar{T})$$

where

$$U_i(x,y,t) = \int_0^{\bar{T}} (\sqrt{v})^{2i-1} e^{-\frac{\omega^2}{v}} dv$$
$$\omega = \frac{1}{\sqrt{2}} \int_x^y \frac{1}{a(u)} du$$

matching

- Recall $\omega = \int_x^y \frac{1}{a(u,t)} du \sim$ "$d(x,y,t)$".
- In Black-Scholes setting $\bar{\omega} = \frac{\log(\frac{y}{x})}{\sigma_{BS}^2}$. In regime $\frac{\omega^2}{\bar{T}} >> 1$, the auxiliary function U_1 (expressible in terms of erfc (complimentary error function)) admits asymptotic expansions

$$U_0(\omega, \bar{T}) = \sim \frac{\bar{T}^{3/2}}{\omega^2(t)} e^{-\frac{\bar{\omega}^2}{\bar{T}}}, U_1 \sim \frac{\bar{T}^{5/2}}{\omega^2(t)} e^{-\frac{\bar{\omega}^2}{\bar{T}}}$$

This leads to following matching:

Matching

$$\overbrace{\sigma_{BS}\sqrt{xy}e^{-\frac{\bar{\omega}^2}{\bar{T}}}\,\bar{T}^{3/2}\frac{1}{\bar{\omega}^2}+\dots}^{\text{Black Scholes price}}$$

$$= \overbrace{\sqrt{a(x,t)a(y,t)}e^{-\frac{\omega^2(t)}{\bar{T}}}\,\bar{T}^{3/2}\frac{1}{\omega^2(t)}+\dots}^{\text{local vol price}}$$

Matching continued: transcendent matching vs algebraic matching

- (Transcendental matching) Exponential contributions on both sides must balance:

  $\Rightarrow$zero-th order exponents of exponentials must match

- (Algebraic Matching) Once zero-th order exponents match, match like powers of $\bar{T}$ on both sides.

Results Transcendental matching leads to Berestycki-Busca-Florent formula, in the time homogeneous case and to a slightly different formula in time inhomogeneous case.

Implied volatility expansion

- Expansion

$$\sigma_{BS}(S_0, K, t, T) = \sigma_{BS}^{(0)}(t) + \overbrace{\sigma_{BS}^{(1)}(t)}^{\frac{d\sigma_{BS}}{dT}|_{T=t}}(T-t) + \dots$$

- (Zero-th order "generalized BBF")

zero order

$$\sigma_{BS}^{(0)}(t) = \frac{\log(\frac{S_0}{K})}{\int_K^{S_0} \frac{1}{a(u,t)} du}$$

Note, to recover optimal expansions, volatility $a(u,\cdot)$ needs to be evaluated at t, not at T, in time inhomogeneous case.

Implied volatility expansion

first order time homogeneous, $r \neq 0$

$$\sigma^{(1)}(S_0, K) \qquad \text{here } \sigma^{(0)} = \text{BBF}$$

$$= \frac{\log\left(\frac{\sqrt{a(K))a(S_0))}}{\sqrt{S_0 K}\log(S_0/K)} \int_K^{S_0} \frac{1}{a(u)} du\right) + r \int_K^{S_0} \frac{1}{(\sigma^{(0)})^2 u} du - r \int_K^{S_0} \frac{u}{a^2(u)} du}{\frac{(\int_K^{S_0} \frac{1}{a(u)} du)^3}{\log(S_0/K)}}$$

Implied volatility expansion

first order time homogeneous, $r \neq 0$

$$\sigma^{(1)}(S_0, K) \qquad \text{here } \sigma^{(0)} = \text{BBF}$$

$$= \frac{\log\left(\frac{\sqrt{a(K))a(S_0))}}{\sqrt{S_0 K}\log(S_0/K)} \int_K^{S_0} \frac{1}{a(u)} du\right) + r\int_K^{S_0} \frac{1}{(\sigma^{(0)})^2 u} du - r\int_K^{S_0} \frac{u}{a^2(u)} du}{\frac{(\int_K^{S_0} \frac{1}{a(u)} du)^3}{\log(S_0/K)}}$$

ATM

$$\Rightarrow \sigma_{\mathrm{BS},1} = \frac{2}{3K}(a_t + au_1) + \frac{1}{12}(\frac{a(K,t)}{K})^3$$

- Compare with Hagan-Woodward formula (for $r = 0$, with $S_{av} = \frac{S_0 + K}{2}$)

$$\sigma^{(0)}\left(1 + \underbrace{\left[\frac{a^2(S_{av})}{24}\left[2\frac{a''(S_{av})}{a(S_{av})} - (\frac{a'(S_{av})}{a(S_{av})})^2 + \frac{1}{S_{av}^2}\right]\right]T}_{\sigma^{(1)}(S_0,K)}\right)$$

Comparisons

Henry-Labordère refinement of Hagan-Woodward

Labordere refinement

$$\sigma^{(0)}\left(1+\left[\frac{1}{24}(\sigma^{(0)})^2+\frac{a^2(S_0)}{4}\left((\frac{a''(S_0)}{a(S_0)})-\frac{1}{2}(\frac{a'(S_0)}{a(S_0)})^2\right)\right]T\right)$$

- (Analytical comparison): Labordère's and Hagan et al.'s σ_1 involves derivatives of the volatility. Optimal σ_1 does not $\Rightarrow$ Better stability properties.
- Labordère σ_1 involves $U_1(x,y)$, first order term in heat kernel expansion. Optimal σ_1 involves only U_0. In fact this pattern holds throughout:

$$\text{Optimal } \sigma_i \quad \text{involves only } U_{i-1}$$

Numerical comparison

Performance in CEV model: $dS = \sigma \sqrt{S\, S_0}\, dZ$

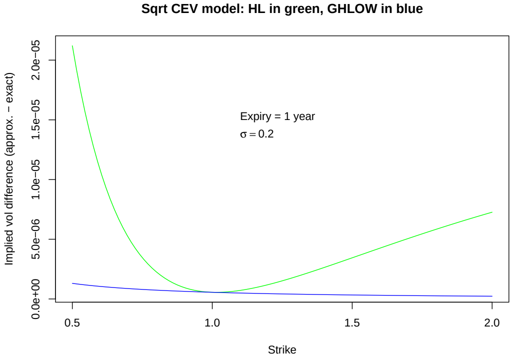
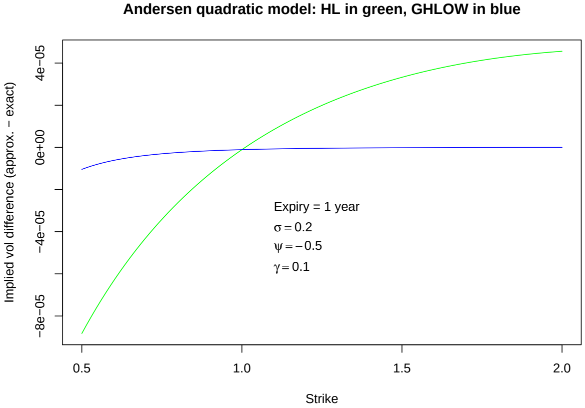


Figure: Comparison CEV, $\beta = \frac{1}{2}$ $\sigma = .2$, $S_0 = 1$

Numerical comparison

Performance in Andersen model:

$$dS = \sigma \left\{ \psi S + (1-\psi) S_0 + \frac{\gamma}{2} \frac{(S-S_0)^2}{S_0} \right\} dZ$$



Tables

Performance in Andersen model:

$$dS = \sigma \left\{ \psi\, S + (1-\psi)\, S_0 + \frac{\gamma}{2}\, \frac{(S-S_0)^2}{S_0} \right\} dZ$$

Strike	$\Delta\sigma_{HL}$	$\Delta\sigma_{GHLOW}$	σ_{exact}	σ_{HL}	σ_{GHLOW}
0.50	8.8E-05	-1.0E-05	31.29%	31.28%	31.29%
0.75	-3.4E-05	-3.0E-06	24.51%	24.50%	24.51%
1.00	-1.1E-06	-1.1E-07	20.03%	20.03%	20.03%
1.25	2.0E-05	-4.3E-07	16.75%	16.75%	16.75%
1.50	3.3E-05	-1.8E-07	14.18%	14.19%	14.18%
1.75	4.1E-05	-7.6E-08	12.09%	12.09%	12.09%
2.00	4.6E-05	-3.2E-08	10.32%	10.33%	10.32%

Table: Quadratic model, $\sigma = .2, T = 1, S_0 = 1, \psi = -.5, \gamma = 1$

Tables

Performance in CEV model:$dS = \sigma \sqrt{S\, S_0}\, dZ$

Strike	$\Delta\sigma_{HL}$	$\Delta\sigma_{GHLOW}$	σ_{exact}	σ_{HL}	σ_{GHLOW}
0.50	2.1E-05	1.3E-06	23.68%	23.68%	23.68%
0.75	3.5E-06	8.0E-07	21.48%	21.48%	21.48%
1.00	5.6E-07	1.1E-07	20.01%	20.01%	20.01%
1.25	1.5E-06	4.2E-07	18.91%	18.91%	18.91%
1.50	3.4E-06	3.3E-07	18.05%	18.05%	18.05%
1.75	5.5E-06	2.7E-07	17.34%	17.34%	17.34%
2.00	7.3E-06	2.3E-07	16.74%	16.74%	16.74%

Table: CEV, $\beta = 1/2, \sigma = .2, T = 1, S_0 = 1, \psi = -.5, \gamma = .1$

Conclusion

- Heat kernel expansion can be used to obtain highly accurate implied volatility.
- Enhanced accuracy of implied volatility expansions due to correct matching after use of off-diagonal heat kernel coefficients.
- Proper expansions involve regimes: No single expansion is best for all regimes!
- Optimal expansions for three factor models a challenge for the future.
- Terms in the expansions correspond to derivatives as function of final time for fixed spot and this can be established rigorously!