Scale functions for spectrally negative Lévy processes and their appearance in economic models

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[1] This is a review talk and not all my own work. The vast majority of what I say leans on collaborative with and independent work of the following individuals F. Hubalek, A. Lambert, R. Loeffen, P. Patie, M.Pistorius, Z. Palmowski, J-F. Renaud, V. Rivero, R. Song, X. Zhou.

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  i.e. the ruin probability with initial position x is $1 - mW(x)$.
- **From now I will call W a scale function.**

Is there a better characterization of scale functions?

- Take its Laplace-Steiltjes transform

$$\begin{aligned}\int_0^\infty e^{-\beta x} W(x) dx &= \frac{1}{m} \int_0^\infty e^{-\beta x} \mathbb{P}_0(-\underline{X}_\infty \leq x) dx \\ &= \frac{1}{\beta m} \mathbb{E}_0(e^{\beta \underline{X}_\infty})\end{aligned}$$

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- The Laplace transform on the right hand side should remind you of the Wiener-Hopf factorization and in fact it has been known for a very long time that

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- In conclusion

$$\int_0^\infty e^{-\beta x} W(x)dx = \frac{1}{\psi(\beta)}$$

i.e. we are just one Laplace inversion away from the ruin probability.

What else can you do with scale functions?

- Define

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- In other words we recover a classical formula of Takács and Zoletaraev

$$\mathbb{P}_x(\tau_a^+ < \tau_0^-) = \frac{W(x)}{W(a)}$$

- Push things a little further by changing measure using the transform $(c \geq 0)$

$$\left.\frac{d\mathbb{P}^c}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{cX_t - \psi(c)t}$$

for $c = \Phi(q)$, the largest root of the equation $\psi(\theta) = q$ then,

$$\mathbb{E}_x(e^{\Phi(q)(X_{\tau_a^+} - x) - q\tau_a^+}; \tau_a^+ < \tau_0^-) = \frac{W_{\Phi(q)}(x)}{W_{\Phi(q)}(a)}$$

where $W_{\Phi(q)}$ plays the role of W under $\mathbb{P}^{\Phi(q)}$.

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- **We call $W^{(q)}$ the q-scale function and we are only ever one inverse Laplace transform away from it.**

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- We have extracted a nice formula for the two sided exit problem and one might hope to see better *fluctuation identities* appearing. Such formulae could easily feed into a number of settings for financial and insurance models.
- **So how far do things go?**

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$$K_x^{(q)}(du,\, dv,\, dy) := \mathbb{E}_x \left(e^{-q\tau}; -X_\tau \in du, X_{\tau-} \in dv, \underline{X}_{\tau-} \in dy \right)$$

for $u, y \geq 0$ and $v \geq y$.

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- Then
$$\begin{aligned} & K_x^{(q)}(du,\, dv,\, dy) \\ = \quad & e^{-\Phi(q)(v-y)}\{W^{(q)\prime}(x-y) - \Phi(q)W^{(q)}(x-y)\}\Pi(du+v)dydv \end{aligned}$$
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- If we are just interested in the undershoot-overshoots (i.e. $q = 0$) then we have

$$K_x(du,\, dv,\, dy) = W'(x-y)\Pi(du+v)dydv$$

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- The evaluation of the latter can be computed from the triple law (or indeed more directly using scale functions) and the optimized over y to give

$$v(x) = K - e^{x-x^*} + qK \int_0^{x-x^*} W^{(q)}(z)dz - (q - \psi(1)) \int_0^{x-x^*} e^{x-x^*-z} W^{(q)}(z)dz$$

where

$$x^* = \log\left\{K \frac{q}{\Phi(q)} \frac{\Phi(q)-1}{q-\psi(1)}\right\}$$

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$$u(x) = \sup_{\tau} \mathbb{E}^{*}(e^{-p\tau} e^{(x \vee \overline{X}_{\tau} - X_{\tau})})$$

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- Again common sense and rigour tells us that the optimal strategy is first passage of $x \vee \overline{X}_\tau - X_\tau$ over a level. One computes in the end

$$u(x) = e^x + qe^x \int_0^{x^* - x} W^{(q)}(z)dz$$

where

$$x^* = \inf\{x \geq 0 : 1 + q \int_0^x W^{(q)}(z)dz \leq qW^{(q)}(x)\}.$$

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- Then, for $q \geq 0$, its Laplace-transform is given by

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- Moreover the net present value of dividends with discount rate $p > 0$ has moments given by

$$\mathbb{E}_x\left(\left\{\int_0^\sigma e^{-pt}dL_t\right\}^n\right) = n!\frac{W^{(pn)}(x)}{W^{(pn)}(a)}\prod_{k=1}^n \frac{W^{(pk)}(a)}{W^{(pk)\prime}(a)}$$

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- **How explicit can one be with scale functions?**

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- The Wiener-Hopf factorization tells us that necessarily $\psi(\beta) = \beta\phi(\beta)$ where

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- Classical theory now identifies $W'(x)dx$ as the renewal measure of the subordinator H about which a lot is known.

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- Instead of choosing the Lévy process (ie. choosing ψ) choose a subordinator (ie. choose ϕ) whose renewal measure is known and build an associated Lévy process by defining its Laplace exponent $\psi(\beta) = \beta\phi(\beta)$.

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- This method gives us a rich supply of scale functions W but not too many concrete examples of q-scale functions $W^{(q)}$. The fundamental difficulty being that the latter two are related via an exponential change of measure involving $\Phi(q)$, the inverse of $\psi(\beta)$ which is not always explicitly available.

Concrete example of the triple law: I

- In this example we build a spectrally negative Lévy process with Laplace exponent

$$\psi(\theta) = \theta\Gamma(\nu + \theta + \lambda)/c\Gamma(\nu + \theta),$$

where $\theta \geq 0$, $c > 0$, $\nu \geq 0$ and $\lambda \in (0, 1)$.

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$$\Pi(x,\infty) = \frac{\lambda}{c^2\Gamma(1-\lambda)} \frac{e^{x(1-\nu)}}{(e^x-1)^{\lambda+1}}.$$
- A straightforward calculation shows that
$$\psi'(0+) = \mathbb{E}(X_1) = \Gamma(\nu+\lambda)/c\Gamma(\nu) \geq 0$$

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$$\Pi(x,\infty) = \frac{\lambda}{c^2\Gamma(1-\lambda)}\frac{e^{x(1-\nu)}}{(e^x-1)^{\lambda+1}}.$$
- A straightforward calculation shows that
$$\psi'(0+) = \mathbb{E}(X_1) = \Gamma(\nu+\lambda)/c\Gamma(\nu) \geq 0$$
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Concrete example of the triple law: I

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- And
$$\mathbb{P}_x(-X_\tau > u, X_{\tau-} \in dv, \underline{X}_{\tau-} \in dy) = \frac{\lambda}{\Gamma(1-\lambda)\Gamma(\lambda)} \frac{e^{-(x-y)\nu}(1-e^{-(x-y)})^{\lambda-1}e^{(u+v)(1-\nu)}}{(e^{(u+v)}-1)^{\lambda+1}}dydv.$$

Concrete example of the triple law: II

- In this example we are able to handle the classical tempered stable process (one sided CGMY)

$$\psi(\theta) = (\theta + c)^{\alpha} - c^{\alpha},$$

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- Then

$$W^{(q)}(x) = e^{-cx} x^{\alpha-1} \mathcal{E}_{\alpha,\alpha}((q + c^\alpha) x^\alpha),$$

where

$$\mathcal{E}_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)}$$

is the two-parameter Mittag-Leffler function.

ctd....

Finally we have the full triple law

$$
\begin{aligned}
&\mathbb{E}_x(e^{-q\tau}; -X_\tau \in du, X_{\tau-} \in dv, \underline{X}_{\tau-} \le y) \\
&\quad = \frac{1}{\Gamma(-\alpha)} e^{((q+c^\alpha)^{1/\alpha} - c)(x-v)} \frac{e^{-c(u+v)}}{(u+v)^{1+\alpha}} \\
&\quad \times \Big\{ e^{-(q+c^\alpha)^{1/\alpha} x} x^{\alpha-1} \mathcal{E}_{\alpha,\alpha}((q+c^\alpha)x^\alpha) \\
&\qquad\qquad - e^{-(q+c^\alpha)^{1/\alpha}(x-y)} (x-y)^{\alpha-1} \mathcal{E}_{\alpha,\alpha}((q+c^\alpha)(x-y)^\alpha) \Big\} \, du dv
\end{aligned}
$$