Affine Transform Analysis and Asset Pricing

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Outline

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Definition

- X Feller Markov process
- State space: convex cone $E \subset \mathbb{R}^d$
- Examples: $E = \mathbb{R}_+^m \times \mathbb{R}^n$, $E = \mathbb{S}_+^m$

Definition

X is affine if the affine transform formula (ATF) holds:

$$\mathbb{E}_x \left[\mathrm{e}^{u \cdot X_t} \right] = \mathrm{e}^{\phi(t,u) + \psi(t,u) \cdot x}, \quad u \in \mathrm{i}\mathbb{R}^d$$

for some continuous functions ϕ, ψ.

Stylized Meta Theorem

Theorem

X is affine if and only if its generator is affine:

$$\mathcal{A}f(x) = (a + \alpha \cdot x) \cdot \partial^2 f(x) + (b + \beta \cdot x) \cdot \partial f(x) - (c + \gamma \cdot x)\, f(x)$$
$$+ \int (f(x+\xi) - f(x) - \partial f(x) \cdot \chi(\xi))\,(m(d\xi) + x \cdot \mu(d\xi))\,.$$

Moreover, the generalized Riccati equations (GREs) hold:

$$\partial_t \phi = F(\psi), \quad \phi(0,u) = 0$$
$$\partial_t \psi = R(\psi), \quad \psi(0,u) = u$$

where

$$F(u) = a \cdot u^2 + b \cdot u - c + \int \left(\mathrm{e}^{u\cdot\xi} - 1 - u \cdot \chi(\xi)\right) m(d\xi)$$
$$R(u) = \alpha \cdot u^2 + \beta \cdot u - \gamma + \int \left(\mathrm{e}^{u\cdot\xi} - 1 - u \cdot \chi(\xi)\right) \mu(d\xi)$$

Discounting: Feynman–Kac

- X affine process
- Short rate: $r(t) = \ell + \lambda \cdot X(t)$

Theorem

The extended process $(X(t), \rho + \int_0^t r(s)\,ds)$ is affine on $E \times \mathbb{R}$ with ATF

$$\mathbb{E}_x \left[\mathrm{e}^{-\int_0^t r(s)ds} \, \mathrm{e}^{u \cdot X(t)} \right] = \mathrm{e}^{\Phi(t,u) + \Psi(t,u) \cdot x}$$

where Φ and Ψ satisfy the extended GREs

$$\partial_t \Phi = F(\Psi) - \ell, \quad \Phi(0,u) = 0$$
$$\partial_t \Psi = R(\Psi) - \lambda \cdot \Psi, \quad \Psi(0,u) = u$$

on the domain of validity $u \in \mathcal{U} \subset \mathbb{C}^d$.

Affine Asset Pricing

- Financial payoff $H(X(t))$ at t
- Price

$$\pi(t,x) = \mathbb{E}_x\left[\mathrm{e}^{-\int_0^t r(s)ds} H(X(t))\right]$$

- Assume

$$H(x) = \int_{\mathbb{R}^q} \mathrm{e}^{(v+\mathrm{i}L\lambda)\cdot x}\widetilde{H}(\lambda)\,d\lambda$$

for some $\widetilde{H} \in L^1(\mathbb{R}^q)$, $v \in \mathcal{U}$, $L \in \mathbb{R}^{d\times q}$, and $q \leq d$

- Then Fubini's theorem gives:

$$\begin{aligned}
\pi(t,x) &= \mathbb{E}_x\left[\mathrm{e}^{-\int_0^t r(s)ds} \int_{\mathbb{R}^q} \mathrm{e}^{(v+\mathrm{i}L\lambda)\cdot X(t)}\widetilde{H}(\lambda)\,d\lambda\right] \\
&= \int_{\mathbb{R}^q} \mathbb{E}_x\left[\mathrm{e}^{-\int_0^t r(s)ds}\mathrm{e}^{(v+\mathrm{i}L\lambda)\cdot X(t)}\right]\widetilde{H}(\lambda)\,d\lambda \\
&= \int_{\mathbb{R}^q} \mathrm{e}^{\Phi(t,v+\mathrm{i}L\lambda)+\Psi(t,v+\mathrm{i}L\lambda)\cdot x}\widetilde{H}(\lambda)\,d\lambda
\end{aligned}$$

Fourier Transform on $L^1(\mathbb{R}^q)$

- Suppose

$$H(x) = \mathrm{e}^{v\cdot x} h(L^\top x)$$

for $v \in \mathcal{U}$ and $h \in L^1(\mathbb{R}^q)$ with Fourier transform

$$\widehat{h}(\lambda) = \int_{\mathbb{R}^q} \mathrm{e}^{-\mathrm{i}\lambda\cdot y} h(y)\, dy \in L^1(\mathbb{R}^q)$$

- Then

$$\widetilde{H}(\lambda) = \frac{1}{(2\pi)^q} \widehat{h}(\lambda)$$

does the job:

$$H(X) = \mathrm{e}^{v\cdot x} \frac{1}{(2\pi)^q} \int_{\mathbb{R}^q} \mathrm{e}^{\mathrm{i}\lambda\cdot L^\top x} \widehat{h}(\lambda)\, d\lambda$$

Examples

- Call option: $K > 0$, $w > 1$

$$\int_{\mathbb{R}} \mathrm{e}^{(w+\mathrm{i}\lambda)y} \frac{K^{-(w-1+\mathrm{i}\lambda)}}{2\pi(w+\mathrm{i}\lambda)(w-1+\mathrm{i}\lambda)} d\lambda = (\mathrm{e}^{y} - K)^{+}$$

- Carr–Madan (99), Hubalek–Kallsen–Krawczyk (06), ...
- Exchange option: choose $K = \mathrm{e}^{z}$, $w > 1$

$$\int_{\mathbb{R}} \frac{\mathrm{e}^{\left(\begin{bmatrix} w \\ 1-w \end{bmatrix} + \mathrm{i}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\lambda\right)\cdot\begin{bmatrix} y \\ z \end{bmatrix}}}{2\pi(w+\mathrm{i}\lambda)(w-1+\mathrm{i}\lambda)} d\lambda = (\mathrm{e}^{y} - \mathrm{e}^{z})^{+}$$

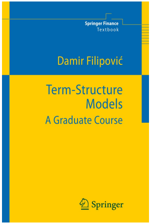
Examples

- Spread option: $w_2 < 0$, $w_1 + w_2 > 1$

$$
\begin{aligned}
&(\mathrm{e}^{y_1} - \mathrm{e}^{y_2} - 1)^+ \\
&= \int_{\mathbb{R}^2} \mathrm{e}^{(w+\mathrm{i}\lambda)\cdot y} \frac{\Gamma(w_1 + w_2 - 1 + \mathrm{i}(\lambda_1 + \lambda_2))\,\Gamma(-w_2 - \mathrm{i}\lambda_2)}{(2\pi)^2\Gamma(w_1 + 1 + \mathrm{i}\lambda_1)}\, d\lambda_1 d\lambda_2
\end{aligned}
$$

- Hurd–Zhou (09)

- Overview of affine asset pricing: e.g. Chapter 10 in



Outline

(T, x)-Bonds

- Sidenius–Piterbarg–Andersen (08), Ehlers–Schönbucher (05,09), Filipović–Overbeck–Schmidt (09)
- Normalized CDO loss process $L(t) = \sum_{s \leq t} \Delta L(s)$
- (T, x)-bond pays $1_{\{L(T) \leq x\}}$ at maturity T
- Time $t \leq T$ price:

$$P(t, T, x) = 1_{\{L(t) \leq x\}} \mathrm{e}^{-\int_t^T f(t,T,x) dx}$$

- Contingent claim $H(L(T))$ decomposed

$$H(L(T)) = H(1) - \int_0^1 H'(x) 1_{\{L(T) \leq x\}} \, dx$$

- Static replication portfolio at $t \leq T$:

$$H(1) P(t, T, 1) - \int_0^1 H'(x) P(t, T, x) \, dx$$

No-arbitrage Conditions

- Exogenous volatility specification

$$df(t,T,x) = a(t,T,x)\,dt + b(t,T,x)\,dW$$

- No-arbitrage: conditions for $f(t,T,x)$-drift

$$a(t,T,x) = b(t,T,x) \cdot \int_t^T b(t,s,x)\,ds$$

and $L(t)$-compensator

$$\nu(t,dx) = -f(t,t,L(t)+dx)$$

- Existence theorem (incl. contagion)

Affine Term Structure

- Affine term structure

$$\int_t^T f(t,s,x)\,ds = A(t,T,x) + B(t,T,x)\cdot Z(t)$$

for some diffusion Z in $E \subset \mathbb{R}^d$

Theorem

Affine term structure holds if and only if Z is affine:

$$\mu(z) = \mu_0 + \sum_{i=1}^d z_i \mu_i, \quad \frac{1}{2}\sigma\cdot\sigma^\top(z) = \nu_0 + \sum_{i=1}^d z_i \nu_i$$

and A, B solve Riccati equations:

$$-\partial_t A(t,T,x) = A'(t,t,x) + \mu_0 \cdot B(t,T,x) - B(t,T,x)\cdot\nu_0\cdot B(t,T,x)$$
$$-\partial_t B_i(t,T,x) = B_i'(t,t,x) + \mu_i \cdot B(t,T,x) - B(t,T,x)\cdot\nu_i\cdot B(t,T,x)$$

with $A(T,T,x) = 0$, $B(T,T,x) = 0$.

Outline

Problems

- Necessary and sufficient conditions on parameters

$$a, \alpha, b, \beta, c, \gamma, m, \mu, \quad (\ell, \lambda)$$

  for existence of X?
- Interplay with state space E?
- Domain of validity $u \in \mathcal{U} \subset \mathbb{C}^d$ where (extended) ATF holds?
- Invariance properties, canonical forms?

- Solved in
  - $E = \mathbb{R}_+^m \times \mathbb{R}^n$: Duffie–Filipović–Schachermayer (03)
  - $E = \mathbb{S}_+^m$: Cuchiero–Filipović–Mayerhofer–Teichmann (09)

Admissibility Conditions

Theorem (Stylized Version)

There exists a unique affine process with parameters a, α, b, β, c, γ, m, μ if and only if

- $a + \alpha \cdot x,\ c + \gamma \cdot x,\ m + \mu \cdot x \geq 0$ *for all* $x \in E$, *and*
- E *is invariant under* X*-dynamics.*

Domain of Validity

- By definition, extended ATF holds for $(u, v) \in \mathrm{i}\mathbb{R}^{d+1}$:

$$\mathbb{E}_x\left[\mathrm{e}^{\,v\int_0^t(\ell+\lambda\cdot X(s))ds}\,\mathrm{e}^{\,u\cdot X(t)}\right] = \mathrm{e}^{\,\phi(t,u,v)+\psi(t,u,v)\cdot x}$$

  where

$$\begin{aligned}\partial_t\phi &= F(\psi) + v\ell, \quad \phi(0,u,v) = 0\\ \partial_t\psi &= R(\psi) + v\lambda, \quad \psi(0,u,v) = u\end{aligned}$$

- Note: $\Phi(t,u) = \phi(t,u,-1)$, $\Psi(t,u) = \psi(t,u,-1)$
- Hence problem reduced to ATF in $\mathbb{R}^d$

Theorem

If $\phi(t,\cdot)$, $\psi(t,\cdot)$ admit analytic extension to 0-star-shaped open $\mathcal{U} \subset \mathbb{R}^d$ then ATF holds for $u \in \mathcal{U} + \mathrm{i}\mathbb{R}^d$.

Domain of Validity

- Maximal domain for GRE: $\mathbb{D} = \{(t,u) \in \mathbb{R}_+ \times \mathbb{R}^d \mid \psi(t,u) \text{ exists}\}$
- Claim 1: $\mathbb{D}(t) = \{u \mid (t,u) \in \mathbb{D}\}$ is 0-star-shaped and

$$\lim_{\theta u \to \partial \mathbb{D}(t)} \|\psi(t, \theta u)\| = \infty$$

- Claim 2: $\mathcal{M}(t) = \mathbb{D}(t)$ where

$$\mathcal{M}(t) = \left\{ u \in \mathbb{R}^d \mid \mathbb{E}_x \left[\mathrm{e}^{\,u \cdot X(t)} \right] < \infty \right\}$$

- Proved for diffusion in $E = \mathbb{R}_+^m \times \mathbb{R}^n$: Glasserman–Kim (09), Filipović–Mayerhofer (09), Filipović (09)
- In general: work in progress (jumps!)

Outline

Applications in Finance

- Let X be $\mathbb{S}_+^m$-valued affine
- Multi-asset model: stochastic correlation

$$d \log S = \left(r\mathbf{1} - \frac{1}{2} X^{\mathrm{diag}} \right) dt + \sqrt{X}\, dW$$

- Nonnegative affine term structure models with stochastically correlated risk factors:

$$r(t) = \ell + \langle \lambda, X(t) \rangle$$

- etc.

Some Related Literature

- Bru (91): Wishart pure diffusion
- Gourieroux–Sufana (03,04): Wishart pure diffusion in Finance
- Barndorff–Nielsen–Stelzer (07): pure-jump matrix O–U subordinator
- Da Fonseca–Grasselli–Tebaldi (07,09), Grasselli–Tebaldi (08)
- Buraschi–Cieslak–Trojani (07), Buraschi–Porchia–Trojani (09), Leippold–Trojani (09)
- Cucherio–Filipović–Mayerhofer–Teichmann (09)
- ...

Characterization and Existence Theorem

Theorem

If X is affine then its generator is affine:

$$\begin{aligned}\mathcal{A}f(x) &= 2\langle x, \nabla\alpha\nabla f(x)\rangle + \langle b + B(x), \nabla f(x)\rangle - (c + \langle\gamma, x\rangle)f(x) \\ &\quad + \int_{\mathbb{S}_+^m} (f(x+\xi) - f(x))\, m(d\xi) \\ &\quad + \int_{\mathbb{S}_+^m} (f(x+\xi) - f(x) - \langle\chi(\xi), \nabla f(x)\rangle)\, M(x, d\xi)\end{aligned}$$

for some admissible parameters $\alpha, b, B(x), c, \gamma, m(d\xi), M(x, d\xi)\ \ldots$

Conversely, for any admissible parameters there exists a unique $\mathbb{S}_+^m$*-valued affine process* X.

Characterization and Existence Theorem

Theorem (contd.)

Moreover, ϕ and ψ in

$$\mathbb{E}_x\left[\mathrm{e}^{-\langle u,x\rangle}\right] = \mathrm{e}^{-\phi(t,u)-\langle\psi(t,u),x\rangle}, \quad u \in \mathbb{S}_+^m$$

satisfy GREs with

$$F(u) = \langle b, u\rangle + c - \int_{\mathbb{S}_+^m} \left(\mathrm{e}^{-\langle u,\xi\rangle} - 1\right) m(d\xi)$$

$$R(u) = -2u\alpha u + B^\top(u) + \gamma - \int_{\mathbb{S}_+^m} \left(\mathrm{e}^{-\langle u,\xi\rangle} - 1 + \langle\chi(\xi), u\rangle\right) \frac{\mu(d\xi)}{\|\xi\|^2 \wedge 1}$$

Admissible Parameters

- $\alpha \in \mathbb{S}_+^m$
- $b \succeq (m-1)\alpha \in \mathbb{S}_+^m$
- $c \in \mathbb{R}_+$, $\gamma \in \mathbb{S}_+^m$
- $\int_{\mathbb{S}_+^m} (\|\xi\| \wedge 1)\, m(d\xi) < \infty$ (FV)
- $M(x, d\xi) = \frac{\langle x, \mu(d\xi)\rangle}{\|\xi\|^2 \wedge 1}$ satisfies
  - $\mu(\{0\}) = 0$ and $\mu(A) \in \mathbb{S}_+^m$ for all $A \in \mathcal{B}(\mathbb{S}_+^m)$
  - $\int_{\mathbb{S}_+^m} \langle \chi(\xi), u\rangle\, M(x, d\xi) < \infty$ for $x, u \in \mathbb{S}_+^m$ with $\langle x, u\rangle = 0$
- $B : \mathbb{S}_+^m \to \mathbb{S}_+^m$ **linear**: $B(x) = \sum_{i,j} x_{ij} \beta^{ij}$ satisfies
  - $\langle B(x), u\rangle - \int_{\mathbb{S}_+^m} \langle \chi(\xi), u\rangle\, M(x, d\xi) \geq 0$ for $x, u \in \mathbb{S}_+^m$ with $\langle x, u\rangle = 0$

Infinite Divisibility

- Additivity: let $X^{(i)}$ be independent affine with parameters $\alpha, b^{(i)}, B(x), c^{(i)}, \gamma, m^{(i)}(d\xi), M(x, d\xi)$, $i = 1, 2$.
- Then $X^{(3)} = X^{(1)} + X^{(2)}$ affine with parameters

$$\alpha, b^{(1)} + b^{(2)}, B(x), c^{(1)} + c^{(2)}, \gamma, m^{(1)}(d\xi) + m^{(2)}(d\xi), M(x, d\xi)$$

Corollary

X is infinitely divisible if and only if $\alpha = 0$ or $m = 1$.

Prototype (e.g. Leippold–Trojani 09)

$$dX = \left(b + NX + XN^\top\right) dt + \sqrt{X}\, dW\, Q + Q^\top dW \sqrt{X} + dJ$$

- $Q \in \mathbb{R}^{m\times m}$: $\alpha = Q^\top Q \in \mathbb{S}^m_+$
- $b \geq (m-1)\alpha \in \mathbb{S}^m_+$ **(!)**
- $B(x) = Nx + xN^\top$: $\langle Nx + xN^\top, u\rangle = 0$ if $\langle u, x\rangle = 0$
- $c = 0$, $\gamma = 0$
- J marked point process with
  - intensity $p + \langle \pi, x\rangle$, $p \in \mathbb{R}_+$, $\pi \in \mathbb{S}^m_+$,
  - jump distribution ν with $\operatorname{supp} \nu \subset \mathbb{S}^m_+$,

  hence $m(d\xi) = p\,\nu(d\xi)$, $M(x, d\xi) = \langle \pi, x\rangle \nu(d\xi)$

$\Rightarrow$ satisfies admissibility conditions

Novelties

- Existence asserted for $b \geq (m-1)\alpha \in \mathbb{S}_+^m$
- General admissible linear drift part $B(x) = \sum_{ij} x_{ij}\beta^{ij}$ allows for

$$B(x)_{ii} = x_{jj}$$

  E.g. volatility of j-th asset determines mean reversion level for volatility of i-th asset (not possible for $Nx + xN^\top$)
- Full generality of jumps

Canonical Form

- Affine property invariant under affine transformations
- There exists $G \in GL(d)$ such that $GXG^\top$ is affine with

$$\alpha = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \mathrm{diag}(b_{11}, \dots, b_{dd})$$

- Useful for efficiently solving GREs (e.g. model calibration!)
- Implies canonical representation for affine short rate models

$$r(t) = \ell + \langle \lambda, X(t) \rangle$$

- For $E = \mathbb{R}_+^m \times \mathbb{R}^n$: see Dai–Singleton (00), Joslin (06), Cheridito–Filipović–Kimmel (09)

Summary

- Overview of affine transform analysis
- Full characterization of $\mathbb{S}_+^m$-valued affine processes
  - Broad mathematical foundation
  - Exhaustive model specification
  - Canonical representation
- Papers available on `www.vif.ac.at`

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