Double Barrier Options in Regime-Switching Hyper-Exponential Jump-Diffusion Models

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Double Barrier Options

Knock-out double barrier options

A knock-out double barrier option on an asset $\{S_t\}_{t\geq 0}$ expires if at any time $t \leq T =$ maturity, the price, S_t, of the underlying leaves the interval (L, U), where $0 < L < U < +\infty$ are the barriers. Otherwise, at $t = T$, the option's owner receives payoff $G(S_T)$, where G is a certain function.

Examples of terminal payoff functions

$G(S) = (S - K)_+ = \max\{S - K, 0\}$, where $K > 0$ is fixed (barrier call);

$G(S) = (K - S)_+$ (barrier put); $G(S) = 1$ (double-no-touch, or DNT).

Continuous vs. discrete monitoring

For discretely monitored options, the expiry condition $S_t \notin (L, U)$ is only checked at a finite collection of times $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$.

Our Approach to Pricing Double Barrier Options

- We work with regime-switching HEJD models and allow an arbitrary (finite) number of states in the modulating Markov chain.
- Option prices are calculated for a (rather fine) uniformly spaced grid of initial log-spot prices (as opposed to one initial spot price).
- This allows us to calculate the deltas and gammas of the option at the points of the same grid using numerical differentiation.
- The prices and sensitivities corresponding to log-spot prices that do not lie on the grid are found using interpolation (the additional computational cost of interpolation is negligible).
- High accuracy is maintained even in the regions where the initial spot price of the underlying is very close to the barrier(s).

Main Contributions of Our Work

Efficiency and Flexibility

Our algorithm is very fast and numerically stable. Allowing an arbitrary number of states opens up the possibility of approximating models with stochastic volatility/interest rate by regime-switching HEJD models.

Theoretical Contribution

We formulate and prove a general result on convergence of Carr's randomization for barrier options under a wide class of strong Markov processes. It includes all regime-switching models with finitely many states where in each state the log-price of the underlying follows a Lévy process that has nonzero diffusion component (e.g., HEJD) or infinite jump activity (e.g., V.G.; NIG; "extended Koponen family"/CGMY/KoBoL), or both.

Acknowledgements and (incomplete) Credits

Discretely monitored barrier options in Lévy-driven models

The definitive work in this area is the article by L. Feng and V. Linetsky.

However, there are situations where monitoring occurs so frequently that it must be treated as occurring continuously, e.g., foreign exchange.

Continuously monitored barrier options in the Black-Scholes model

These were studied by many authors. The single barrier case was pioneered by R.C. Merton (1973), while the double barrier case was pioneered by N. Kunimoto and M. Ikeda (1992).

Continuously monitored barrier options in Kou's model

S.G. Kou, H. Wang (2002, 2003, 2004), A. Lipton (2002), A. Sepp (2004).

Acknowledgements and Credits, continued

Continuously monitored barrier options in HEJD models

S.B. (2006, perpetual case); M. Jeannin and M. Pistorius (2007, barrier and first-touch digitals; single state); P. Carr and J. Crosby (2008, double barrier; regime switching case).

Approximations of other jump-diffusion models by HEJD

M. Jeannin and M. Pistorius (2007); S. Asmussen, D. Madan and M. Pistorius (2007); J. Crosby, N. Le Saux and A. Mijatović (2009).

Acknowledgements

Numerical results reproduced in the works by Jeannin-Pistorius and Carr-Crosby were very useful for benchmarking. We also particularly benefited from correspondence with Peter Carr and John Crosby.

Hyper-Exponential Jump-Diffusions (HEJD processes)

- Parameters: volatility $\sigma > 0$, drift $\mu \in \mathbb{R}$, and four collections of positive real numbers $\{\alpha_j^+\}_{j=1}^{n^+}$, $\{c_j^+\}_{j=1}^{n^+}$, $\{\alpha_k^-\}_{k=1}^{n^-}$ and $\{c_k^-\}_{k=1}^{n^-}$.
- Let $W = \{W_t\}_{t\geq 0}$ be a Brownian motion with volatility σ and drift μ.
- For all $1 \leq j \leq n^+$ and all $1 \leq k \leq n^-$, construct pure jump processes $Z_j^+ = \{Z_j^+(t)\}_{t\geq 0}$ and $Z_k^- = \{Z_k^-(t)\}_{t\geq 0}$ with jump densities $\nu_j^+(dy) = c_j^+\alpha_j^+ e^{-\alpha_j^+ y}\mathbb{1}_{(0,+\infty)}(y)\,dy$ and $\nu_k^-(dy) = c_k^-\alpha_k^- e^{\alpha_k^- y}\mathbb{1}_{(-\infty,0)}(y)\,dy$, respectively, so that the processes W, Z_j^+, Z_k^- are all independent.
- Set $X_t = W_t + \sum_{j=1}^{n^+} Z_j^+(t) + \sum_{k=1}^{n^-} Z_k^-(t)$. This is a HEJD.
- The characteristic exponent of X is a *rational* function, given by

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi - \sum_{j=1}^{n^+} \frac{ic_j^+\xi}{\alpha_j^+ - i\xi} - \sum_{k=1}^{n^-} \frac{ic_k^-\xi}{\alpha_k^- + i\xi}$$

Further comments on HEJD models

- The case $n^+ = n^- = 0$ is the Black-Scholes model (1973).
- The case $n^+ = n^- = 1$ is the model introduced by S.G. Kou (2002).
- S. Levendorskiĭ (2004) constructed HEJD processes and found explicit formulas for their WHF (the name "HEJD" was introduced later).
- S. Asmussen, F. Avram, M. Pistorius (2004) used closely related, but more general Lévy processes (with "negative jumps of phase-type").
- Increasing $n^\pm$ allows one to approximate many other Lévy processes by HEJD (cf. the talk of J. Crosby later at this workshop).
- We consider a *regime-switching model* with finite state modulating Markov chain, specified by giving transition rates $(\lambda_{jk} \geq 0)_{1 \leq j \neq k \leq m}$.
- Increasing m, we can approximate HEJD models with stochastic volatility/interest rates by regime-switching HEJD models.

Some Numerical Examples

Table: Prices of DNT options on Cable (STG/USD) under a HEJD process with 7 double exponential summands. The initial spot price is $S = 2.006$ in all cases. J. Crosby reports computational times of 1–1.5 seconds per option price.

Maturity (in years)	Domestic (USD) interest rate	Foreign (GBP) interest rate	Lower barrier	Upper barrier	Our price	Carr-Crosby price	Relative difference	CPU time (seconds)
0.085	0.0537	0.0589	1.95	2.05	0.87546	0.87547	-9.30e-06	0.094
0.170	0.0539	0.0591	1.95	2.05	0.72668	0.72665	3.43e-05	0.094
0.258	0.0539	0.0597	1.95	2.05	0.58146	0.58140	1.08e-04	0.078
0.337	0.0539	0.0601	1.95	2.05	0.46872	0.46870	2.99e-05	0.094
0.419	0.0539	0.0604	1.95	2.05	0.37261	0.37265	-9.64e-05	0.109
0.507	0.0539	0.0607	1.95	2.05	0.29070	0.29076	-2.04e-04	0.125
0.756	0.0538	0.0613	1.95	2.05	0.14239	0.14247	-6.15e-04	0.172
1.005	0.0536	0.0618	1.95	2.05	0.06943	0.06952	-1.30e-03	0.234
0.085	0.0537	0.0589	1.97	2.04	0.78752	0.78755	-4.11e-05	0.047
0.258	0.0539	0.0597	1.97	2.04	0.38150	0.38152	-6.59e-05	0.063
0.019	0.0535	0.0589	1.98	2.03	0.93403	0.93400	2.33e-05	0.046
0.085	0.0537	0.0589	1.98	2.03	0.66446	0.66440	9.14e-05	0.047

An Example with Two States

Table: Prices of double barrier foreign exchange options in a regime-switching HEJD model with two states and 4 double exponential summands in each state

Initial state	1	2	1	2
Transition rates	1.85/1.1	1.85/1.1	10/10	10/10
Vanilla call	4.40036	4.00855	4.40383	4.33705
Vanilla put	2.84040	2.52717	2.81875	2.76598
Digital call	0.45971	0.43212	0.45313	0.44872
Digital put	0.38852	0.36614	0.38332	0.37970

Option parameters (same for both states): $S_0 = 220$, $r_{dom} = 0.046$, $r_{for} = 0.051$, $L = 195$, $U = 250$, $T = 0.9$, $K = 218$. Rebate: $R_L = R_U = 0.25$, paid at maturity if either barrier is crossed.

Table: Prices obtained by Ambrose, Carr and Crosby in the same setup

Initial state	1	2	1	2
Transition rates	1.85/1.1	1.85/1.1	10/10	10/10
Vanilla call	4.39899	4.00910	4.40469	4.33799
Vanilla put	2.83956	2.52770	2.81620	2.76183
Digital call	0.46099	0.43320	0.45554	0.45108
Digital put	0.38718	0.36507	0.38284	0.37919

Table: Relative differences between our prices and A-C-C prices

Initial state	1	2	1	2
Transition rates	1.85/1.1	1.85/1.1	10/10	10/10
Vanilla call	0.00031	-0.00014	-0.00020	-0.00022
Vanilla put	0.00030	-0.00021	0.00091	0.00150
Digital call	-0.00278	-0.00249	-0.00529	-0.00523
Digital put	0.00346	0.00293	0.00125	0.00135

Overview of Our Pricing Method

- Carr's randomization is applied to reduce to a sequence of pricing problems for double barrier options with infinite time horizon.
- Convergence of Carr's randomization is proved formally.
- A perpetual pricing problem with regime switching is reduced to the case with 1 state using a quickly convergent iteration procedure.
- The perpetual pricing problem under a (single state) HEJD model is solved using the operator form of the Wiener-Hopf method (developed by S. Boyarchenko and S. Levendorskiĭ).
- Explicit formulas for WH factors allow us to express the latter solution in semi-analytic form (reduce to calculating the action of a single convolution operator with exponentially decaying kernel).

What is Carr's Randomization?

- Carr's randomization (a.k.a. "Canadization") was originally discovered (P. Carr, 1998) as a probabilistic interpretation of the "analytic method of lines" (used by P. Carr and D. Faguet, 1996).
- Carr proposed to approximate a finite-lived option pricing problem by replacing a deterministic maturity date T with a suitably chosen random maturity date whose mean is equal to T.
- When this random maturity is a sum of independent exponentially distributed maturity dates, the new pricing problem often reduces to a sequence of perpetual pricing problems, which are easier to solve.
- We believe that this idea has a very wide scope of applications. For the time being, the efficiency of Carr's randomization for American and barrier options has been well documented in the literature.

Double Barrier Options in Regime-Switching Models

- For simplicity we consider constant riskless rate $r > 0$.
- $\mathbb{X} = (Y_t, X_t)_{t\geq 0}$ temporally homogeneous Markov process with state space $\{1, 2, \ldots, m\} \times \mathbb{R}$, constructed using pairwise independent Lévy processes $X^{(j)} = (X_t^{(j)})_{t\geq 0}$ $(1 \leq j \leq m)$ and a continuous time Markov chain $Y = (Y_t)_{t\geq 0}$ defined by transition rates $(\lambda_{jk})_{1\leq j\neq k\leq m}$.
- Price of the underlying: $S_t = \exp(c_{Y_t} X_t + d_{Y_t})$ $(c_j > 0,\ d_j \in \mathbb{R})$.
- In each state $1 \leq j \leq m$, have log-barriers $-\infty < h_-^j < h_+^j < \infty$ and terminal payoff function $g^j(x)$ defined for $h_-^j < x < h_+^j$.
- The option expires worthless if for some $t \leq T =$ maturity we have $X_t \notin (h_-^j, h_+^j)$, where $j = Y_t$. Otherwise at $t = T$ the owner of the option receives payoff $g^j(e^{X_t})$, where $j = Y_T$.

Carr's Randomization for Barrier Options: Setup

- We will write $\vec{h}_\pm = (h^j_\pm)^m_{j=1}$ and $\vec{g} = (g^j)^m_{j=1}$.
- $V_j(x, T; \vec{h}_\pm; \vec{g})$ = value function of the option in state j (the no-arbitrage price of the option above assuming $\mathbb{X}$ starts at (j, x)).
- $v_j(x; q; \vec{h}_\pm; \vec{g})$ = value function of the knock-out continuous cash flow $\{e^{-qt} g^{Y_t}(X_t)\}_{t\geq 0}$ in state j, where $q > 0$ is the killing rate.
- Important remark: suppose $q = r + \Delta^{-1}$ for some $\Delta > 0$. Then $\Delta^{-1} \cdot v_j(x; q; \vec{h}_\pm; \vec{g})$ can be interpreted as the value function of a *finite lived* option with *random* maturity date $T \sim \operatorname{Exp} \Delta^{-1}$.
- We will write $\vec{V} = (V_j)^m_{j=1}$, $\vec{v} = (v_j)^m_{j=1}$ (vector-valued functions).

Carr's Randomization via Backward Induction

Step I. Choose a partition, $\mathcal{P}$, of the interval $[0, T]$. Thus $\mathcal{P}$ is a finite collection of points $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$, where N is a positive integer. (Typically, $t_s = sT/N$ for all s.)

Step II. For every $0 \le s \le N-1$, set $\Delta_s = t_{s+1} - t_s$ and $q_s = r + \Delta_s^{-1}$.

Step III. Put $V_j^N(x) = g^j(x)$ for all $1 \le j \le m$.

Step IV. In a cycle with respect to $s = N-1, N-2, \dots, 1, 0$, calculate

$$\vec{V}^s(x) = \Delta_s^{-1} \cdot \vec{v}(x; q_s; \vec{h}_\pm; \vec{V}^{s+1}).$$

Step V. Put $\vec{V}_{\mathcal{P}}(x, T; \vec{h}_\pm; \vec{g}) = \vec{V}^0(x)$, where $\vec{V}^0(x)$ is obtained at the end of the cycle in Step IV. Then $\vec{V}_{\mathcal{P}}$ is *Carr's randomization approximation* to $\vec{V}(x, T; \vec{h}_\pm; \vec{g})$, defined by the partition $\mathcal{P}$.

Convergence of Carr's randomization

Theorem. Suppose that each Lévy process $X^{(j)}$ has either nonzero diffusion component or infinite jump activity (or both). Assume also that each $g^j(x)$ is a bounded continuous function on (h^j_-, h^j_+). Then for fixed j and x, we have $V_{\mathcal{P},j}(x, T; \vec{h}_\pm; \vec{g}) \to V_j(x, T; \vec{h}_\pm; \vec{g})$ as $\mathrm{mesh}(\mathcal{P}) \to 0$, where $\mathrm{mesh}(\mathcal{P}) = \max_{0\le s\le N-1} \Delta_s$.

Important ingredient in the proof

Theorem. Under the same assumptions, for fixed j and x, $V_j(x, T; \vec{h}_\pm; \vec{g})$ is continuous as a function of T.

The assumption on $X^{(j)}$ is used in the following way: it is equivalent to requiring that $\mathbb{P}[X^{(j)}_t = a] = 0$ for any $t > 0$, $1 \le j \le m$ and $a \in \mathbb{R}$.

Perpetual Pricing Problem: Iterative Procedure

- Problem: calculate $v_j(x) := v_j(x; q; \vec{h}_\pm; \vec{g})$ for $1 \le j \le m$.
- The functions $v_j(x)$ solve the following system of PIDE:

$$\begin{cases} (q - L_j)v_j(x) - \sum_{k=1}^m \lambda_{jk} v_k(x) = g^j(x), & h_-^j < x < h_+^j; \\ v_j(x) = 0, & x \le h_-^j \quad \text{or} \quad x \ge h_+^j, \end{cases}$$

where L_j is the infinitesimal generator of the Lévy process $X^{(j)}$.

- We construct a sequence of approximations $\vec{v}^0, \vec{v}^1, \vec{v}^2, \ldots$ to $\vec{v}$.
- Put $\Lambda_j = -\sum_{k \ne j} \lambda_{jk}$. Set $\vec{v}^0 = 0$ and for $n = 1, 2, \ldots$ solve

$$\begin{cases} (q + \Lambda_j - L_j)v_j^n(x) = g^j(x) + \sum_{k \ne j} \lambda_{jk} v_k^{n-1}(x), & h_-^j < x < h_+^j; \\ v_j^n(x) = 0, & \text{otherwise.} \end{cases}$$

- This is a perpetual pricing problem in a Lévy model with one state.

Normalized EPV operators of a Lévy process

- Next goal: explain how to price knock-out continuous cash flows with one or two barriers in a 1-dimensional Lévy model.
- The *supremum* and *infimum* processes of $X = \{X_t\}_{t\geq 0}$ are

$$\overline{X}_t = \sup_{0\leq s\leq t} X_s \quad \text{and} \quad \underline{X}_t = \inf_{0\leq s\leq t} X_s.$$

- The normalized expected present value (EPV) operators are

$$(\mathcal{E}_q f)(x) = \mathbb{E}\left[\int_0^\infty qe^{-qt} f(x + X_t)\, dt\right],$$

$$(\mathcal{E}_q^+ f)(x) = \mathbb{E}\left[\int_0^\infty qe^{-qt} f(x + \overline{X}_t)\, dt\right],$$

$$(\mathcal{E}_q^- f)(x) = \mathbb{E}\left[\int_0^\infty qe^{-qt} f(x + \underline{X}_t)\, dt\right].$$

Wiener-Hopf Factorization (WHF)

- Let $T_q \sim \operatorname{Exp} q$ denote an exponentially distributed random variable with mean q^{-1} that is independent of the process $X = \{X_t\}_{t\geq 0}$.
- Probability form of the WHF formula:

$$\mathbb{E}\left[e^{i\xi X_{T_q}}\right] = \mathbb{E}\left[e^{i\xi \overline{X}_{T_q}}\right] \cdot \mathbb{E}\left[e^{i\xi \underline{X}_{T_q}}\right] \qquad \forall\, \xi \in \mathbb{R}.$$

- The last identity follows from the following facts:
  (1) we have $X_{T_q} = \overline{X}_{T_q} + (X_{T_q} - \overline{X}_{T_q})$;
  (2) the random variables $\overline{X}_{T_q}$ and $X_{T_q} - \overline{X}_{T_q}$ are independent (deep!);
  (3) the random variables $\underline{X}_{T_q}$ and $X_{T_q} - \overline{X}_{T_q}$ are identical in law;
  (4) the characteristic function of the sum of two independent random variables is equal to the product of their characteristic functions.

Two Other WHF Formulas

- Define the *Wiener-Hopf factors* $\phi_q^{\pm}(\xi)$ (for $\xi \in \mathbb{R}$) by the formulas

$$\phi_q^{+}(\xi) = \mathbb{E}\left[e^{i\xi \overline{X}_{T_q}}\right], \qquad \phi_q^{-}(\xi) = \mathbb{E}\left[e^{i\xi \underline{X}_{T_q}}\right].$$

- $\phi_q^{\pm}(\xi)$ admit analytic continuation without zeroes into the upper/lower half plane.
- They are related to the normalized EPV operators $\mathcal{E}_q^{\pm}$ via

$$\mathcal{E}_q^{\pm}\left(e^{i\xi x}\right) = \phi_q^{\pm}(\xi) \cdot e^{i\xi x} \qquad \forall \, \xi \in \mathbb{R}.$$

- One can verify directly using the definitions that

$$\mathcal{E}_q\left(e^{i\xi x}\right) = q \cdot (q + \psi(\xi))^{-1} \cdot e^{i\xi x} \qquad \forall \, \xi \in \mathbb{R}.$$

- Analytic form of the WHF formula: $q \cdot (q + \psi(\xi))^{-1} = \phi_q^{+}(\xi)\phi_q^{-}(\xi)$.
- Operator form of the WHF formula: $\mathcal{E}_q = \mathcal{E}_q^{+}\mathcal{E}_q^{-} = \mathcal{E}_q^{-}\mathcal{E}_q^{+}$.

The Wiener-Hopf Method for One Barrier

- Now we return to pricing a continuous cash flow $\{e^{-qt}g(\ln S_t)\}_{t\ge 0}$.
- Consider the down-and-out case; let $0 < L < \infty$ be the barrier.
- Write $x = \ln S_0$ and $h_- = \ln L$. The flow is terminated as soon as the price of the underlying $S_t = e^{x+X_t}$ reaches or falls below $L = e^{h_-}$.
- The price of this down-and-out continuous cash flow equals

$$v_{down-and-out}(x;q;h_-;g) = q^{-1}\cdot \mathcal{E}_q^-\left(\mathbb{1}_{(h_-,+\infty)}(x)\cdot(\mathcal{E}_q^+ g)(x)\right).$$

- Similar formula in the up-and-out case ($U = e^{h_+}$ is the upper barrier):

$$v_{up-and-out}(x;q;h_+;g) = q^{-1}\cdot \mathcal{E}_q^+\left(\mathbb{1}_{(-\infty,h_+)}(x)\cdot(\mathcal{E}_q^- g)(x)\right).$$

The Wiener-Hopf Method for Two Barriers

- Consider two barriers, $0 < L < U < +\infty$. Put $h_- = \ln L$, $h_+ = \ln U$.
- Value of a knock-out cash flow $\{e^{-qt} g(\ln S_t)\}_{t \geq 0}$ with barriers (L, U):

$$v(x; q; h_\pm; g) = G^0(x) - G^1_+(x) - G^1_-(x) + G^2_+(x) + G^2_-(x)$$
$$- G^3_+(x) - G^3_-(x) + G^4_+(x) + G^4_-(x) - \cdots$$

- To find the terms on the RHS, first calculate $G^0(x) = q^{-1} \cdot (\mathcal{E}_q g)(x)$.
- Next, use the formulas

$$G^0_+(x) = G^0(x)\big|_{[h_+, +\infty)}, \qquad G^0_-(x) = G^0(x)\big|_{(-\infty, h_-]},$$

$$G^n_+(x) = \mathcal{E}^-_q \Big(\mathbb{1}_{(-\infty, h_-]}(x) \cdot \big((\mathcal{E}^-_q)^{-1} G^{n-1}_- \big)(x) \Big) \qquad \forall\, n \geq 1,$$

$$G^n_-(x) = \mathcal{E}^+_q \Big(\mathbb{1}_{[h_+, +\infty)}(x) \cdot \big((\mathcal{E}^+_q)^{-1} G^{n-1}_+ \big)(x) \Big) \qquad \forall\, n \geq 1.$$

Normalized EPV Operators in the Black-Scholes Model

- Introduce two types of integral operators:

$$(I_\beta^+ f)(x) = \int_0^\infty \beta e^{-\beta y} f(x+y)\, dy \qquad (\beta > 0),$$

$$(I_\beta^- f)(x) = \int_{-\infty}^0 (-\beta) e^{-\beta y} f(x+y)\, dy \qquad (\beta < 0).$$

- Assume that $X = \{X_t\}_{t \geq 0}$ is a BM with volatility σ and drift μ.
- Denote by $\beta^- < 0 < \beta^+$ the roots of the *characteristic equation*

$$\frac{\sigma^2}{2}\beta^2 + \mu\beta - q = 0.$$

- Then $\phi_q^\pm(\xi) = \beta^\pm \cdot (\beta^\pm - i\xi)^{-1}$ and $\mathcal{E}_q^\pm = I_{\beta^\pm}^\pm$.
- Also: $\mathcal{E}_q = (\beta^+ - \beta^-)^{-1} \cdot \left(-\beta^- I_{\beta^+}^+ + \beta^+ I_{\beta^-}^-\right)$.
- This allows us to calculate the action of $\mathcal{E}_q^\pm$, $\mathcal{E}_q$ very efficiently.

Numerical Realization of the Operators $I_\beta^\pm$

Finite element approximation (as in A. Eydeland's talk)

Given a function $f(x)$, approximate it with a piecewise linear function. For piecewise linear functions, the action of $I_\beta^\pm$ can be calculated explicitly.

Precise description of the setup

Choose a grid $\vec{x} = (x_\ell)_{\ell=0}^M$ of points in $\mathbb{R}$, where $x_\ell = x_0 + \ell \cdot \Delta$ for all $0 \leq \ell \leq M$ and $\Delta > 0$ is fixed. For $0 \leq \ell \leq M-1$, use the approximation

$$f(x) \approx f_\ell + \Delta^{-1} \cdot (f_{\ell+1} - f_\ell) \cdot (x - x_\ell), \qquad x_\ell \leq x \leq x_{\ell+1},$$

where $f_\ell = f(x_\ell)$. Further, approximate $f(x)$ by zero outside of $[x_0, x_M]$. The error is controlled by the size of $f''(x)$ for $x_\ell < x < x_{\ell+1}$ (assuming it exists) and by the size of $f(x)$ for $x < x_0$ and $x > x_M$.

Numerical Realization of $I_\beta^\pm$ using $O(M)$ Operations

Fix $\beta > 0$, and let I_ℓ^+ denote the approximation to $(I_\beta^+ f)(x_\ell)$ constructed above. The values I_ℓ^+ can be computed inductively as follows:

- Set $I_M^+ = 0$.
- In a cycle with respect to $\ell = M, M-1, \dots, 2, 1$, calculate

$$I_{\ell-1}^+ = e^{-\beta\Delta} \cdot I_\ell^+ + \frac{e^{-\beta\Delta} - 1 + \beta\Delta}{\beta\Delta} \cdot f_{\ell-1} + e^{-\beta\Delta} \cdot \frac{e^{\beta\Delta} - 1 - \beta\Delta}{\beta\Delta} \cdot f_\ell.$$

Next, let $\beta < 0$, and let I_ℓ^- denote the approximation to $(I_\beta^- f)(x_\ell)$ constructed above. The values I_ℓ^- can be computed inductively as follows:

- Set $I_0^- = 0$.
- In a cycle with respect to $\ell = 1, 2, \dots, M$, calculate

$$I_\ell^- = e^{\beta\Delta} \cdot I_{\ell-1}^- + \frac{1 + \beta\Delta - e^{\beta\Delta}}{\beta\Delta} \cdot f_\ell + e^{\beta\Delta} \cdot \frac{1 - \beta\Delta - e^{-\beta\Delta}}{\beta\Delta} \cdot f_{\ell-1}.$$

Normalized EPV Operators for HEJD Processes

- Let $X = \{X_t\}_{t\geq 0}$ be a HEJD with parameters $\sigma > 0$, μ, $\{\alpha_j^+\}_{j=1}^{n^+}$, $\{c_j^+\}_{j=1}^{n^+}$, $\{\alpha_k^-\}_{k=1}^{n^-}$ and $\{c_k^-\}_{k=1}^{n^-}$. The characteristic exponent of X is

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi - \sum_{j=1}^{n^+} \frac{ic_j^+\xi}{\alpha_j^+ - i\xi} - \sum_{k=1}^{n^-} \frac{ic_k^-\xi}{\alpha_k^- + i\xi}.$$

- Let $\{\beta_j^+\}_{j=1}^{n^++1}$ and $\{\beta_k^-\}_{k=1}^{n^-+1}$ be the positive and negative roots of the *characteristic equation* $q + \psi(-i\beta) = 0$.
- For suitable constants a_j^+, a_k^-, b_j^+, b_k^- (given by explicit formulas),

$$\mathcal{E}_q^+ = \sum_{j=1}^{n^++1} a_j^+ I_{\beta_j^+}^+, \qquad \mathcal{E}_q^- = \sum_{k=1}^{n^-+1} a_k^- I_{\beta_k^-}^-,$$

$$\mathcal{E}_q = \sum_{j=1}^{n^++1} b_j^+ I_{\beta_j^+}^+ + \sum_{k=1}^{n^-+1} b_k^- I_{\beta_k^-}^-.$$

Barrier Options in HEJD Models

Up-and-out and down-and-out options

The numerical calculation of

$$v_{down-and-out}(x;q;h_-;g) = q^{-1} \cdot \mathcal{E}_q^- \left(\mathbb{1}_{(h_-,+\infty)}(x) \cdot (\mathcal{E}_q^+ g)(x)\right)$$

and $$v_{up-and-out}(x;q;h_+;g) = q^{-1} \cdot \mathcal{E}_q^+ \left(\mathbb{1}_{(-\infty,h_+)}(x) \cdot (\mathcal{E}_q^- g)(x)\right)$$

can be realized very efficiently using the formulas on the last two slides

Knock-out options with two barriers

In the formula $v(x;q;h_\pm;g) = G^0(x) + \sum_{n=1}^{\infty}(-1)^n \cdot (G_+^n(x) + G_-^n(x))$,

the infinite sum can be evaluated *in closed form*.

In both cases, combining Carr's randomization with the Wiener-Hopf method yields fast, accurate and numerically stable algorithms.

Wiener-Hopf Factors for HEJD

Characteristic exponent and the Wiener-Hopf factors of a HEJD process:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi - \sum_{j=1}^{n^+} \frac{ic_j^+\xi}{\alpha_j^+ - i\xi} - \sum_{k=1}^{n^-} \frac{ic_k^-\xi}{\alpha_k^- + i\xi} \qquad (\sigma > 0),$$

$$\phi_q^+(\xi) = \left(\prod_{j=1}^{n^+} \frac{\alpha_j^+ - i\xi}{\alpha_j^+}\right) \cdot \left(\prod_{j=1}^{n^+ + 1} \frac{\beta_j^+}{\beta_j^+ - i\xi}\right) = \sum_{j=1}^{n^+ + 1} \frac{a_j^+ \beta_j^+}{\beta_j^+ - i\xi},$$

$$\phi_q^-(\xi) = \left(\prod_{k=1}^{n^-} \frac{\alpha_k^- + i\xi}{\alpha_k^-}\right) \cdot \left(\prod_{k=1}^{n^- + 1} \frac{\beta_k^-}{\beta_k^- - i\xi}\right) = \sum_{k=1}^{n^- + 1} \frac{a_k^- \beta_k^-}{\beta_k^- - i\xi},$$

where $\{\beta_j^+\}_{j=1}^{n^+ + 1}$ and $\{\beta_k^-\}_{k=1}^{n^- + 1}$ are the positive and negative roots of the characteristic equation $q + \psi(-i\beta) = 0$, and the formulas for a_j^+ and a_k^- are given on the next slide.

Normalized EPV Operators for HEJD

With the notation of the previous slide, we have (for any $q > 0$)

$$\mathcal{E}_q^+ = \sum_{j=1}^{n^+ +1} a_j^+ I_{\beta_j^+}^+, \quad \mathcal{E}_q^- = \sum_{k=1}^{n^- +1} a_k^- I_{\beta_k^-}^-, \quad \mathcal{E}_q = \sum_{j=1}^{n^+ +1} b_j^+ I_{\beta_j^+}^+ + \sum_{k=1}^{n^- +1} b_k^- I_{\beta_k^-}^-,$$

where the integral operators $I_\beta^\pm$ were defined earlier,

$$a_j^+ = \left(\prod_{\ell=1}^{n^+} \frac{\alpha_\ell^+ - \beta_j^+}{\alpha_\ell^+} \right) \cdot \left(\prod_{\substack{\ell \neq j \\ 1 \le \ell \le n^+ +1}} \frac{\beta_\ell^+}{\beta_\ell^+ - \beta_j^+} \right),$$

$$a_k^- = \left(\prod_{\ell=1}^{n^-} \frac{\alpha_\ell^- + \beta_k^-}{\alpha_\ell^-} \right) \cdot \left(\prod_{\substack{\ell \neq k \\ 1 \le \ell \le n^- +1}} \frac{\beta_\ell^-}{\beta_\ell^- - \beta_k^-} \right),$$

$$b_j^+ = a_j^+ \cdot \phi_q^-(-i\beta_j^+) \qquad \text{and} \qquad b_k^- = a_k^- \cdot \phi_q^+(-i\beta_k^-).$$

Double Barrier K.O. Payoff Streams under HEJD

- Given: a HEJD $X = \{X_t\}_{t\geq 0}$ so that $S_t = S_0 e^{X_t}$; log-barriers $h_- < h_+$; a bounded measurable function $g(x)$ on (h_-, h_+); and a killing rate $q > 0$.
- Let $\vec{G}^0_+$ and $\vec{G}^0_-$ be column vectors of size $n^- + 1$ and $n^+ + 1$ with entries

$$(\vec{G}^0_+)_k = q^{-1} \cdot b^-_k \cdot (I^-_{\beta^-_k} g)(h_+), \qquad (\vec{G}^0_-)_j = q^{-1} \cdot b^+_j \cdot (I^+_{\beta^+_j} g)(h_-).$$

- Introduce matrices $A^\pm$ of size $(n^\pm + 1) \times (n^\mp + 1)$ with entries

$$A^+_{jk} = \frac{a^+_j \beta^+_j}{\beta^+_j - \beta^-_k} \cdot \frac{e^{-\beta^+_j (h_+ - h_-)}}{\phi^+_q(-i\beta^-_k)} \qquad \text{and} \qquad A^-_{kj} = \frac{a^-_k \beta^-_k}{\beta^-_k - \beta^+_j} \cdot \frac{e^{\beta^-_k (h_+ - h_-)}}{\phi^-_q(-i\beta^+_j)}$$

- Put $B = (I - A^+ A^-)^{-1}$ and $C = (I - A^- A^+)^{-1}$, and calculate the vectors

$$\vec{V}^+ = A^- \cdot B \cdot (\vec{G}^0_- - A^+ \vec{G}^0_+), \qquad \vec{V}^- = A^+ \cdot C \cdot (\vec{G}^0_+ - A^- \vec{G}^0_-).$$

- Then for all $h_- < x < h_+$, we have

$$v_{k.o.}(x; q; h_\pm; g) = q^{-1} \cdot (\mathcal{E}_q g)(x) - \sum_{j=1}^{n^+ + 1} \vec{V}^-_j \cdot e^{\beta^+_j (x - h_-)} - \sum_{k=1}^{n^- + 1} \vec{V}^+_k \cdot e^{\beta^-_k (x - h_+)}.$$