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Credit value adjustment for credit default swaps via the structural default model

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We present a multi-dimensional jump-diffusion version of a structural default model and show how to use it in order to value the credit value adjustment for a credit default swap. We develop novel analytical and numerical methods for solving the corresponding boundary value problem with a special emphasis on the role of negative asset value jumps. Using recent market data, we show that under realistic assumptions credit value adjustment greatly reduces the value of a credit default swap sold by a risky counterparty compared with one sold by a non-risky counterparty. We identify features having the biggest impact on credit value adjustment: namely, default correlation and spread volatility.

1 INTRODUCTION

1.1 Motivation

Credit default swaps (CDSs) are fundamental building blocks of more complicated credit instruments such as synthetic collateralized debt obligations (CDOs). Typically their prices are readily available; they are used as inputs for more complicated models. For some time now it has been realized that, in order to value CDSs properly, counterparty effects have to be taken into account. However, conventional tools used for valuing the credit value adjustment (CVA) for over-the-counter (OTC) derivatives are not adequate for CDSs due to the pronounced correlation effect. The key goal of this article is to present a novel valuation methodology for the valuation of CVA for CDSs. As a prerequisite for achieving this goal, we build a new multi-name structural default model (SDM).

Recall that a typical CDS is an OTC contract between a protection buyer and a protection seller that insures the protection buyer against a possible default of

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the reference name in exchange for periodic payments up to the termination of the contract, which occurs when the CDS expires or the reference name defaults, whichever comes first. Cash flows to the protection buyers from protection sellers and vice versa are called default legs and coupon legs (CLs), respectively.

When a protection buyer buys a CDS protection from a risky protection seller, it has to account for two types of exposure. First, there is market risk due to the changes in the mark-to-market (MtM) value of the corresponding CDS caused by changes in credit spread and interest rate. Second, there is credit risk due to the fact that the protection seller can default prior to CDS termination. During the life of a CDS contract, a realized loss due to the counterparty exposure can arise when the protection seller defaults before the reference name, provided that the MtM of the CDS is positive for the protection buyer, since only a fraction of the MtM value of the existing CDS contract can be recovered. If the MtM of the CDS is negative to the protection buyer, this CDS is unwound at market price.

Since the protection buyer realizes positive MtM gains when the credit quality of the reference name deteriorates (because the probability of receiving protection payment increases), the realized loss due to protection seller default is particularly big if the credit quality of the reference name and the protection seller are positively correlated and deteriorate simultaneously. In order to account for the correlation effect in a meaningful way, we introduce a novel multi-name SDM based on correlated jump-diffusion processes. In this model, a typical default of a single name is triggered by a sudden drop in the asset value followed by its gradual deterioration, which is a pattern observed in practice, in particular, in the case of financial institutions. In order to calculate CVA for CDSs, we combine the dynamics for asset values for the reference name and the protection seller and describe it via two correlated jump-diffusion processes.

We show that CVA (which we define as the maximal expected loss for the protection buyer) can be interpreted as a two-dimensional down-and-in digital option with the down barrier being triggered when the value of protection seller assets crosses its default barrier and the option rebate being determined by the value of the reference name assets at the time of barrier crossing. Under realistic assumptions, CVA can be very significant, reducing the present value of the default leg of the corresponding CDS by 5–30%. Naturally, the magnitude of this reduction is sensitive to the choice of both volatility and correlation for the corresponding dynamics. Our results indicate that CVA for OTC CDSs should be viewed as a first-order effect.

At inception the spread of a CDS contract is set in such a way that the MtM value of the contract is zero, so that the option underlying CVA is at-the-money. Accordingly, its value is highly sensitive to the volatility of the asset values for both the reference name and the protection seller as well as to their correlation. Thus, in order to evaluate CVA accurately, we need to model the correlated dynamics of the reference name and the protection seller in a convincing manner.

Between 2001 and 2007 the notional value of outstanding CDSs grew by a factor of 100 and the outstanding notional amount of credit derivatives has become huge. Due to the turbulence in the credit markets, the counterparty problem has become critical for the financial industry. This has resulted in a dramatic shrinkage of the market. According to the most recent International Swaps and Derivatives Association survey published on April 22, 2009 (see www.isda.org), the notional value of outstanding CDSs decreased to US$39 trillion as of December 31, 2008 from US$55 trillion in the middle of the year and US$62 trillion as of December 31, 2007. For comparison, the notional value of outstanding interest rate derivatives is about US$382 trillion, while the notional value of equity derivatives is about US$10 trillion. Recently, in order to improve market transparency and liquidity, more than 2,000 banks, hedge funds and asset managers trading CDSs agreed to a "Big Bang Protocol". Regulators and legislators have repeatedly called for an industry overhaul since the swaps were blamed for playing a pivotal role in the collapse of Lehman Brothers and the disintegration of AIG. All this makes the evaluation and hedging of CVA for CDSs vital for the financial system as a whole.

1.2 Prior research

Merton (1974) introduces the original SDM. He assumes that a firm's asset value is driven by a lognormal diffusion and the firm defaults at the time of debt maturity if the notional of the debt exceeds the asset value. His ideas were extended by several researchers, notably by Black and Cox (1976), who propose the idea of the continuous default barrier. In the early 2000s, when CDS contracts gained popularity and became liquid, it was realized that the classical SDM based on lognormal dynamics cannot explain high short-term CDS spreads traded in the market. This is due to the fact that in such a model a default event is predictable, so that the probability of a non-distressed firm defaulting in the near term is close to zero. To account for this effect, the SDM has been extended to incorporate curvilinear barriers (see, for example, Hyer *et al* (1998); Hull and White (2001)), sudden jumps in a firm's value (see, for example, Zhou (2001a); Hilberink and Rogers (2002); Lipton (2002); Lipton *et al* (2007); Sepp (2004, 2006)) or default barrier uncertainty (see, for example, Finger *et al* (2002)).

Two-dimensional extensions of SDM are studied by Zhou (2001b), Hull and White (2001), Haworth *et al* (2006), Blanchet-Scalliet and Patras (2008) and several other researchers, who consider bivariate correlated lognormal dynamics for two firms and derive analytical formulas for their joint survival. Multi-dimensional extensions of the SDM are described by JPMorgan (1997), Li (2000) and many others, who use Gaussian copulas to describe correlated default times for a portfolio of names. More recently a version of a multi-dimensional SDM with jumps has been proposed by Kiesel and Scherer (2007), who develop a mixture of semi-analytical and Monte Carlo tools for its calibration and usage.

This article is devoted specifically to counterparty risk. This topic has been investigated by numerous researchers, for instance, Jarrow and Turnbull (1995), Jarrow and Yu (2001), Leung and Kwok (2005) and Crepey *et al* (2009) develop the so-called reduced-form default model and analyze the counterparty risk in its framework; Hull and White (2001) and Blanchet-Scalliet and Patras (2008) model the evolution of the reference name and the protection seller by two correlated Brownian motions and analyze bivariate default effects; Turnbull (2005) derives model-free upper and lower bounds for the counterparty exposure; Pykhtin and Zhu (2006) and Gregory (2009) use the Gaussian copula approach to study defaults of the reference name and the protection seller; Brigo and Chourdakis (2008) consider correlated dynamics of the credit spreads. By necessity, this list is not exhaustive and could easily be extended.

1.3 New findings

We propose a new SDM based on jump-diffusion dynamics for a single name and extend it to the multi-dimensional case. As an asset value driver we use an additive process, which is a jump-diffusion process with time-inhomogeneous increments (see Sato (1999)). Instead of assuming that the default barrier is time inhomogeneous in order to make the SDM match the CDS spread curve, we consider a piecewise constant jump intensity and calibrate it and other relevant parameters to the CDS spread curve and other market observables.

Using this model, we develop a novel approach to valuing CVA for CDSs. Our approach is dynamic in nature and takes into account both CDS spread volatilities for reference names and protection sellers and their correlation. The approaches proposed by Leung and Kwok (2005), Pykhtin and Zhu (2006) and Gregory (2009), among others, do not account for spread volatility and, as a result, may underestimate CVA. Blanchet-Scalliet and Patras (2008) consider a conceptually similar approach; however, they restrict themselves to the analytically solvable bivariate lognormal dynamics with constant parameters; as a result, their model cannot fit the term structure of CDS spreads implied by the market, thus introducing a bias in CVA valuation. In contrast, our model can be fitted to any reasonable term structure of CDS spreads and market prices of CDS and equity options. Hull and White (2001) use Monte Carlo simulations of correlated Brownian motions; in contrast, we assume jump-diffusion dynamics, which are potentially more realistic for default modeling, and we use robust semi-analytical and numerical methods for model calibration and CVA valuation.

Our approach is based on solving partial integro-differential equations (PIDEs) with non-local integral terms. We develop robust fast fourier transform (FFT) and PIDE-based methods for model calibration via forward induction and instrument pricing via backward induction, which are applicable in one, two and, potentially, three dimensions. Although the FFT-based method is efficient and easy to implement, it requires a uniform grid and a large number of discretization steps. In contrast, the numerical PIDE method provides greater flexibility and tends to be more stable. We greatly increase the efficacy of the latter method by deriving explicit recursive formulas for the computation of the convolution terms for discrete and exponential jump distributions.

The article is organized as follows. In Sections 2 and 3 we introduce our SDM. In Section 4 we present the valuation problem for CVA for CDS contracts. In Section 5 we provide an illustration of model calibration and CVA evaluation based on a recent market snapshot. Conclusions are formulated in Section 6.

Of necessity this paper is short. A detailed exposition of our results will be presented in Lipton and Sepp (2009).

2 SINGLE-NAME STRUCTURAL MODEL

In this section we introduce the SDM for a single name. We assume that the default and counterparty risk can be hedged, so that we work with the pricing measure denoted by $\mathbb{Q}$. We also assume a risk-free deterministic rate of return $r(t)$ and denote by $D(t,T)$ the corresponding discount factor for a risk-free cash flow at time T.

We start with the firm's asset value dynamics. We denote the corresponding asset value by $a(t)$. We assume that $a(t)$ is driven by the following dynamics under $\mathbb{Q}$:

$$\mathrm{d}a(t) = (r(t) - \varsigma(t) - \lambda(t)\kappa)a(t)\,\mathrm{d}t + \sigma(t)a(t)\,\mathrm{d}W(t) + (\mathrm{e}^{j} - 1)a(t)\,\mathrm{d}N(t) \quad (1)$$

Here $r(t)$ is the interest rate, $\varsigma(t)$ is the dividend rate, $W(t)$ is a standard Brownian motion, $N(t)$ is a Poisson process independent of $W(t)$, $\sigma(t)$ is a deterministic volatility, $\lambda(t)$ is jump arrival intensity, j is the jump amplitude, which is a random variable with probability density function $\varpi(j)$, and κ is the jump compensator:

$$\kappa = \mathbb{E}\{\mathrm{e}^{j} - 1\} = \int_{-\infty}^{\infty} \mathrm{e}^{j}\varpi(j)\,\mathrm{d}j - 1 \quad (2)$$

To reduce the number of free parameters, we concentrate on one-parameter probability density functions defined on the negative semi-axis. We consider the following two specifications:

1) Non-random discrete negative jumps with a known size $-\nu$, $\nu > 0$:

$$\varpi(j) = \delta(j + \nu), \qquad \kappa = \mathrm{e}^{-\nu} - 1 \quad (3)$$

where $\delta(x)$ is the Dirac delta function.

2) Random exponentially distributed jumps with parameter ν, $\nu > 0$:

$$\varpi(j) = \begin{cases} \nu\mathrm{e}^{\nu j}, & j \leqslant 0 \\ 0, & j > 0 \end{cases}, \qquad \kappa = -\frac{1}{\nu + 1} \quad (4)$$

In our experience, for one-dimensional marginal dynamics the choice of the jump size distribution has no appreciable impact on the model behavior; however, for the joint correlated dynamics this choice becomes crucial, as we demonstrate below.

The cornerstone assumption of SDMs is that the firm defaults when its value per share becomes less than a fraction of its debt per share. In our approach, which is similar to that of Lipton (2002) and Finger *et al* (2002), we assume that the default barrier of the firm is a deterministic function of time given by:

$$l(t) = E(t)l(0) \tag{5}$$

where:

$$E(t) = \exp\biggl(\int_0^t (r(t') - \varsigma(t') - \tfrac{1}{2}\sigma^2(t'))\,\mathrm{d}t'\biggr) \tag{6}$$

with $l(0) = RL(0)$, where R is the average recovery of the firm's liabilities and $L(0)$ is its total debt per share. The convexity term $\frac{1}{2}\sigma^2(t)$ reflects the fact that the barrier is flat for the logarithm of the asset value rather than for the asset value itself (as suggested by Zhou (2001b) and Haworth *et al* (2006)).

We fix a time horizon T and assume that the monitoring of the default barrier is either continuous or discrete. In the case of continuous monitoring, the default event can occur at any time t, $0 < t \leqslant T$ (Black and Cox (1976)). In the case of discrete monitoring occurring on a schedule $\{t_m^{\mathrm{d}}\}_{m=1,\dots,\bar{m}}$, $0 < t_1^{\mathrm{d}} < \cdots < t_{\bar{m}}^{\mathrm{d}} \leqslant T$, the default event can occur only at monitoring times t_m^{d} (Hull and White (2001); Lipton *et al* (2007)). If discrete monitoring is frequent (say, weekly monitoring), then both assumptions are equivalent, provided that the model is calibrated properly. From the computational standpoint, the second assumption is easier to handle, especially if Monte Carlo simulations are used. Continuous monitoring is much easier to treat via analytical methods; however, in many cases these methods are too restrictive. Below we briefly describe robust numerical methods for model calibration and pricing applicable for both continuous and discrete default monitoring.

We consider the firm equity price per share $s(t)$ and, following Stamicar and Finger (2006), assume that it is given by:

$$s(t) = \begin{cases} a(t) - l(t), & t < \tau \\ 0, & t \geqslant \tau \end{cases} \tag{7}$$

where τ is the default time.

We find the total debt per share $L(0)$ from the balance sheet as the ratio of the firm's total liabilities and the number of common shares outstanding. The average recovery R is estimated from the bonds and CDS quotes data. Then we compute the firm's default barrier, $l(0)$, as $l(0) = RD(0)$. At time $t = 0$, $s(0)$ is specified by the market price of the equity share. Accordingly, the initial asset value is given by:

$$a(0) = s(0) + l(0) \tag{8}$$

It is important to note that $\sigma(t)$ is the volatility of the asset value. The volatility of the equity price, $\sigma^{\mathrm{eq}}(t)$, is approximately related to $\sigma(t)$ by:

$$\sigma^{\mathrm{eq}}(t) = \left(1 + \frac{l(t)}{s(t)}\right)\sigma(t) \tag{9}$$

As a result, for fixed $\sigma(t)$ the equity volatility increases as the spot price $s(t)$ decreases, creating the leverage effect frequently observed in the equity market (see, for example, Rubinstein (1983)).

The solution of the stochastic differential equation (1) can be written as a product of a deterministic part and a stochastic exponent as follows:

$$a(t) = E(t)\mathrm{e}^{x(t)}l(0) \tag{10}$$

where the stochastic factor $x(t)$ is driven by the following dynamics under $\mathbb{Q}$:

$$\left.\begin{aligned} \mathrm{d}x(t) &= -\lambda(t)\kappa\,\mathrm{d}t + \sigma(t)\,\mathrm{d}w(t) + j\,\mathrm{d}N(t) \\ x(0) &= \ln\left(\frac{a(0)}{l(0)}\right) \equiv \xi \end{aligned}\right\} \tag{11}$$

Here $x(0)$ represents the "relative distance" of the asset value from the default barrier. In the current formulation, the default event occurs at the first time τ when $x(\tau)$ becomes non-positive. Here τ is continuous in the case of continuous monitoring and discrete in the case of discrete monitoring. The default barrier is fixed at zero and the default event is determined only by the dynamics of the stochastic driver $x(t)$.

The equity price per share $s(t)$ can be written as follows:

$$s(t) = \begin{cases} E(t)(\mathrm{e}^{x(t)} - 1)l(0), & t < \tau \\ 0, & t \geqslant \tau \end{cases} \tag{12}$$

3 MULTI-NAME STRUCTURAL MODEL

Consider N firms and assume that their asset values are driven by the following stochastic differential equations:

$$\begin{aligned} \mathrm{d}a_i(t) = (r(t) - \varsigma_i(t) - \kappa_i\lambda_i(t))a_i(t)\,\mathrm{d}t + \sigma_i(t)a_i(t)\,\mathrm{d}W_i(t) \\ + (\mathrm{e}^{j_i} - 1)a_i(t)\,\mathrm{d}N_i(t) \end{aligned} \tag{13}$$

where:

$$\kappa_i = \mathbb{E}\{\mathrm{e}^{j_i} - 1\} \tag{14}$$

Default boundaries have the form:

$$l_i(t) = E_i(t)l_i(0)$$

where:

$$E_i(t) = \exp\left(\int_0^t (r(t') - \varsigma_i(t') - \tfrac{1}{2}\sigma^2(t'))\,\mathrm{d}t'\right) l_i(0) \tag{15}$$

In log coordinates with:

$$x_i(t) = \ln\left(\frac{a_i(t)}{l_i(t)}\right) \tag{16}$$

we have:

$$\mathrm{d}x_i(t) = -\kappa_i \lambda_i(t)\,\mathrm{d}t + \sigma_i(t)\,\mathrm{d}W_i(t) + j_i\,\mathrm{d}N_i(t), \qquad x_i(0) = \ln\left(\frac{a_i(0)}{l_i(0)}\right) \equiv \xi_i \tag{17}$$

The corresponding default boundaries are now flat:

$$x_i = 0 \tag{18}$$

We correlate diffusions in the usual way and assume that:

$$\mathrm{d}W_i(t)\,\mathrm{d}W_j(t) = \rho_{ij}(t)\,\mathrm{d}t \tag{19}$$

We correlate jumps following the Marshall and Olkin (1967) idea. Let $\Pi^{(N)}$ be the set of all subsets of N names except for the empty subset $\{\varnothing\}$ and let π be its typical member. With every π we associate a Poisson process $N_\pi(t)$ with intensity $\lambda_\pi(t)$, and we represent $N_i(t)$ as follows:

$$N_i(t) = \sum_{\pi \in \Pi^{(N)}} 1_{\{i \in \pi\}} N_\pi(t) \tag{20}$$

$$\lambda_i(t) = \sum_{\pi \in \Pi^{(N)}} 1_{\{i \in \pi\}} \lambda_\pi(t) \tag{21}$$

Thus, we assume that there are both collective and idiosyncratic jump sources.

We now formulate a typical pricing equation in the positive cone $R_+^{(N)}$. We have:

$$\partial_t U(t, \vec{x}) + \mathcal{L}^{(N)} U(t, \vec{x}) = \chi(t, \vec{x}) \tag{22}$$

$$U(t, \vec{x}_{0,k}) = \phi_{0,k}(t, \vec{y}), \qquad U(t, \vec{x}_{\infty,k}) = \phi_{\infty,k}(t, \vec{y}) \tag{23}$$

$$U(T, \vec{x}) = \psi(\vec{x}) \tag{24}$$

where $\vec{x}$, $\vec{x}_{0,k}$, $\vec{x}_{\infty,k}$, $\vec{y}_k$ are N- and $(N-1)$-dimensional vectors, respectively:

$$\left.\begin{aligned}
\vec{x} &= (x_1, \dots, x_k, \dots, x_N) \\
\vec{x}_{0,k} &= \Big(x_1, \dots, \underset{k}{0}, \dots, x_N\Big) \\
\vec{x}_{\infty,k} &= \Big(x_1, \dots, \underset{k}{\infty}, \dots, x_N\Big) \\
\vec{y}_k &= (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_N)
\end{aligned}\right\} \tag{25}$$

Here $\chi(t,\vec{x})$, $\phi_{0,k}(t,\vec{y})$, $\phi_{\infty,k}(t,\vec{y})$, $\psi(\vec{x})$ are known quantities that are contract specific. The corresponding integro-differential operator $\mathcal{L}^{(N)}$ can be written in the form:

$$\mathcal{L}^{(N)}f(\vec{x}) = \frac{1}{2}\sum_i \sigma_i^2\partial_i^2 f(\vec{x}) + \sum_{i,j,j>i}\rho_{ij}\sigma_i\sigma_j\partial_i\partial_j f(\vec{x}) - \sum_i \kappa_i\lambda_i\partial_i f(\vec{x}) - \upsilon f(\vec{x}) + \sum_{\pi\in\Pi^{(N)}}\lambda_\pi\prod_{i\in\pi}\mathcal{J}_i f(\vec{x}) \tag{26}$$

where $\upsilon = r + \sum_{\pi\in\Pi^{(N)}}\lambda_\pi$.

For negative exponential jumps:

$$\mathcal{J}_i f(\vec{x}) = \nu_i\int_{-x_i}^{0} f(x_1,\dots,x_i+j_i,\dots,x_N)\mathrm{e}^{\nu_i j_i}\,\mathrm{d}j_i \tag{27}$$

For discrete negative jumps:

$$\mathcal{J}_i f(\vec{x}) = \boldsymbol{H}(x_i-\nu_i)f(x_1,\dots,x_i-\nu_i,\dots,x_N) \tag{28}$$

where $\boldsymbol{H}$ is the Heaviside function.

The corresponding adjoint operator is:

$$\mathcal{L}^{(N)\dagger}g(\vec{x}) = \frac{1}{2}\sum_i \sigma_i^2\partial_i^2 g(\vec{x}) + \sum_{i,j,j>i}\rho_{ij}\sigma_i\sigma_j\partial_i\partial_j g(\vec{x}) + \sum_i \kappa_i\lambda_i\partial_i g(\vec{x}) - \upsilon g(\vec{x}) + \sum_{\pi\in\Pi^{(N)}}\lambda_\pi\prod_{i\in\pi}\mathcal{J}_i^\dagger g(\vec{x}) \tag{29}$$

where:

$$\mathcal{J}_i^\dagger g(\vec{x}) = \nu_i\int_0^\infty g(x_1,\dots,x_i-j_i,\dots,x_N)\mathrm{e}^{\nu_i j_i}\,\mathrm{d}j_i \tag{30}$$

or:

$$\mathcal{J}_i^\dagger g(\vec{x}) = g(x_1,\dots,x_i+\nu_i,\dots,x_N) \tag{31}$$

It is easy to check that in both cases:

$$\int_{R_+^{(N)}}[\mathcal{J}_i f(\vec{x})g(\vec{x}) - f(\vec{x})\mathcal{J}_i^\dagger g(\vec{x})]\,\mathrm{d}\vec{x} = 0 \tag{32}$$

Now we can formulate Green's formula adapted to the problem under consideration. To this end we introduce Green's function $G(t,\vec{x})$, or, more explicitly, $G(t,\vec{x};0,\vec{\xi})$, such that:

$$\partial_t G(t,\vec{x}) - \mathcal{L}^{(N)\dagger}G(t,\vec{x}) = 0 \tag{33}$$

$$G(t,\vec{x}_{0k}) = 0, \qquad G(t,\vec{x}_{\infty k}) = 0 \tag{34}$$

$$G(0,\vec{x}) = \delta(\vec{x}-\vec{\xi}) \tag{35}$$

Integration by parts yields:

$$U(0,\vec{\xi}) = \int_{R_+^{(N)}} \psi(\vec{x})G(T,\vec{x})\,\mathrm{d}\vec{x} + \sum_k \int_0^T \int_{R_+^{(N-1)}} \phi_{0,k}(t,\vec{y})g_k(t,\vec{y})\,\mathrm{d}t\,\mathrm{d}\vec{y} - \int_0^T \int_{R_+^{(N)}} \chi(t,\vec{x})G(t,\vec{x})\,\mathrm{d}t\,\mathrm{d}\vec{x} \tag{36}$$

where:

$$g_k(t,\vec{y}) = \partial_k G\Big(t, x_1, \dots, \underset{k}{0}, \dots, x_N\Big) \tag{37}$$

This remarkable formula shows that instead of solving the backward pricing problem with non-homogeneous right-hand side and boundary conditions, we can solve the forward propagation problem for Green's function with homogeneous right-hand side and boundary conditions.

4 CREDIT VALUE ADJUSTMENT FOR CREDIT DEFAULT SWAPS

We are now in a position to formulate the pricing problems of interest. We start with a CDS on a reference name, 1, which is sold by a non-risky protection seller to a non-risky protection buyer. For the sake of brevity, we consider only exponential jumps. Assuming for simplicity that the coupon is paid continuously at the rate ι and that the default boundary is continuous, we write the pricing equation for the present value of a CDS for the protection buyer as follows:

$$\partial_t U^{(1)}(t,x_1) + \mathcal{L}^{(1)}U^{(1)}(t,x_1) = \chi(t,x_1) \tag{38}$$

$$U^{(1)}(t,0) = \phi_0(t), \qquad U^{(1)}(t,\infty) = \phi_\infty(t) \tag{39}$$

$$U^{(1)}(T,x_1) = 0 \tag{40}$$

Here:

$$\chi(t,x_1) = \iota - (1-\hat{R}_1)\lambda_1 \mathrm{e}^{-\nu_1 x_1} \tag{41}$$

$$\phi_0(t) = (1-R_1), \qquad \phi_\infty(t) = -\iota \int_t^T D(t,t')\,\mathrm{d}t' \tag{42}$$

$$\hat{R}_1 = \nu_1 R_1 \int_{-\infty}^0 \mathrm{e}^{(1+\nu_1)j_1}\,\mathrm{d}j_1 = \frac{\nu_1 R_1}{1+\nu_1} \tag{43}$$

where $\hat{R}_1$ is the conditional recovery rate and $D(t,t')$ is the discount factor. We emphasize that $\hat{R}_1 < R_1$ due to the jump effects.

The corresponding problem for Green's function has the form:

$$\partial_t G(t,x_1) - \mathcal{L}^{(1)\dagger}G(t,x_1) = 0 \tag{44}$$

$$G(t,0) = 0, \qquad G(t,\infty) = 0 \tag{45}$$

$$G(0,x_1) = \delta(x_1 - \xi_1) \tag{46}$$

When parameters are constant, this problem can be solved via the direct and inverse Laplace transform (see Lipton (2002); Sepp (2004)).

The direct Laplace transform $G(t, x_1) \to \hat{G}(p, x_1)$ yields:

$$\begin{aligned} \tfrac{1}{2}\sigma_1^2 \hat{G}_{x_1 x_1} + \kappa_1 \lambda_1 \hat{G}_{x_1} - (v_1 + p)\hat{G} \\ +\lambda_1 \nu_1 \int_0^\infty \hat{G}(p, x_1 + j_1) \mathrm{e}^{-\nu_1 j_1}\, \mathrm{d}j_1 = -\delta(x_1 - \xi_1) \end{aligned} \tag{47}$$

$$\hat{G}(p, 0) = 0, \qquad \hat{G}(p, \infty) = 0 \tag{48}$$

The corresponding forward characteristic equation has the form:

$$\tfrac{1}{2}\sigma_1^2 \psi^2 + \kappa_1 \lambda_1 \psi - (v_1 + p) - \frac{\lambda_1 \nu_1}{\psi - \nu_1} = 0 \tag{49}$$

This equation has three roots of which two are positive and one negative. We denote them by $-\psi_i$.

The function $\hat{G}(p, x_1)$ has the form:

$$\hat{G}(p, x_1) = \begin{cases} C_3 \mathrm{e}^{-\psi_3(x_1 - \xi_1)}, & x_1 \geqslant \xi_1 \\ D_1 \mathrm{e}^{\psi_1 \xi_1}[\mathrm{e}^{-\psi_1 x_1} - \mathrm{e}^{-\psi_3 x_1}] + D_2 \mathrm{e}^{\psi_2 \xi_1}[\mathrm{e}^{-\psi_2 x_1} - \mathrm{e}^{-\psi_3 x_1}], & x_1 < \xi_1 \end{cases} \tag{50}$$

where:

$$\left.\begin{aligned} D_1 &= -\frac{2}{\sigma_1^2} \frac{(\nu_1 + \psi_1)}{(\psi_1 - \psi_2)(\psi_1 - \psi_3)}, \qquad D_2 = -\frac{2}{\sigma_1^2} \frac{(\nu_1 + \psi_2)}{(\psi_2 - \psi_1)(\psi_2 - \psi_3)} \\ C_3 &= \frac{2}{\sigma_1^2}\left[\frac{(\mathrm{e}^{(\psi_1 - \psi_3)\xi} - 1)(\nu_1 + \psi_1)}{(\psi_1 - \psi_2)(\psi_1 - \psi_3)} + \frac{(\mathrm{e}^{(\psi_2 - \psi_3)\xi} - 1)(\nu_1 + \psi_2)}{(\psi_2 - \psi_1)(\psi_2 - \psi_3)}\right] \end{aligned}\right\} \tag{51}$$

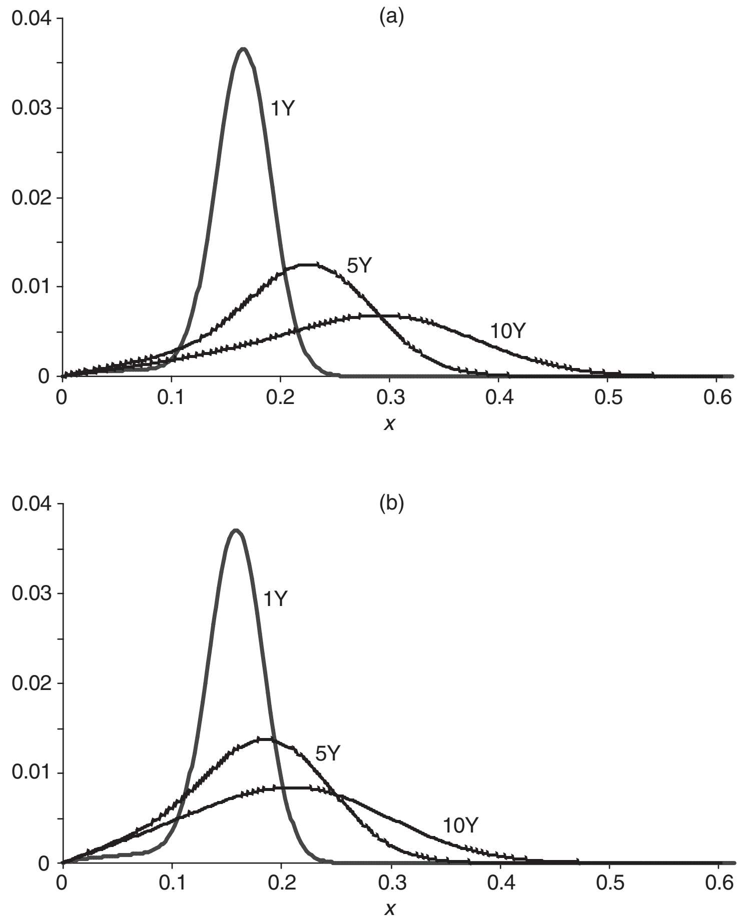
The inverse Laplace transform yields $G(t, x_1)$. We use the Stehfest algorithm and write:

$$G(t, x_1) = \frac{\ln 2}{t} \sum_{k=1}^{N} (-1)^k \operatorname{St}_k^N \hat{G}\left(k \frac{\ln 2}{t}, x_1\right) \tag{52}$$

where St_k^N are the Stehfest coefficients (Stehfest (1970)). We typically choose $N = 20$. For small t, inversion can be numerically unstable unless computation is performed with high accuracy. Once $G(t, x_1)$ is known, all the relevant instruments, including CDSs, equity options, etc, can be evaluated without too much effort. In Figure 1 on the next page, we show $G(t, x_1)$ for a model with representative parameters at $t = 1\mathrm{Y}, 5\mathrm{Y}, 10\mathrm{Y}$.

When parameters are time dependent we need to use numerical methods in order to solve the pricing problem. Time and space are discretized in the usual way. A judicious combination of finite differences for partial and ordinary differential equations allows us to build a fast and accurate numerical scheme. Only the integral

FIGURE 1 Analytically computed probability density function for $x(t)$ conditional on no default for the constant parameter case: $\xi = 0.15$, $\sigma = 2.5\%$, $\lambda = 10\%$, $r = 0$, (a) $\nu = 5.0$ and (b) $\nu = 10.0$ exponential jump.



term requires special treatment, the rest of the scheme is standard (see, for example, Lipton (2001)). Specifically, let:

$$I_\nu(x) = \nu \int_{-x}^{0} f(x+j)\mathrm{e}^{\nu j}\,\mathrm{d}j = \nu \int_{0}^{x} \mathrm{e}^{-\nu(x-y)} f(y)\,\mathrm{d}y \tag{53}$$

We can then calculate I_ν recursively on the spatial grid:

$$I_\nu(x) = \mathrm{e}^{-\nu\delta x} I_\nu(x-\delta x) + \frac{1-(1+\nu\delta x)\mathrm{e}^{-\nu\delta x}}{\nu\delta x} f(x-\delta x) + \frac{-1+\nu\delta x+\mathrm{e}^{-\nu\delta x}}{\nu\delta x} f(x) \tag{54}$$

with $I_\nu(0) = 0$ (Lipton and Sepp (2009)). This is a special trick for exponential jumps – it does not work for more familiar Gaussian jumps. For discrete jumps a simple interpolation routine is sufficient.

Alternatively, we can use the discrete Fourier transform (DFT) to solve the pricing problem. Without going into detail, we present the corresponding method symbolically. In this method, we expand the computational domain and calculate one time step evolution by applying the DFT algorithm twice and write:

$$U^{(1)}(t-\delta t,\xi) = \boldsymbol{P}(\mathrm{IFFT}(\mathrm{FFT}(U^{(1)}(t,x)) \odot \mathrm{FFT}(G(t,x;t-\delta t,\xi)))) \tag{55}$$

where FFT and IFFT stand for the directional and inverse FFT, respectively, $\odot$ denotes element-wise multiplication and $\boldsymbol{P}$ denotes projection on the positive semi-axis. The algorithm is made feasible by the fact that $\mathrm{FFT}(G(t,x;t-\delta t,\xi))$ can be computed analytically (Lipton *et al* (2007)).

For a CDS on a reference name, 1, which is sold by a risky protection seller, 2, to a non-risky protection buyer, we write the pricing equation as follows:

$$\partial_t U^{(2)}(t,x_1,x_2) + \mathcal{L}^{(2)} U^{(2)}(t,x_1,x_2) = \chi(t,x_1,x_2) \tag{56}$$

$$\left.\begin{aligned} U^{(2)}(t,0,x_2) &= \phi_{0,1}(t,x_2), \qquad U^{(2)}(t,\infty,x_2) = \phi_{\infty,1}(t,x_2) \\ U^{(2)}(t,x_1,0) &= \phi_{0,2}(t,x_1), \qquad U^{(2)}(t,x_1,\infty) = \phi_{\infty,2}(t,x_1) \end{aligned}\right\} \tag{57}$$

$$U^{(2)}(T,x_1,x_2) = 0 \tag{58}$$

Here:

$$\begin{aligned} &\chi(t,x_1,x_2) \\ &\qquad = \iota - \lambda_{\{1\}}\upsilon_1(t,x_1,x_2) - \lambda_{\{2\}}\upsilon_2(t,x_1,x_2) - \lambda_{\{1,2\}}\upsilon_{1,2}(t,x_1,x_2) \end{aligned} \tag{59}$$

$$\left.\begin{aligned} \upsilon_1(t,x_1,x_2) &= (1-\hat{R}_1)\mathrm{e}^{-\nu_1 x_1} \\ \upsilon_2(t,x_1,x_2) &= \tilde{U}^{(1)}(t,x_1)\mathrm{e}^{-\nu_2 x_2} \end{aligned}\right\} \tag{60}$$

$$\begin{aligned} \upsilon_{1,2}(t,x_1,x_2) = (1-\hat{R}_1)&\mathrm{e}^{-\nu_1 x_1}(1-\mathrm{e}^{-\nu_2 x_2}) \\ &+ \nu_1 \mathrm{e}^{-\nu_2 x_2}\int_{-x_1}^{0} \tilde{U}^{(1)}(t,x_1+j_1)\mathrm{e}^{\nu_1 j_1}\,\mathrm{d}j_1 \\ &\qquad + (1-\hat{R}_1)\hat{R}_2\mathrm{e}^{-\nu_1 x_1-\nu_2 x_2} \end{aligned} \tag{61}$$

$$\tilde{U}^{(1)}(t,x_1) = \hat{R}_2 U_+^{(1)}(t,x_1) + U_-^{(1)}(t,x_1) \tag{62}$$

$$\phi_{0,1}(t,x_2) = 1-R_1, \qquad \phi_{\infty,1}(t,x_2) = -\iota\int_t^T D(t,t')\,\mathrm{d}t' \tag{63}$$

$$\phi_{0,2}(t,x_1) = R_2 U_+^{(1)}(t,x_1) + U_-^{(1)}(t,x_1), \qquad \phi_{\infty,2}(t,x_1) = U^{(1)}(t,x_1) \tag{64}$$

and $x_+ = \max(x,0)$, $x_- = \min(x,0)$. We define CVA as the difference between $U^{(1)}$ and $U^{(2)}$:

$$\mathrm{CVA} = U^{(1)} - U^{(2)} \tag{65}$$

A numerical method for solving the above problem is given by Lipton and Sepp (2009). We discuss some representative examples below.

Mutual dependence for two names is specified as follows. It is assumed that standard Brownian motions $W_1(t)$ and $W_2(t)$ are correlated with parameter ρ:

$$\mathrm{d}W_1(t)\,\mathrm{d}W_2(t) = \rho\,\mathrm{d}t \tag{66}$$

collective jumps occur with intensity $\lambda_{\{1,2\}}(t)$:

$$\lambda_{\{1,2\}}(t) = \max\{\rho, 0\}\min\{\lambda_1(t), \lambda_2(t)\} \tag{67}$$

and idiosyncratic jumps occur with complementary intensities $\lambda_{\{1\}}(t)$ and $\lambda_{\{2\}}(t)$, specified as follows:

$$\lambda_{\{1\}}(t) = \lambda_1(t) - \lambda_{\{1,2\}}(t), \qquad \lambda_{\{2\}}(t) = \lambda_2(t) - \lambda_{\{1,2\}}(t) \tag{68}$$

Expressing correlation structure in terms of one parameter ρ is advantageous for calibration of the model. After the calibration to marginal dynamics is completed for each firm and the set of the firm's volatilities, jump sizes and intensities is obtained, we estimate the parameter ρ by fitting the model spread of a first-to-default swap to a given market quote. If such a quote is not available, we use historical data.

It is clear that the default time correlations are closely connected to instantaneous correlations of the firm's value dynamics. Using the postulated bivariate dynamics given by Equation (13), we calculate the instantaneous correlations between the firms in question under the assumption of discrete jumps, $\rho_{12}^{\mathrm{dis}}(t)$, and exponential jumps, $\rho_{12}^{\exp}(t)$:

$$\rho_{12}^{\mathrm{dis}}(t) = \frac{\rho\sigma_1(t)\sigma_2(t) + \lambda_{\{1,2\}}(t)\nu_1\nu_2}{\sqrt{\sigma_1^2(t) + \lambda_1(t)\nu_1^2}\sqrt{\sigma_2^2 + \lambda_2(t)\nu_2^2}} \tag{69}$$

$$\rho_{12}^{\exp}(t) = \frac{\rho\sigma_1(t)\sigma_2(t) + \lambda_{\{1,2\}}(t)/\nu_1\nu_2}{\sqrt{\sigma_1^2(t) + 2\lambda_1(t)/\nu_1^2}\sqrt{\sigma_2^2(t) + 2\lambda_2(t)/\nu_2^2}} \tag{70}$$

In the limit of high systemic intensity $\lambda_{\{1,2\}} \sim \lambda_1(t) \sim \lambda_2(t)$, we have:

$$\rho_{12}^{\mathrm{dis}} \sim 1, \qquad \rho_{12}^{\exp} \sim \tfrac{1}{2} \tag{71}$$

Thus, if jumps are exponential, the correlation cannot approach unity. In our experiments with different firms, we compute implied Gaussian correlations from model spreads for first-to-default swaps referencing different names. Typically, the maximal implied Gaussian correlation that can be achieved in our model is about 90% for the model with negative discrete jumps and about 50% for the model with negative exponential jumps, provided that marginal dynamics are calibrated to match the term structure of CDS spreads and CDS option volatilities. Accordingly, using negative exponential jumps for modeling the joint dynamics of strongly correlated firms belonging to one industry group is not advisable.

TABLE 1 Market data for December 8, 2009.

	XYZ	ZYX
$s(0)$	36.49	8.47
$L(0)$	604.11	353.07
R	40%	40%
$l(0)$	241.64	141.22
$a(0)$	278.13	149.70
ξ	0.1406	0.0582
Jump size	0.1406	0.0582
Jump size	0.0703	0.0291
σ	0.0262	0.0113

TABLE 2 CDS spread ι, survival probability Q, and the values of the default leg and the risky annuity for XYZ and ZYX.

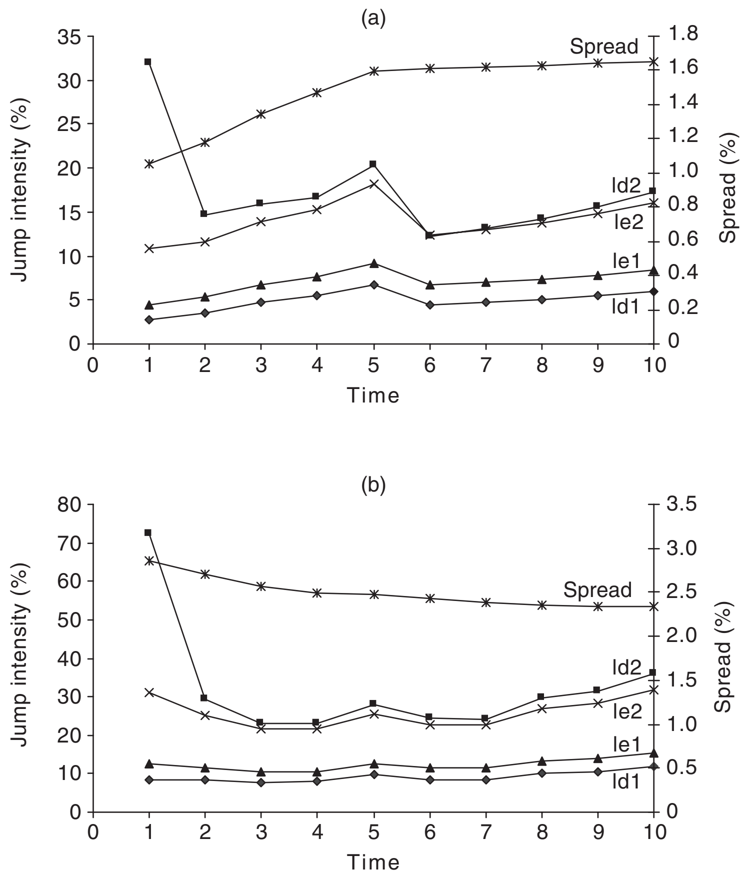
	ι (%)		Q (%)		DL		RA	
T	XYZ	ZYX	XYZ	ZYX	XYZ	ZYX	XYZ	ZYX
1Y	1.05	2.86	98.3	95.3	0.0104	0.0279	0.991	0.977
2Y	1.18	2.71	96.1	91.4	0.0232	0.0518	1.963	1.910
3Y	1.34	2.57	93.5	88.0	0.0391	0.0721	2.911	2.807
4Y	1.47	2.49	90.6	85.6	0.0562	0.0915	3.832	3.670
5Y	1.60	2.48	87.4	81.4	0.0754	0.1117	4.722	4.501
6Y	1.61	2.43	85.0	78.6	0.0901	0.1286	5.584	5.300
7Y	1.62	2.38	82.7	75.9	0.1032	0.1446	6.422	6.073
8Y	1.63	2.36	80.3	73.2	0.1180	0.1609	7.237	6.817
9Y	1.64	2.34	78.0	70.6	0.1318	0.1766	8.029	7.537
10Y	1.65	2.33	75.8	68.0	0.1451	0.1919	8.799	8.229

5 AN EXAMPLE OF MODEL CALIBRATION AND CVA VALUATION

For illustration, we use market data for two representative financial institutions: XYZ and ZYX. In Table 1 we provide a market data snapshot for December 8, 2009; it is used to calculate the initial absolute and relative asset values $v(0)$ and ξ. In Table 2 we provide the term structure of CDS spreads for XYZ and ZYX, their survival probabilities implied from these spreads via a reduced-form model with piecewise constant hazard rate and the corresponding default leg and risky annuity (RA) ($\text{CL} = \iota\,\text{RA}$). For details of this procedure see, for example, Jarrow and Turnbull (1995). For simplicity, we assume quarterly coupon payments and a discount factor of 1.

We compare jump-diffusion models with discrete and exponential jumps and consider two choices for the jump sizes. First, we set the expected jump size to

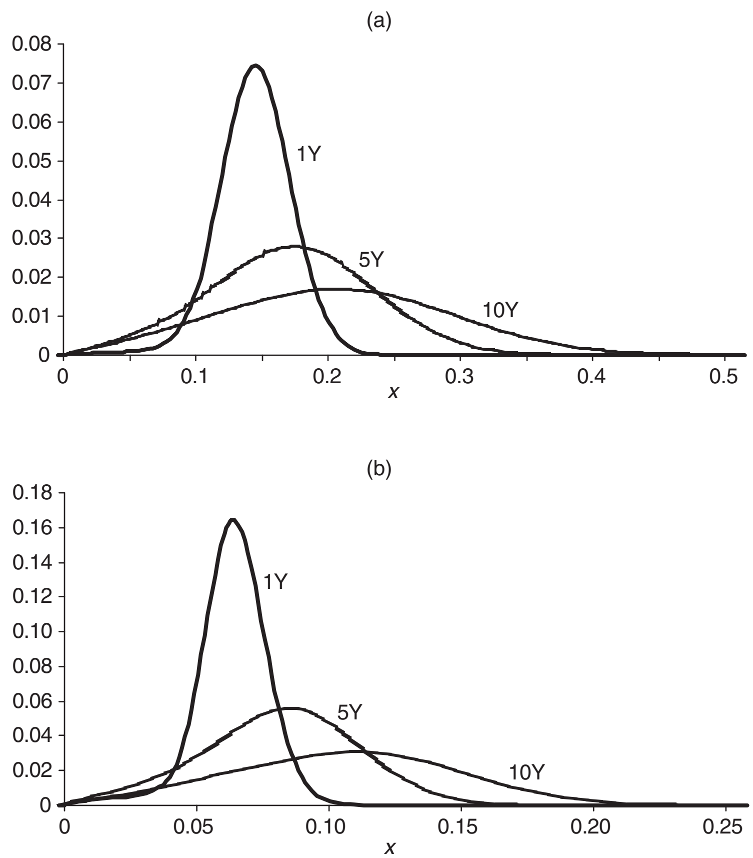
FIGURE 2 Spreads (right axis) and calibrated jump intensities (left axis) for (a) XYZ and (b) ZYX.



the initial relative value of the firm, so that $\nu \equiv \nu_1 = x(0)$ in the model with discrete jumps and $\nu \equiv 1/\nu_1 = x(0)$ in the model with exponential jumps. Second, we set the jump amplitude to half the previous choice: $\nu \equiv \nu_2 = \frac{1}{2}x(0)$ and $\nu \equiv 1/\nu_2 = \frac{1}{2}x(0)$, respectively. The asset volatility σ is chosen via Equation (9) in such a way that the diffusion fraction of equity volatility is 20%. We assume discrete weekly default monitoring. We use a finite-difference solver for model calibration and counterparty charge evaluation.

All in all we have four different models for each firm. Each of these models is calibrated to the term structure of CDS spreads given in Table 2 on the preceding page using forward induction. Following this we use the abbreviations le1 and le2 for the model with exponential jumps and ld1 and ld2 for the model with discrete jumps, respectively.

FIGURE 3 The probability density function of the stochastic driver $x(t)$ conditional on no default: (a) XYZ; (b) ZYX.

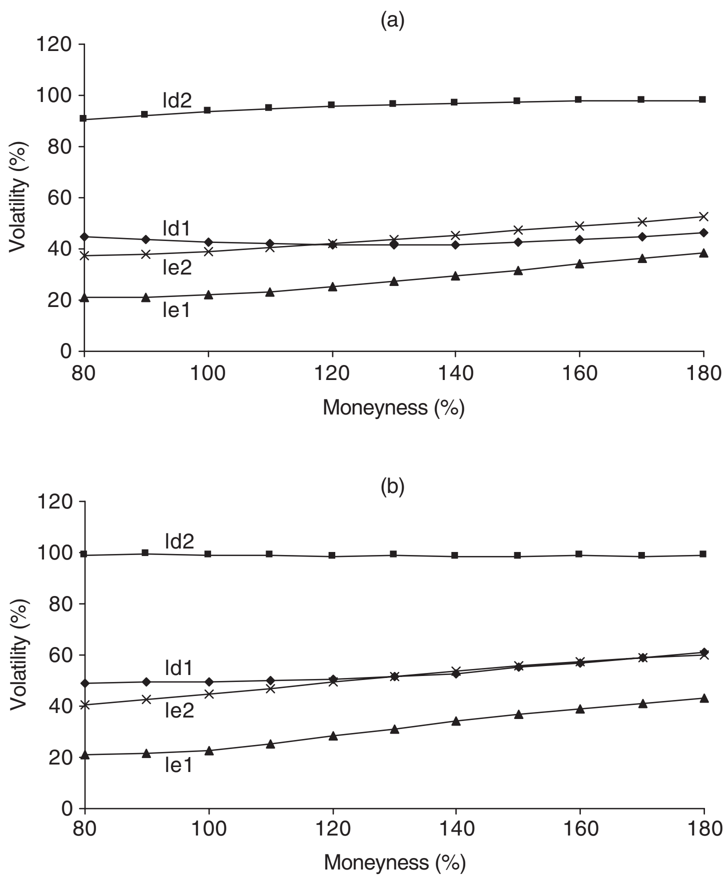


In Figure 2 on the facing page we show calibrated intensities. We see that when the jump size is large, $\nu = x(0)$, the first jump leads to the crossing of the default barrier with high probability for both stochastic and deterministic jumps, so that jump intensities are close in both cases. For the smaller jump size, the model with exponential jumps has a higher intensity rate because exponential jumps have higher variance, and thus the model expects the jumps to be more frequent in order to match market data.

In Figure 3 we show the implied density of the driver $x(t)$ for the calibrated model with large exponential jumps at $t = 1\text{Y}, 5\text{Y}, 10\text{Y}$. We see that even for $t = 1\text{Y}$ there is a possibility of default in the short term.

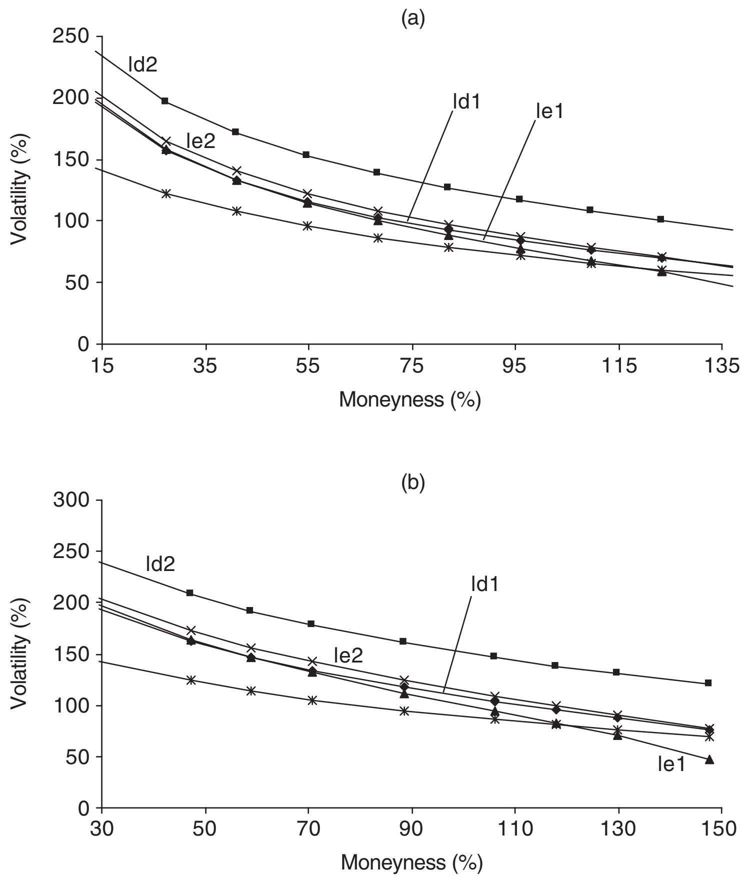
In Figure 4 on the next page we show the implied lognormal CDS option volatility for a one-year payer option on a five-year CDS contract calculated via a version of

FIGURE 4 Black CDS option volatility for a one-year option on a five-year CDS: (a) XYZ; (b) ZYX.



the Black futures formula (see, for example, Hull and White (2003); Schönbucher (2004)). This formula assumes that the five-year spread ι at the option expiry is lognormal with volatility σ. We plot the model implied volatility as a function of the moneyness, K/f, where f is the forward CDS spread at option expiry. We observe that the model implied lognormal volatility σ exhibits a positive skew. This effect is in line with the fact that option market makers charge an extra premium for out-of-the-money CDS options, because the spread volatility is expected to increase when the spread itself increases. Comparing models with different jump sizes, we see that when the jump size is large, the first jump leads to the crossing of the default barrier with high probability, so the spread volatility is low; when the jump size is small, the asset value is expected to have several jumps before crossing the default barrier, which results in higher spread volatility. Comparing models with different

FIGURE 5 Market and model implied Black–Scholes volatilities for six-month equity put options: (a) XYZ; (b) ZYX.

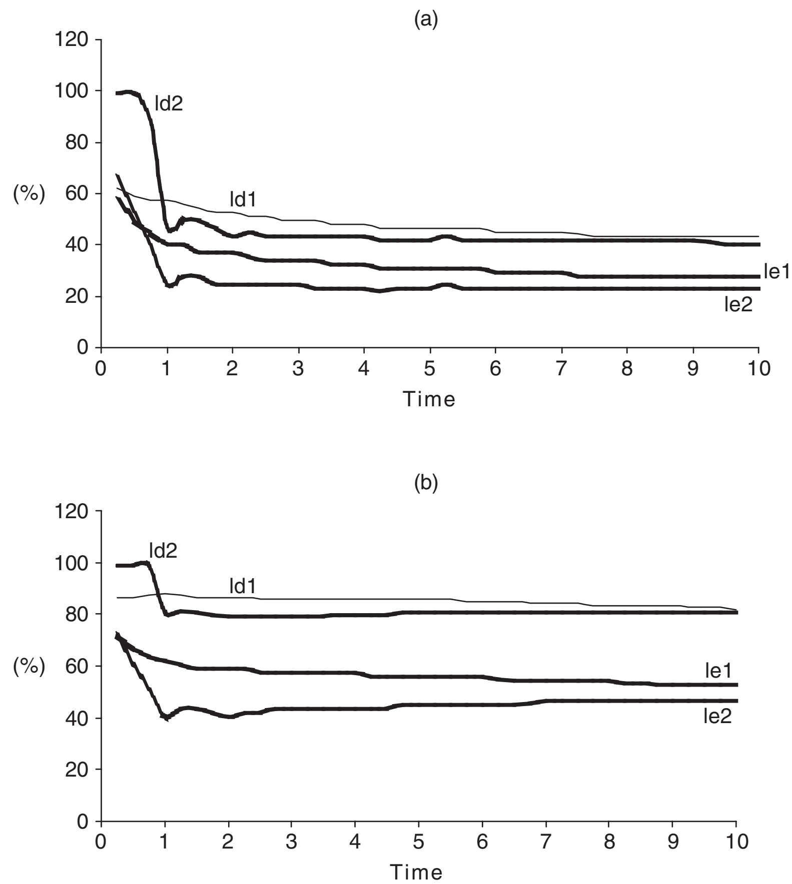


jump distributions, we see that the model with discrete jumps implies higher spread volatility than the one with exponential jumps. The reason is that, in the former case, the firm is expected to have more jumps before crossing the barrier, so the realized volatility has to be higher.

In Figure 5 we plot the model implied volatility skew for equity put options with a maturity of six months as a function of moneyness, K/s. We see that the model implies a pronounced skew in line with the one observed in the market. The skew appears to be less sensitive to the choice of the distribution. For the same reason as before, smaller jump size causes higher volatility.

In Figure 6 on the next page we illustrate the model implied Gaussian correlation, which is inferred by equating the fair spread of first-to-default swap referencing XYZ and ZYX computed using the calibrated model to the fair spread computed

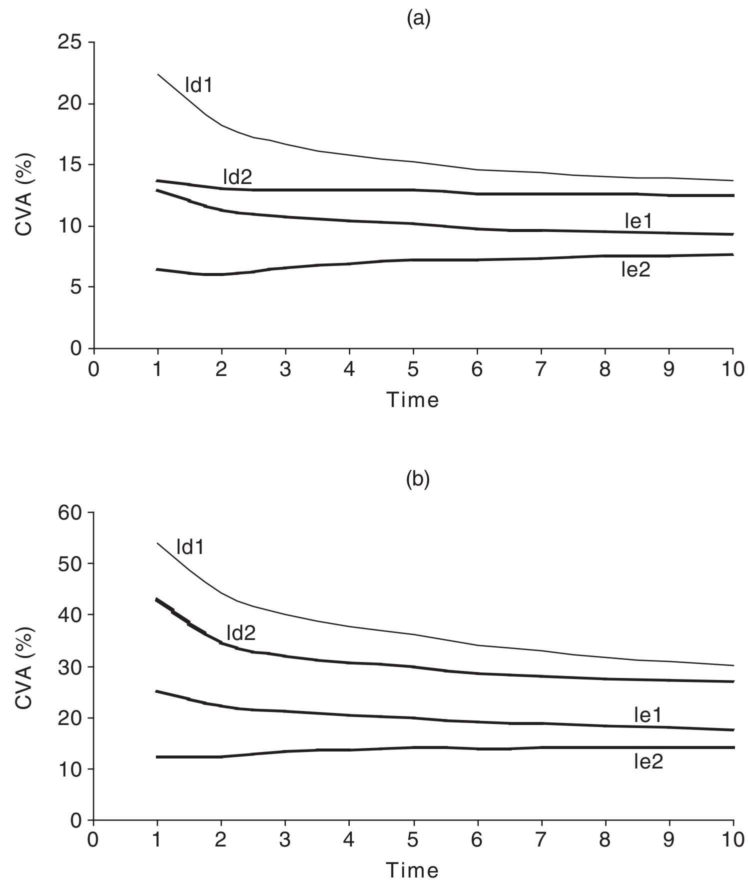
FIGURE 6 Implied Gaussian correlations for different instantaneous correlation specifications: (a) $\rho = 0.5$; (b) $\rho = 0.99$.



using the Gaussian copula with an appropriate correlation. We consider two choices for the correlation parameter: $\rho = 0.5$ and $\rho = 0.99$ (see Equations (66) and (67)). We see that, in line with Equation (69), the model with exponential jumps produces lower implied correlation than the one with discrete jumps. In general the correlation is smaller when the jumps are smaller; however, this effect is less pronounced for the model with discrete jumps.

In Figure 7 on the facing page we show CVA for par CDSs with XYZ as the reference name and ZYX as the protection seller as a function of maturity using CDS spreads from Table 2 on page 137 and two model correlation parameters: $\rho = 0.5$ and $\rho = 0.99$. We show CVA normalized by the present value of the default leg of the corresponding CDSs. We observe the following effects. First, higher correlation

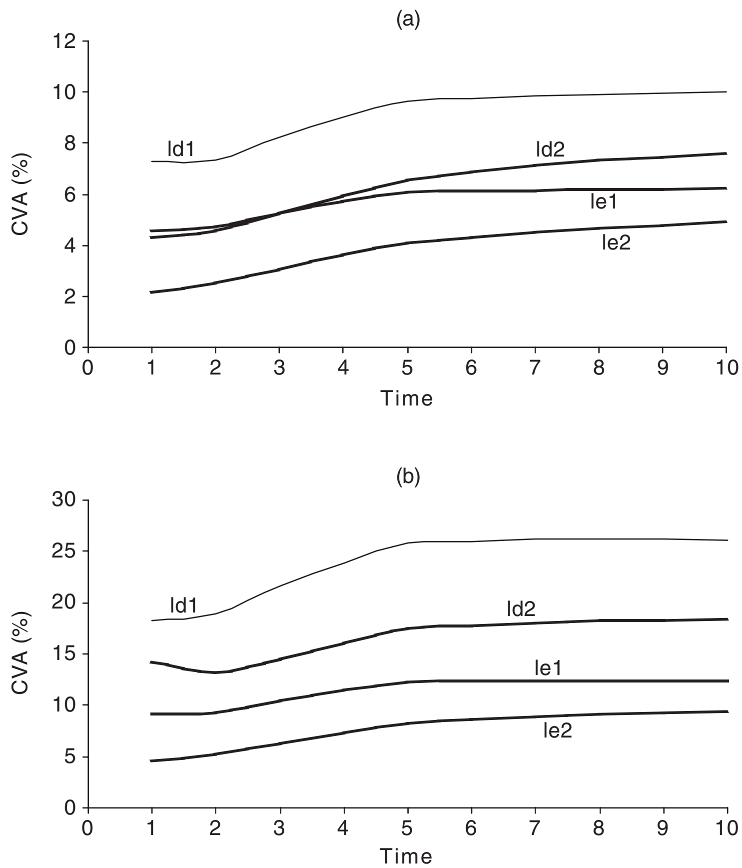
FIGURE 7 Credit value adjustment for CDSs on XYZ underwritten by ZYX normalized by the value of the default leg for different instantaneous correlation specifications: (a) $\rho = 0.5$; (b) $\rho = 0.99$.



results in higher CVA, because, conditional on protection seller default, the loss of protection is expected to be higher. Second, higher jump size results in higher CVA, because the model with higher jumps implies higher correlation (see Figure 5 on page 141). Third, discrete jumps result in higher CVA than exponential jumps, because the former imply higher correlation and higher CDS spread volatility. We see that for moderate correlation, $\rho = 0.5$, the model with large discrete jumps implies CVA of the order of 10–15% of the present value of the default leg, while for high correlation, $\rho = 0.99$, its magnitude grows to 30–40% of the default leg.

In Figure 8 on the next page we show CVA for par CDSs with ZYX as the reference name and XYZ as the protection seller. We observe the same effects as before. We also see that CVA is not symmetric. It might be expected that a riskier protection

FIGURE 8 Credit value adjustment for CDSs on ZYX underwritten by XYZ normalized by the value of the default leg for different instantaneous correlation specifications: (a) $\rho = 0.5$; (b) $\rho = 0.99$.



seller implies higher CVA; however, for high correlation the opposite effect might be observed. The reason being that in cases when a safer protection seller defaults right before the riskier reference name, the realized loss is higher than that in the opposite case.

6 CONCLUSIONS

We present a multi-dimensional extension of the Merton (1974) model, with the joint dynamics of asset values driven by a multi-dimensional jump-diffusion process. Applying FFT- and PIDE-based methods, we develop a forward induction procedure for the model calibration and a backward induction procedure for the valuation of credit instruments in one and two dimensions.

We consider two types of jump size distributions: namely, negative discrete and negative exponential. We show that both jump specifications result in an adequate fit to the market data for individual names. However, for the joint bivariate dynamics, the model with exponential negative jumps produces a noticeably lower correlation between names than the model with discrete negative jumps. Based on this observation and given a high level of the default correlation among financial institutions (above 50%), the model with constant negative jumps, despite being simple, seems to provide a more realistic description of default correlations and, thus, is preferable for the evaluation of CVA. Our empirical analysis shows that CVA is largely determined by the correlation between the reference name and the protection seller, and their spread volatility, with higher correlation and volatility resulting in higher adjustment. Thus, by accounting for both of these factors, the model provides a robust estimate of the counterparty risk.

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