WIENER-HOPF FACTORIZATION AND DISTRIBUTION OF EXTREMA FOR A FAMILY OF LÉVY PROCESSES

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In this paper we introduce a ten-parameter family of Lévy processes for which we obtain Wiener-Hopf factors and distribution of the supremum process in semi-explicit form. This family allows an arbitrary behavior of small jumps and includes processes similar to the generalized tempered stable, KoBoL and CGMY processes. Analytically it is characterized by the property that the characteristic exponent is a meromorphic function, expressed in terms of beta and digamma functions. We prove that the Wiener-Hopf factors can be expressed as infinite products over roots of certain transcendental equation and the density of the supremum process can be computed as an exponentially converging infinite series. In several special cases when the roots can be found analytically we are able to identify the Wiener-Hopf factors and distribution of the supremum in closed form. In general case we prove that all the roots are real and simple, we provide localization results and asymptotic formulas which allow an efficient numerical evaluation. We also derive a convergence acceleration algorithm for infinite products and a simple and efficient procedure to compute the Wiener-Hopf factors for complex values of parameters. As a numerical example we discuss computation of the density of the supremum process.

1. Introduction.

Wiener-Hopf factorization is a powerful tool in the study of various functionals of a Lévy process, such as extrema of the process, first passage time and the overshoot, the last time the extrema was achieved, etc. These results are very important from the theoretical point of view, for example they can be used to prove general theorems about short/long time behavior, see [5], [17] and [13]. Howevery in recent years there is also a growing interest in applications of Wiener-Hopf factorization, for example in Mathematical Finance, where the above mentioned functionals are being used to describe the payoff of a contract and the corresponding probability distribution is used to compute its price (see [2], [6], [16] and [18]).

Let us summarize one of the most important results on Wiener-Hopf factorization. Assume that X_t is a one-dimensional real-valued Lévy process started from $X_0 = 0$ and defined by a triple (μ, σ, ν): the drift $\mu \in \mathbb{R}$, volatility $\sigma \geq 0$ and the jump measure $\nu(x)$ which is given by a nonnegative function defined on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}} \min\{1, x^2\}\nu(x)dx < \infty$. The characteristic exponent $\Psi(z)$ is defined by

$$\mathbb{E}\left[e^{izX_t}\right] = \exp(-t\Psi(z)),$$

and the Lévy-Khintchine representation (see [5]) tells us that $\Psi(z)$ can be expressed in terms of the generating triple (μ, σ, ν) as follows

$$\Psi(z) = -i\mu z + \frac{\sigma^2 z^2}{2} - \int_{\mathbb{R}} \left(e^{izx} - 1 - izx\right)\nu(x)dx. \tag{1}$$

We define extrema processes $M_t = \sup\{X_s : s \leq t\}$ and $N_t = \inf\{X_s : s \leq t\}$, introduce an exponential random variable $\tau(q)$ with parameter $q > 0$, which is independent of the process X_t, and use the following notation for

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characteristic functions of $M_{\tau(q)}$, $N_{\tau(q)}$:

$$\phi_q^+(z) = \mathbb{E}\left[e^{izM_{\tau(q)}}\right], \quad \phi_q^-(z) = \mathbb{E}\left[e^{izN_{\tau(q)}}\right].$$

The Wiener-Hopf factorization states that the random variables $M_{\tau(q)}$ and $X_{\tau(q)} - M_{\tau(q)}$ are independent, random variables $N_{\tau(q)}$ and $X_{\tau(q)} - M_{\tau(q)}$ have the same distribution, and for $z \in \mathbb{R}$ we have

$$\begin{aligned} \frac{q}{\Psi(z)+q} &= \mathbb{E}\left[e^{izX_{\tau(q)}}\right] \\ &= \mathbb{E}\left[e^{izM_{\tau(q)}}\right]\mathbb{E}\left[e^{iz(X_{\tau(q)}-M_{\tau(q)})}\right] = \phi_q^+(z)\phi_q^-(z). \end{aligned} \tag{2}$$

Moreover, random variable $M_{\tau(q)}$ $[N_{\tau(q)}]$ is infinitely divisible, positive [negative] and has zero drift. There also exist several integral representation for $\phi_q^{\pm}$ in terms of $\mathbb{P}(X_t \in dx)$ (see [5], [17] or [13]) or in terms of $\Psi(z)$ ([4], [15]).

The integral expressions for the Wiener-Hopf factors $\phi_q^{\pm}$ are quite complicated, however in the case of stable process it is possible to obtain explicit formulas for a dense class of parameters, see [10]. It is remarkable that in some cases we can compute Wiener-Hopf factors explicitly with the help of factorization identity (2). As an example, let us consider the case when the jump measure ν is of phase-type. Phase-type distribution (see [1]) can be defined as the distribution of the first passage time of a finite state continuous time Markov chain. A Lévy process X_t whose jumps are phase-type distributed enjoys the following analytical property: its characteristic function $\Psi(z)$ is a rational function. Thus function $q(\Psi(z)+q)^{-1}$ is also a rational function, therefore it has a finite number of zeros/poles in the complex plane $\mathbb{C}$. And here is the main idea: since the random variable $M_{\tau(q)}$ $[N_{\tau(q)}]$ is positive [negative], its characteristic function must be analytic and have no zeros in $\mathbb{C}^+$ $[\mathbb{C}^-]$, where

$$\mathbb{C}^+ = \{z \in \mathbb{C} \; : \; \text{Im}(z) > 0\}, \;\; \mathbb{C}^- = \{z \in \mathbb{C} \; : \; \text{Im}(z) < 0\}, \;\; \bar{\mathbb{C}}^{\pm} = \mathbb{C}^{\pm} \cup \mathbb{R}.$$

Thus we can *uniquely* identify $\phi_q^+(z)$ $[\phi_q^-(z)]$ as a rational function, which has value one at $z=0$ and whose poles/zeros coincide with poles/zeros of $q(\Psi(z)+q)^{-1}$ in $\mathbb{C}^+$ $[\mathbb{C}^-]$.

While Lévy processes with phase-type jumps are very convenient objects to work with and one can implement efficient numerical schemes, there are some unresolved difficulties. One of them is that by definition phase-type distribution has a smooth density on $[0,\infty)$, in particular we can not have a singularity at zero in the jump measure. If we approximate a jump measure with singularity by a sequence of phase-type measures, the degree of rational function $\Psi(z)$ would go to infinity and the above algorithm for computing Wiener-Hopf factors would quickly become unfeasible.

In this paper we address this problem and discuss Wiener-Hopf factorization for processes whose jump measure can have a singularity of arbitrary order at zero. The main idea is quite simple: if characteristic exponent $\Psi(z)$ is *meromorphic* in $\mathbb{C}$ and if we have sufficient information about zeros/poles of $\Psi(z)+q$, we can still use the factorization identity (2) essentially in the same way as in the case of phase-type distributed jumps, except that all the finite products will be replaced by infinite products and we have to carefully treat the convergence. The main analytical tools will be asymptotic expansion of solutions to $\Psi(z)+q=0$ and asymptotic results for infinite products.

The paper is organized as follows: in section 2 we introduce a simple example of a compound Poisson process, whose jump measure is $\nu(x) = e^{\alpha x}\text{sech(x)}$. This process actually does not belong to the β-family, but we have decided to include it since in this case we are able to obtain closed form expression for the Wiener-Hopf factors and density of $M_{\tau(q)}$. Also in this simple case we introduce many ideas which will be used in other sections. In section 3 we introduce a very special member of the β-family: a Lévy process X_t (whose jump part is qualitatively similar to the normal inverse Gaussian process) with jumps of infinite variation and jump measure $\nu(x) = e^{\alpha x}\text{cosech(x/2)}^2$. This process is quite unique because its characteristic exponent $\Psi(z)$ is expressed in terms of hyperbolic functions, thus all the formulas are simpler and stronger results can be proved. We derive the localization results and asymptotic expansion for the solutions of $\Psi(z)+q=0$, prove that all of them are real and simple, obtain explicit formulas for sums of inverse powers of these solutions and finally obtain semi-explicit formulas for Wiener-Hopf factors and distribution of supremum $M_{\tau(q)}$. In section 4 we define the ten-parameter β-family of Lévy processes and derive formulas for characteristic exponent and prove results similar to the ones in section 3. Section 5 deals with numerical issues: we discuss acceleration of convergence of infinite products and introduce an efficient method to compute roots of $\Psi(z)+q$ for q complex. As an example we compute the distribution of supremum process M_t.

2. A compound Poisson process. In this section we study a compound Poisson process X_t, defined by a Lévy measure

$$\nu(x) = \frac{e^{\alpha x}}{\cosh(x)}$$

The characteristic exponent of X_t is given by

$$\Psi(z) = -\int\limits_{\mathbb{R}} \left(e^{ixz}-1\right)\nu(x)dx = \frac{\pi}{\cos\left(\frac{\pi}{2}\alpha\right)} - \frac{\pi}{\cosh\left(\frac{\pi}{2}(z-i\alpha)\right)}, \tag{3}$$

and the above integral can be computed with the help of [12]. Our main result in this section is the following theorem, which provides closed-form expressions for the Wiener-Hopf factors and the density of $M_{\tau(q)}$.

THEOREM 1. *Assume that $q>0$. Define*

$$\begin{aligned} \eta &= \frac{2}{\pi}\arccos\left(\frac{\pi}{q+\pi\sec\left(\frac{\pi}{2}\alpha\right)}\right) \\ p_0 &= \frac{\Gamma\left(\frac14(1-\alpha)\right)\Gamma\left(\frac14(3-\alpha)\right)}{\Gamma\left(\frac14(\eta-\alpha)\right)\Gamma\left(\frac14(4-\eta-\alpha)\right)}. \end{aligned} \tag{4}$$

Then for $\mathrm{Im}(z) > i(\alpha-\eta)$ *we have*

$$\phi_q^+(z) = p_0\frac{\Gamma\left(\frac14(\eta-\alpha-iz)\right)\Gamma\left(\frac14(4-\eta-\alpha-iz)\right)}{\Gamma\left(\frac14(1-\alpha-iz)\right)\Gamma\left(\frac14(3-\alpha-iz)\right)} \tag{5}$$

We have $\mathbb{P}(M_{\tau(q)}=0)=p_0$ *and the density* $p^M(q,x)=\frac{d}{dx}\mathbb{P}(M_{\tau(q)}<x)$ *is given by*

$$\begin{aligned} p^M(q,x) &= \frac{2p_0}{\pi}\cot\left(\frac{\pi\eta}{2}\right) \\ &\times \left[\frac{\Gamma\left(\frac{1+\eta}{4}\right)\Gamma\left(\frac{3+\eta}{4}\right)}{\Gamma\left(\frac{\eta}{2}\right)} e^{(\alpha-\eta)x}{}_2F_1\left(\frac{1+\eta}{4},\frac{3+\eta}{4};\frac{\eta}{2};e^{-4x}\right)\right. \\ &- \left.\frac{\Gamma\left(\frac{5-\eta}{4}\right)\Gamma\left(\frac{7-\eta}{4}\right)}{\Gamma\left(\frac{4-\eta}{2}\right)} e^{(\alpha-4+\eta)x}{}_2F_1\left(\frac{5-\eta}{4},\frac{7-\eta}{4};\frac{4-\eta}{2};e^{-4x}\right)\right] \end{aligned} \tag{6}$$

where ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function. If $q=0$ and $\alpha<0$ equation (4) implies $\eta=\mid\alpha\mid$, and formulas (5) and (6) are still valid. In this case the random variable $M_{\tau(0)}$ should be interpreted as $M_\infty=$ $\sup\{X_s : s\geq 0\}$.

First we will state and prove the following lemma, which will be used repeatedly in this paper. It is a variant of the Wiener-Hopf argument, which we have borrowed from the proof of lemma 45.6 in [17] .

LEMMA 2. *Assume we have two functions* $f^+(z)$ *and* $f^-(z)$*, such that* $f^{\pm}(0) = 1$*,* $f^{\pm}(z)$ *is analytic in* $\mathbb{C}^{\pm}$*, continuous and has no roots in* $\bar{\mathbb{C}}^{\pm}$ *and* $z^{-1}\ln(f^{\pm}(z)) \to 0$ *as* $z \to \infty$*,* $z \in \bar{\mathbb{C}}^{\pm}$*. If*

$$\frac{q}{\Psi(z)+q} = f^+(z)f^-(z), \qquad z \in \mathbb{R} \tag{7}$$

then $f^{\pm}(z) = \phi_q^{\pm}(z)$*.*

PROOF. We define function $F(z)$ as

$$F(z) = \begin{cases} \frac{\phi_q^-(z)}{f^-(z)}, & \text{if} \quad z \in \bar{\mathbb{C}}^- \\ \frac{f^+(z)}{\phi_q^+(z)}, & \text{if} \quad z \in \bar{\mathbb{C}}^+ \end{cases}$$

Function $F(z)$ is well defined for z real due to (7) and (2). Using properties of $\phi_q^{\pm}$ and $f^{\pm}$ we conclude that $F(z)$ is analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$ and continuous on $\mathbb{C}$, therefore by Morera's theorem (see [9]) it must be analytic in the entire complex plane. Moreover, by construction function $F(z)$ has no zeros in $\mathbb{C}$, thus its logarithm is also an entire function. All that is left to do is to prove that $\ln(F(z))$ is constant.

Using integration by parts and formula (1) one could prove the following result: if ξ is an infinitely divisible positive random variable with no drift and $\Psi_{\xi}(z)$ is its characteristic exponent, then $z^{-1}\Psi_{\xi}(z) \to 0$ as $z \to \infty$, $z \in \bar{\mathbb{C}}^+$ (this statement is similar to proposition 2 in [5]). Thus

$$z^{-1}\ln(\phi_q^{\pm}(z)) \to 0, \quad z \to \infty, \quad z \in \bar{\mathbb{C}}^{\pm}.$$

Since functions $f^{\pm}$ also satisfy the above conditions, we find that $z^{-1}\ln(F(z)) \to 0$ as $\mid z \mid \to \infty$ in the entire complex plane, and by using Cauchy's estimates (see [9]) we obtain that $\ln(F(z))$ is constant. The value of this constant is easily seen to be zero, since $f^{\pm}(0) = \phi_q^{\pm}(0) = 1$. □

Proof of theorem 1: Using expression (3) for $\Psi(z)$ we find that function $q(\Psi(z)+q)^{-1}$ has simple zeros at $\{i(1+\alpha+4n)i, i(3+\alpha+4n)\}$ and simple poles at $\{i(\alpha+\eta+4n), i(\alpha-\eta+4n)\}$, where $n \in \mathbb{Z}$ and η is defined by formula (4). Next we check that $\mid \alpha \mid < \eta < 1$ and define function $f^+(z)$ as product over all poles/zeros lying in $\mathbb{C}^-$

$$f^+(z) = \prod_{n=0} \frac{\left(1 - \frac{iz}{4n+1-\alpha}\right)\left(1 - \frac{iz}{4n+3-\alpha}\right)}{\left(1 - \frac{iz}{4n+\eta-\alpha}\right)\left(1 - \frac{iz}{4n+4-\eta-\alpha}\right)} \tag{8}$$

and similarly $f^-(z)$ as product over zeros/poles in $\mathbb{C}^+$. It is easy to see that the product converges uniformly on compact subsets of $\mathbb{C} \setminus i\mathbb{R}$ since each term is $1 + O(n^{-2})$. The fact that $f^+(z)$ is equal to the right side of formula (5) can be seen by applying the following result from [11]

$$\prod_{n=0}^{\infty} \frac{1 + \frac{x}{n+a}}{1 + \frac{x}{n+b}} = \frac{\Gamma(a)\Gamma(b+x)}{\Gamma(b)\Gamma(a+x)}. \tag{9}$$

The formula for $f^-(z)$ is identical to (5) with (z, α) replaced by $(-z, -\alpha)$.

Now we will prove that $f^{\pm} = \phi_q^{\pm}$. First, using reflection formula for gamma function (see [12]), one can check that for $z \in \mathbb{R}$ function $f^{\pm}$ satisfy factorization identity (7). Next, using the following asymptotic formula from [11]

$$\frac{\Gamma(a+x)}{\Gamma(b+x)} = x^{a-b} + O(x^{a-b-1}) \tag{10}$$

we conclude that $z^{-1}\ln(f^{\pm}(z)) \to 0, \quad z\to\infty, \quad z\in\bar{\mathbb{C}}^{\pm}$, thus all the conditions of lemma 2 are satisfied and we conclude that $f^{\pm}(z)=\phi_q^{\pm}(z)$.

To derive formula (6) for the density of $M_{\tau(q)}$ we use equations (5) and (10) to find that $\mathbb{E}\left[e^{-\zeta M_{\tau(q)}}\right] = \phi_q^+(i\zeta)\to p_0$ as $\zeta\to\infty$, where p_0 is given by (4). This implies that distribution of $M_{\tau(q)}$ has an atom at $x=0$ (which should not be surprising since X_t is a compound Poisson process), and $\mathbb{P}(M_{\tau(q)}=0)=p_0$. The density of $M_{\tau(q)}$ can be computed by the inverse Fourier transform

$$\begin{aligned} p^M(q,x) &= \frac{1}{2\pi}\int\limits_{\mathbb{R}}\left[\phi_q^+(z)-p_0\right]e^{-ixz}dz \\ &= \frac{p_0}{2\pi}\int\limits_{\mathbb{R}}\left[\frac{\Gamma\left(\frac14(\eta-\alpha-iz)\right)\Gamma\left(\frac14(4-\eta-\alpha-iz)\right)}{\Gamma\left(\frac14(1-\alpha-iz)\right)\Gamma\left(\frac14(3-\alpha-iz)\right)}-1\right]e^{-ixz}dz. \end{aligned}$$

Formula (6) is obtained from the above expression by replacing the contour of integration by $ic+\mathbb{R}$, letting $c\to-\infty$ and evaluating the residues at $z\in\{-i(4n+\eta-\alpha), -i(4n+4-\eta-\alpha)\}$ for $n\ge 0$. Evaluating the residues can be made easier by using reflection formula for gamma function, see [12]. □

Remark 1. There are other examples of measures $\nu(x)$, which have finite total mass (and thus define a process with a finite intensity of jumps), and for which the characteristic exponent is a simple meromorphic function. These are two examples based on theta functions (see [12]):

$$\begin{aligned} \nu_1(x) &= 2\sum_{n\ge0}e^{-((n+\frac12)^2+\alpha)x} = e^{-\alpha x}\theta_2\left(0\mid\frac{ix}{\pi}\right), \\ \nu_2(x) &= 2\sum_{n\ge0}e^{-(n^2+\alpha)x} = e^{-\alpha x}\left(1+\theta_3\left(0\mid\frac{ix}{\pi}\right)\right). \end{aligned}$$

These two jump densities are defined on $x>0$, they decay exponentially as $x\to+\infty$ and behave as $x^{-\frac12}$ as $x\to0^+$, thus the total mass is finite. The Fourier transform of these functions can be computed explicitly as

$$\begin{aligned} \int\limits_0^{\infty}e^{ixz}\nu_1(x)dx &= \frac{\pi}{\sqrt{\alpha-iz}}\tanh\left(\pi\sqrt{\alpha-iz}\right), \\ \int\limits_0^{\infty}e^{ixz}\nu_2(x)dx &= \frac{\pi}{\sqrt{\alpha-iz}}\coth\left(\pi\sqrt{\alpha-iz}\right)+\frac{1}{\alpha-iz}. \end{aligned}$$

Unfortunately equation $\Psi(z)+q=0$ can not be solved explicitly, however these processes could be treated using methods presented in the next sections.

3. A process with jumps of infinite variation.

In this section we study a Lévy process X_t, defined by a triple (μ,σ,ν), where the jump measure $\nu(x)$ is given by

$$\nu(x)=\frac{e^{\alpha x}}{\left[\sinh(\frac{x}{2})\right]^2}$$

with $\mid\alpha\mid<1$. The jump part of X_t is similar to the normal inverse Gaussian process (see [3], [8]), as it is also a process of infinite variation, the jump measure decays exponentially as $|x|\to\infty$ and has a $O(x^{-2})$ singularity at $x=0$.

By definition process X_t has three parameters. However if we want to achieve greater generality for modeling purposes, we could introduce two additional scaling parameters a and $b>0$ and define a process $Y_t=aX_{bt}$, thus obtaining a five parameter family of Lévy processes.

PROPOSITION 3. *The characteristic exponent of X_t is given by*

$$\Psi(z) = \frac{\sigma^2 z^2}{2} + i\rho z + 4\pi(z - i\alpha)\coth\left(\pi(z - i\alpha)\right) - 4\gamma, \tag{11}$$

where

$$\gamma = \pi\alpha\cot\left(\pi\alpha\right), \quad \rho = 4\pi^2\alpha + \frac{4\gamma(\gamma-1)}{\alpha} - \mu.$$

PROOF. The series $\left[\sinh\left(\frac{x}{2}\right)\right]^{-2} = 4\sum\limits_{n\geq1} ne^{-nx}$ converges uniformly on (ϵ,∞) for every $\epsilon > 0$, thus

$$\begin{aligned}\int\limits_0^{\infty}\left(e^{izx} - 1 - izx\right)\frac{e^{\alpha x}}{\sinh(\frac{x}{2})^2}dx &= 4\sum_{n\geq1}\frac{n}{n-\alpha-iz} - \frac{n}{n-\alpha} - \frac{inz}{(n-\alpha)^2} \\ &= 4\sum_{n\geq1}\frac{\alpha+iz}{n-\alpha-iz} - \frac{\alpha+iz}{n-\alpha} - \frac{i\alpha z}{(n-\alpha)^2}\end{aligned}$$

The integral in the Lévy-Khintchine representation (1) for $\Psi(z)$ can now be computed as

$$\begin{aligned}&\int\limits_0^{\infty}\left(e^{izx} - 1 - izx\right)\frac{e^{\alpha x}}{\sinh(x)^2}dx + \int\limits_0^{\infty}\left(e^{-izx} - 1 + izx\right)\frac{e^{-\alpha x}}{\sinh(x)^2}dx \\ = \; & 8(\alpha+iz)^2\sum_{n\geq1}\frac{1}{n^2-(\alpha+iz)^2} - 8(\alpha+iz)\alpha\sum_{n\geq1}\frac{1}{n^2-\alpha^2} \\ & -4i\alpha z\sum_{n\in\mathbb{Z}}\frac{1}{(n-\alpha)^2} + \frac{4iz}{\alpha}\end{aligned}$$

To complete the proof we need to use the following well known series expansions (see [12])

$$\begin{aligned}\coth(\pi x) &= \frac{1}{\pi x} + \frac{2x}{\pi}\sum_{n\geq1}\frac{1}{x^2+n^2}, \\ 1 + \cot(\pi x)^2 &= \operatorname{cosec}(\pi x)^2 = \frac{1}{\pi^2}\sum_{n\in\mathbb{Z}}\frac{1}{(n-x)^2}.\end{aligned}$$

□

Note that it is impossible to find solutions to $\Psi(z) + q = 0$ explicitly in the general case, even though the characteristic exponent $\Psi(z)$ is quite simple. It is remarkable that in some special cases, when $\sigma = 0$ and parameters μ, α and q satisfy certain conditions, we can still obtain closed form results. Below we present such an example:

PROPOSITION 4. *Assume that $\sigma = \alpha = 0$. Define*

$$\eta = \frac{1}{\pi}\text{arcctg}\left(\frac{\mu}{4\pi}\right). \tag{12}$$

Then Wiener-Hopf factor $\phi_q^+(z)$ can be computed in closed form when $q = 4$:

$$\phi_4^+(z) = \frac{\Gamma(\eta - iz)}{\Gamma(\eta)\Gamma(1-iz)}. \tag{13}$$

The density of $M_{\tau(4)}$ is given by

$$p^M(4,x) = \frac{d}{dx}\,\mathbb{P}(M_{\tau(4)} < x) = \frac{\sin(\pi\eta)}{\pi}\left(e^x - 1\right)^{-\eta}. \tag{14}$$

The proof of proposition 4 is identical to the proof of theorem 1. It would be very interesting to find a probabilistic explanation of the fact that we can compute distribution of $M_{\tau(q)}$ in closed form only for a single value of parameter q.

The following theorem is one of the main results in this section. It describes various properties of solutions to equation $\Psi(z)+q=0$, which will be used later to compute Wiener-Hopf factors and the distribution of the supremum process.

THEOREM 5. *Assume that $q>0$.*

(i) Equation $\Psi(i\zeta)+q=0$ has infinitely many solutions, all of which are real and simple. They are located as follows:

$$\begin{aligned} &\zeta_0^- \in (\alpha-1,0) \\ &\zeta_0^+ \in (0,\alpha+1) \\ &\zeta_n \in (n+\alpha, n+\alpha+1), \quad n\geq 1 \\ &\zeta_n \in (n+\alpha-1, n+\alpha), \quad n \leq -1 \end{aligned} \tag{15}$$

(ii) If $\sigma\neq 0$ we have as $n\to\pm\infty$

$$\begin{aligned} \zeta_n &= (n+\alpha)+\frac{8}{\sigma^2}(n+\alpha)^{-1} \\ &- \frac{8}{\sigma^2}\left(\frac{2\rho}{\sigma^2}+\alpha\right)(n+\alpha)^{-2}+O(n^{-3}). \end{aligned} \tag{16}$$

(iii) If $\sigma=0$ we have as $n\to\pm\infty$

$$\begin{aligned} \zeta_n &= (n+\alpha+\omega_0)+c_0(n+\alpha+\omega_0)^{-1} \\ &- \frac{c_0}{\rho}\left(4\gamma-q-4\pi^2c_0\right)(n+\alpha+\omega_0)^{-2}+O(n^{-3}), \end{aligned} \tag{17}$$

where

$$c_0=-\frac{4(4\gamma-q+\alpha\rho)}{16\pi^2+\rho^2}, \quad \omega_0=\frac{1}{\pi}\text{arcctg}\left(\frac{\rho}{4\pi}\right)-1.$$

(iv) Function $q(\Psi(z)+q)^{-1}$ can be factorized as follows

$$\frac{q}{\Psi(z)+q}=\frac{1}{\left(1+\frac{iz}{\zeta_0^+}\right)\left(1+\frac{iz}{\zeta_0^-}\right)}\prod_{|n|\geq 1}\frac{1+\frac{iz}{n+\alpha}}{1+\frac{iz}{\zeta_n}} \tag{18}$$

where the infinite product converges uniformly on the compact subsets of the complex plane excluding zeros/poles of $\Psi(z)+q$.

First we need to prove the following technical result.

LEMMA 6. *Assume that α and β are not equal to a negative integer, and $b_n=O(n^{-\epsilon_1})$ for some $\epsilon_1>0$ as $n\to\infty$. Then*

$$\prod_{n\geq 0}\frac{1+\frac{z}{n+\alpha}}{1+\frac{z}{n+\beta+b_n}}\approx Cz^{\beta-\alpha},$$

as $z\to\infty$, $|\arg(z)|<\pi-\epsilon_2<\pi$, where $C=\frac{\Gamma(\alpha)}{\Gamma(\beta)}\prod_{n\geq 0}\left(1+\frac{b_n}{n+\beta}\right)$.

Proof. First we rewrite the infinite product as

$$\prod_{n\geq 0}\frac{1+\frac{z}{n+\alpha}}{1+\frac{z}{n+\beta+b_n}} = \prod_{n\geq 0}\frac{1+\frac{z}{n+\alpha}}{1+\frac{z}{n+\beta}}\prod_{n\geq 0}\frac{1+\frac{z}{n+\beta}}{1+\frac{z}{n+\beta+b_n}} \tag{19}$$
$$= C\frac{\Gamma(\beta+z)}{\Gamma(\alpha+z)}\prod_{n\geq 0}\frac{z+n+\beta}{z+n+\beta+b_n}$$

The product of gamma functions gives us the leading asymptotic term $z^{\beta-\alpha}$ due to (10). Now we need to prove that the last infinite product in (19) converges to one as $z\to\infty$ such that $|\arg(z)|<\pi-\epsilon_2<\pi$. We take the logarithm of this product and estimate it as

$$\left|\sum_{n\geq 1}\ln\left(\frac{z+n+\beta}{z+n+\beta+b_n}\right)\right| = \left|\sum_{n\geq 1}\ln\left(1+\frac{\beta_n}{z+n+\beta}\right)\right|$$
$$\leq \sum_{n\geq 1}\ln\left(1+\frac{|\beta_n|}{|z+n+\beta|}\right)$$
$$\leq \sum_{n\geq 1}\frac{|\beta_n|}{|z+n+\beta|}\leq A\sum_{n\geq 1}\frac{1}{n^{\epsilon_1}|z+n+\beta|},$$

where we have used the fact that $\ln(1+x)<x$ and $|b_n|<An^{-\epsilon_1}$ for some $A>0$. Since $|\arg(z)|<\pi-\epsilon_2<\pi$ we have for z sufficiently large $|z+n+\beta|>\max\{1,|n-|z+\beta||\}$. Let $m=[|z+\beta|]$, where $[x]$ denotes the integer part of x. Then

$$\sum_{n\geq 1}\frac{1}{n^{\epsilon_1}|z+n+\beta|}<\sum_{n=1}^{m}\frac{1}{n^{\epsilon_1}(m+1-n)}+\sum_{n=m+1}^{\infty}\frac{1}{n^{\epsilon_1}(n-m)}$$

The first series in the right side of above inequality converges to zero as $m\to\infty$, since

$$\sum_{n=1}^{m}\frac{1}{n^{\epsilon_1}(m+1-n)} = \sum_{n=1}^{[\sqrt{m}]}\frac{1}{n^{\epsilon_1}(m+1-n)}+\sum_{n=[\sqrt{m}]+1}^{m}\frac{1}{n^{\epsilon_1}(m+1-n)}$$
$$< \frac{[\sqrt{m}]}{m+1-[\sqrt{m}]}+m^{-\epsilon_1/2}\sum_{n=[\sqrt{m}]+1}^{m}\frac{1}{(m+1-n)}$$
$$< \frac{[\sqrt{m}]}{m+1-[\sqrt{m}]}+m^{-\epsilon_1/2}\ln(m).$$

The second series can be rewritten as $\sum_{n=1}^{\infty}(n+m)^{-\epsilon_1}n^{-1}$ and we see that it is a convergent series of positive terms, where each term converges to zero as $m\to\infty$. By considering its partial sums it is easy to prove that the series itself must converge to zero as $m\to\infty$. □

Proof of theorem 5. First we will prove localization result (15). We rewrite equation $\Psi(i\zeta)+q=0$ as

$$4\pi(\zeta-\alpha)\cot(\pi(\zeta-\alpha))-(\rho+\mu)\zeta-4\gamma=\frac{\sigma^2\zeta^2}{2}-\mu\zeta-q. \tag{20}$$

Note that we have separated the jump part of $\Psi(z)$ on the left side and the diffusion part on the right side of equation (20). See Figure 1, where the jump part is represented by black line and diffusion part by grey line.

The left side of equation (20) is zero at $x=0$ and goes to $-\infty$ as $x\nearrow\alpha+1$ or $x\searrow\alpha-1$ (see Figure 1). The right side is negative at $x=0$ and continuous, thus we have at least one solution $\zeta_0^+\in(0,\alpha+1)$ and at

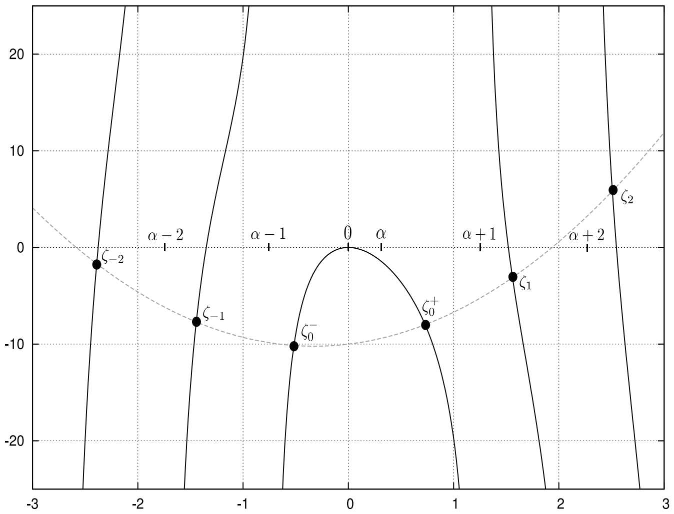

FIG 1. *Illustration to the proof of theorem 5.*

least one solution $\zeta_0^- \in (\alpha-1,0)$. In fact it is easy to prove that we have *exactly* one solution on each of these intervals, since $4\pi(\zeta-\alpha)\cot(\pi(\zeta-\alpha))$ is a concave function on $(\alpha-1,\alpha+1)$, while $\frac{\sigma^2}{2}\zeta^2-\mu\zeta-q$ is convex.

Next, for $n\neq 0$ we have

$$\begin{aligned}4\pi(\zeta-\alpha)\cot(\pi(\zeta-\alpha)) &\nearrow +\infty \text{ as } \zeta\nearrow\alpha-n, \quad \zeta\searrow\alpha+n,\\ 4\pi(\zeta-\alpha)\cot(\pi(\zeta-\alpha)) &\searrow -\infty \text{ as } \zeta\searrow\alpha-n, \quad \zeta\nearrow\alpha+n,\end{aligned}$$

thus there must exist at least one zero ζ_n on each interval $(n+\alpha,n+\alpha+1)$, $(n+\alpha-1,n+\alpha)$.

Next we will prove the asymptotic expansion (16). Since we have assumed that $\sigma\neq 0$ we can rewrite equation (20) as

$$\begin{aligned}\frac{1}{\pi}\tan(\pi(\zeta-\alpha)) &= \frac{4(\zeta-\alpha)}{\frac{\sigma^2}{2}\zeta^2+\rho\zeta+4\gamma-q}\\ &= \frac{8}{\sigma^2}\zeta^{-1}-\frac{8}{\sigma^2}\left(\frac{2}{\rho}+\alpha\right)\zeta^{-2}+O(\zeta^{-3})\end{aligned} \tag{21}$$

The right side of (21) is small when ζ is large, thus the solution to (20) should be close to the solution of $\tan(\pi(\zeta-\alpha))=0$, which implies

$$\zeta = n+\alpha+\omega, \tag{22}$$

and $\omega\to 0$ as $n\to\infty$. We expand

$$\frac{1}{\pi}\tan(\pi(\zeta-\alpha)) = \frac{1}{\pi}\tan(\pi\omega) = \omega+O(\omega^3),$$

use the fact that $\zeta^{-1}=(n+\alpha)^{-1}-\omega(n+\alpha)^{-2}+O(n^{-3})$ which follows from (22), and rewrite (21) as

$$\omega+O(\omega^3) = \frac{8}{\sigma^2}\left((n+\alpha)^{-1}+\omega(n+\alpha)^{-2}\right)-\frac{8}{\sigma^2}\left(\frac{2}{\rho}+\alpha\right)(n+\alpha)^{-2}+O(n^{-3}).$$

The asymptotic expansion (16) follows easily from the above formula.

If $\sigma = 0$, equation (21) has to be modified as follows:

$$\begin{aligned}\frac{1}{\pi}\tan(\pi(\zeta-\alpha)) &= \frac{4(\zeta-\alpha)}{\rho\zeta+4\gamma-q} = \frac{4}{\rho} - \frac{4}{\rho^2}(4\gamma-q+\alpha\rho)\zeta^{-1} \\ &+ \frac{4(4\gamma-q)}{\rho^3}(4\gamma-q+\alpha\rho)\zeta^{-2} + O(\zeta^{-3})\end{aligned} \tag{23}$$

As before, for ζ large the solution of (20) should be close to the solution of $\frac{1}{\pi}\tan(\pi(\zeta-\alpha)) = \frac{4}{\rho}$, thus we define ω as

$$\zeta = n + \alpha + \frac{1}{\pi}\arctan\left(\frac{4\pi}{\rho}\right) - 1 + \omega = n + \alpha + \omega_0 + w, \tag{24}$$

and again $\omega \to 0$ as $n \to \infty$. Next we use addition formula for $\tan(\cdot)$ and find that

$$\begin{aligned}\frac{1}{\pi}\tan(\pi(\zeta-\alpha)) &= \frac{1}{\pi}\tan\left(\arctan\left(\frac{4\pi}{\rho}\right) + \pi\omega\right) \\ &= \frac{1}{\pi}\frac{\frac{4\pi}{\rho} + \tan(\pi\omega)}{1 - \frac{4\pi}{\rho}\tan(\pi\omega)} = \frac{4}{\rho}\frac{1+\frac{\rho}{4}\omega}{1-\frac{4\pi^2}{\rho}\omega} + O(\omega^3) \\ &= \frac{4}{\rho} + \frac{1}{\rho^2}\left(16\pi^2+\rho^2\right)\omega + \frac{4\pi^2}{\rho^3}\left(16\pi^2+\rho^2\right)\omega^2 + O(\omega^3).\end{aligned}$$

From (24) we obtain $\zeta^{-1} = (n+\alpha+\omega_0)^{-1} - \omega(n+\alpha+\omega_0)^{-2} + O(n^{-3})$, thus (23) can be rewritten as

$$\begin{aligned}&\left(16\pi^2+\rho^2\right)\left[\omega + \frac{4\pi^2}{\rho}\omega^2\right] = -4(4\gamma-q+\alpha\rho) \\ &\times\left((n+\alpha+\omega_0)^{-1} - \omega(n+\alpha+\omega_0)^{-2}\right) \\ &+\frac{4(4\gamma-q)}{\rho}(4\gamma-q+\alpha\rho)(n+\alpha+\omega_0)^{-2} + O(n^{-3}) + O(\omega^3)\end{aligned}$$

and from this equation we obtain the second asymptotic expansion (17).

Now we are ready to prove the factorization identity (18) and the fact that all the zeros are simple and that there are no other zeros except for the ones described in (15). Let us introduce the following entire functions

$$P(z) = \frac{\alpha\sinh(\pi(z-i\alpha))}{\sin(\pi\alpha)(z-i\alpha)}, \qquad Q(z) = q^{-1}(\Psi(z)+q)P(z). \tag{25}$$

By definition $q(\Psi(z)+q)^{-1} = P(z)/Q(z)$. Function $P(z)$ is entire and can be represented as the following infinite product (see [12]):

$$P(z) = \prod_{n\geq 1}\left(1+\frac{iz}{n+\alpha}\right)\left(1+\frac{iz}{-n+\alpha}\right). \tag{26}$$

Next, using (11) we check that $Q(z) = 0$ if and only if $\Psi(z)+q = 0$. We have proved already that the zeros of $\Psi(i\zeta)+q$ include $\zeta_n, \zeta_0^{\pm}$, however some of them might have order greater than one, also there might exist other roots (real and/or complex). Let us denote the set of these unaccounted roots (counting with multiplicity) as $\mathfrak{Z}$. Using asymptotic expansions given by equations (21) and (23) one can easily prove that $\mathfrak{Z}$ is a finite set (possibly empty). Since function $Q(z)$ is an entire function of exponential type, there exists a constant $c_1 \in \mathbb{C}$ such that $Q(z)$ can be represented as the following infinite product (see [14]):

$$\begin{aligned}Q(z) &= \exp(c_1 z)\left(1+\frac{iz}{\zeta_0^+}\right)\left(1+\frac{iz}{\zeta_0^-}\right) \\ &\times \prod_{z_k\in\mathfrak{Z}}\left(1+\frac{iz}{z_k}\right)\prod_{|n|\geq 1}\left(1+\frac{iz}{\zeta_n}\right)\exp\left(-\frac{iz}{\zeta_n}\right).\end{aligned}$$

As the next step we rearrange the infinite product in the above formula and obtain

$$\begin{aligned} Q(z) &= \exp(c_2 z)\left(1+\frac{iz}{\zeta_0^+}\right)\left(1+\frac{iz}{\zeta_0^-}\right) \\ &\quad \times \prod_{z_k\in\mathfrak{Z}}\left(1+\frac{iz}{z_k}\right)\prod_{n\geq 1}\left(1+\frac{iz}{\zeta_n}\right)\left(1+\frac{iz}{\zeta_{-n}}\right), \end{aligned} \tag{27}$$

for some other constant $c_2\in\mathbb{C}$, where the infinite product converges absolutely since each term is $1+O(n^{-2})$ as $n\to\infty$. Combining equations (25), (26) and (27) we find that for all $z\in\mathbb{C}$

$$\frac{q}{\Psi(z)+q}=\frac{\exp(c_2 z)}{\left(1+\frac{iz}{\zeta_0^+}\right)\left(1+\frac{iz}{\zeta_0^-}\right)}\prod_{z_k\in\mathfrak{Z}}\frac{1}{1+\frac{iz}{z_k}}\prod_{|n|\geq 1}\frac{1+\frac{iz}{n+\alpha}}{1+\frac{iz}{\zeta_n}}. \tag{28}$$

First let us prove that $c_2=0$. Denote the left side of equation (28) as $F_1(z)$ and right side as $F_2(z)$. Since $\Psi(z)$ is a characteristic exponent, it must be $O(z^2)$ as $z\to\infty$, $z\in\mathbb{R}$, thus $z^{-1}\ln(F_1(z))\to 0$ as $z\to\infty$, $z\in\mathbb{R}$. Using lemma 6 we find that the $z^{-1}\ln(F_2(z))\to c_2$ as $z\to\infty$ $z\in\mathbb{R}$, thus $c_2=0$.

All that is left to do it to prove that $\mathfrak{Z}=\emptyset$. The main tool is again lemma 6. Assuming that $\sigma\neq 0$ and using asymptotic expansion (16) and lemma 6 we find that the infinite product in equation (28) converges to a constant as $z\to\infty$, $z\in\mathbb{R}$. Thus function $F_2(z)\approx A_2 z^{-2-M}$ where M is equal to the number of elements in the set $\mathfrak{Z}$. However function $F_1(z)\approx A_1 z^{-2}$ as $z\to\infty$, $z\in\mathbb{R}$, thus $M=0$ and the set $\mathfrak{Z}$ must be empty. In the case $\sigma\neq 0$ the proof is identical, except that both $F_1(z)$ and $F_2(z)$ behave like Az^{-1}, which can be established by asymptotic expression for ζ_n given in (17) and lemma 6. □

Theorem 5 provides us with all the information about the zeros of $\Psi(z)+q$ that we will need later to prove results about Wiener-Hopf factors and perform numerical computations. However we can also compute explicitly the sums of inverse powers of zeros. These results can be useful for checking the accuracy, but more importantly, for approximating the smallest solutions $\zeta_0^{\pm}$. We assume that $\alpha\neq 0$ and define for $m\geq 0$

$$S_m=\alpha^{-m-1}+(\zeta_0^-)^{-m-1}+(\zeta_0^+)^{-m-1}+\sum_{n\geq 1}\left[\zeta_n^{-m-1}+\zeta_{-n}^{-m-1}\right]. \tag{29}$$

Asymptotic expansions (16) and (17) guarantee that the series converges absolutely for $m\geq 0$, thus the sequence $\{S_m\}_{m\geq 0}$ is correctly defined.

LEMMA 7. *The sequence $\{S_m\}_{m\geq 0}$ can be computed using the following recurrence relation:*

$$S_k=-\frac{1}{b_0}\left[(k+1)b_{k+1}+\sum_{m=0}^{k-1}S_m b_{k-m}\right],\quad k\geq 0$$

where coefficients $\{b_n\}_{n\geq 0}$ are defined as

$$\begin{aligned} b_{2n} &= \frac{(-1)^{n-1}\pi^{2n-1}}{(2n)!}\left[n(2n-1)\alpha\sigma^2+\pi^2\alpha(q+8n)-2n\gamma\rho\right] \\ b_{2n+1} &= \frac{(-1)^n\pi^{2n}}{(2n+1)!}\left[n(2n+1)\frac{\gamma\sigma^2}{\pi}\right. \\ &\qquad \left.-\pi(4\pi^2\alpha^2+4\gamma^2-\gamma q)+\pi(2n+1)(4\gamma+\alpha\rho)\right] \end{aligned} \tag{30}$$

Proof. This statement is just an application of the following general result. Assume that we have an entire function $h(z)$ which can be expressed as an infinite product over the set of its zeros $\mathfrak{Z}$:

$$h(z)=\prod_{z_k\in\mathfrak{Z}}\left(1-\frac{z}{z_k}\right)$$

Taking derivative of $\ln(h(z))$ we find

$$h'(z)=-h(z)\sum_{z_k\in\mathfrak{Z}}(z_k-z)^{-1}=-h(z)\sum_{m\ge0}\left[\sum_{z_k\in\mathfrak{Z}}z_k^{-m-1}\right]z^m,$$

and recurrence relation for $\sum_{z_k\in\mathfrak{Z}}z_k^{-m-1}$ is obtained by expanding $h(z)$ and $h'(z)$ in Maclaurin series, multiplying two series in the right side and comparing the coefficients in front of z^m. The statement of lemma 7 follows by considering an entire function

$$h(z)=q\pi(z-\alpha)Q(iz),$$

where $Q(z)$ is defined by (25). Function $h(z)$ has zeros at $\{\alpha,\zeta_0^{\pm},\zeta_n\}$ and one can check that the Maclaurin expansion is given by $h(z)=\sum_{n\ge0}b_nz^n$ where coefficients b_n are defined in (30). □

Finally we can state and prove our main results: expressions for Wiener-Hopf factors and density of $M_{\tau(q)}$.

THEOREM 8. *For $q>0$*

$$\phi_q^-(z)=\frac{1}{1+\frac{iz}{\zeta_0^+}}\prod_{n\ge1}\frac{1+\frac{iz}{n+\alpha}}{1+\frac{iz}{\zeta_n}},\quad \phi_q^+(z)=\frac{1}{1+\frac{iz}{\zeta_0^-}}\prod_{n\le-1}\frac{1+\frac{iz}{n+\alpha}}{1+\frac{iz}{\zeta_n}} \tag{31}$$

Infinite products converge uniformly on compact subsets of $\mathbb{C}\setminus i\mathbb{R}$. The density of $M_{\tau(q)}$ is given by

$$p^M(q,x)=\frac{d}{dx}\,\mathbb{P}(M_{\tau(q)})<x)=-c_0^-\zeta_0^-e^{\zeta_0^-x}-\sum_{k\ge1}c_k^-\zeta_{-k}e^{\zeta_{-k}x} \tag{32}$$

where

$$c_0^-=\prod_{n\le-1}\frac{1-\frac{\zeta_{-k}}{n+\alpha}}{1-\frac{\zeta_{-k}}{\zeta_n}},\qquad c_k^-=\frac{1-\frac{\zeta_{-k}}{-k+\alpha}}{1-\frac{\zeta_{-k}}{\zeta_0^-}}\prod_{n\le-1,\,n\ne-k}\frac{1-\frac{\zeta_{-k}}{n+\alpha}}{1-\frac{\zeta_{-k}}{\zeta_n}}. \tag{33}$$

Proof. Expressions (31) for Wiener-Hopf factors are obtained using factorization identity (18) and lemmas 2 and 6. Expression (32) for the density of $M_{\tau(q)}$ is derived by computing the inverse Fourier transform via residues. □

Remark 2. Theorem 8 remains true for $q=0$ if $\mu<0$. In this case $\mathbb{E}X_1<0$ and $M_{\tau(q)}\to M_\infty$ and $N_{\tau(q)}\to-\infty$ as $q\to0^+$. From the analytical point of view we have $\zeta_0^-<0$ and $\zeta_0^+=0$, see Figure 1 and equation (20). If $q=\mu=0$ then $\mathbb{E}X_1=0$ and the process X_t is a martingale, thus $M_\infty=N_\infty=\infty$, which is expressed analytically by the fact that $\zeta_0^+=\zeta_0^-=0$.

4. A family of Lévy processes.

Definition 3. We define a β-family of Lévy processes by the generating triple (μ, σ, ν), where the jump density is defined as

$$\nu(x) = c_1 \frac{e^{-\alpha_1 \beta_1 x}}{(1 - e^{-\beta_1 x})^{\lambda_1}} \mathbf{I}_{\{x>0\}} + c_2 \frac{e^{\alpha_2 \beta_2 x}}{(1 - e^{\beta_2 x})^{\lambda_2}} \mathbf{I}_{\{x<0\}} \tag{34}$$

and parameters satisfy $\alpha_i > 0$, $\beta_i > 0$, $c_i \geq 0$ and $\lambda_i \in (0, 3)$.

The β-family is quite rich: in particular by controlling parameters λ_i we can obtain an arbitrary behavior of small jumps, parameters α_i and β_i are responsible for the tail of the jump density (which is always exponential). Parameters c_i control the total "intensity" of positive/negative jumps. The β-family is similar to the generalized tempered stable family (see [8])

$$\nu(x) = c_+ \frac{e^{-\alpha_+ x}}{x^{\lambda_+}} \mathbf{I}_{\{x>0\}} + c_- \frac{e^{\alpha_- x}}{|x|^{\lambda_-}} \mathbf{I}_{\{x<0\}}.$$

In fact we can obtain the above measure as the limit of measures in β-family. If we set $c_1 = c_+ \beta^{\lambda_+}$, $c_2 = c_- \beta^{\lambda_-}$, $\alpha_1 = \alpha_+ \beta^{-1}$, $\alpha_2 = \alpha_- \beta^{-1}$, $\beta_1 = \beta_2 = \beta$ and let $\beta \to 0^+$ we see that the jump measure defined in (34) will converge to the jump measure of the generalized tempered stable process. Next, when $\lambda_1 = \lambda_2$ the processes in β-family are similar to the tempered stable processes, also called KoBoL processes in [6]. If we restrict the parameters even further $c_1 = c_2$, $\lambda_1 = \lambda_2$ and $\beta_1 = \beta_2$ so that the small positive/negative jumps have the same behavior, while large jumps which are controlled by α_i may be different, we obtain a process very similar to the CGMY family in [7]. Finally, if $c_i = 4$, $\beta_i = 1/2$, $\lambda_i = 2$ and $\alpha_1 = 1 - \alpha$ and $\alpha_2 = 1 + \alpha$ we obtain the process X_t discussed in section 3.

In the following proposition we derive a formula for the characteristic exponent $\Psi(z)$ for processes in the β-family. This function is expressed in terms of beta and digamma functions, which justifies the name of the family.

Proposition 9. *If $\lambda_i \in (0, 3) \setminus \{1, 2\}$ then*

$$\begin{aligned} \Psi(z) &= \frac{\sigma^2 z^2}{2} + i\rho z \\ &- \frac{c_1}{\beta_1} \mathrm{B}\left(\alpha_1 - \frac{iz}{\beta_1}; 1 - \lambda_1\right) - \frac{c_2}{\beta_2} \mathrm{B}\left(\alpha_2 + \frac{iz}{\beta_2}; 1 - \lambda_2\right) + \gamma \end{aligned} \tag{35}$$

where

$$\begin{aligned} \gamma &= \frac{c_1}{\beta_1} \mathrm{B}\left(\alpha_1; 1 - \lambda_1\right) + \frac{c_2}{\beta_2} \mathrm{B}\left(\alpha_2; 1 - \lambda_2\right) \\ \rho &= \frac{c_1}{\beta_1^2} \mathrm{B}\left(\alpha_1; 1 - \lambda_1\right) \left(\psi(1 + \alpha_1 - \lambda_1) - \psi(\alpha_1)\right) \\ &- \frac{c_2}{\beta_2^2} \mathrm{B}\left(\alpha_2; 1 - \lambda_2\right) \left(\psi(1 + \alpha_2 - \lambda_2) - \psi(\alpha_2)\right) - \mu \end{aligned}$$

If λ_1 or $\lambda_2 \in \{1, 2\}$ the characteristic exponent can be computed using the following two integrals:

$$\begin{aligned} &\int_0^\infty (e^{ixy} - 1 - ixy) \frac{e^{-\alpha \beta x}}{1 - e^{-\beta x}} dx = \\ &-\frac{1}{\beta} \left[\psi\left(\alpha - \frac{iy}{\beta}\right) - \psi(\alpha) \right] + -\frac{iy}{\beta^2} \psi'(\alpha) \end{aligned} \tag{36}$$

$$\int_0^\infty (e^{ixy}-1-ixy)\frac{e^{-\alpha\beta x}}{(1-e^{-\beta x})^2}dx = -\frac{1}{\beta}\left(1-\alpha+\frac{iy}{\beta}\right)\left[\psi\left(\alpha-\frac{iy}{\beta}\right)-\psi(\alpha)\right]-\frac{iy(1-\alpha)}{\beta^2}\psi'(\alpha) \tag{37}$$

Proof. First we assume that $\lambda\in(0,1)$. Performing change of variables $u=\exp(-\beta x)$ we find that

$$\begin{aligned}\beta\int_0^\infty \left(e^{ixz}-1-ixz\right)\frac{e^{-\alpha\beta x}}{(1-e^{-\beta x})^\lambda}dx &= \mathrm{B}\left(\alpha-\frac{iz}{\beta};1-\lambda\right)-\mathrm{B}(\alpha;1-\lambda)\\ &- z\left[\frac{d}{dz}\mathrm{B}\left(\alpha-\frac{iz}{\beta};1-\lambda\right)\right]_{z=0}\end{aligned}$$

and we obtain the desired result (35). The left side of the above equation is analytic in λ for $\mathrm{Re}(\lambda)<3$, the right side is analytic and well defined for $\mathrm{Re}(\lambda)<3$, $\lambda\neq\{1,2\}$, thus by analytic continuation they should be equal for $\lambda\in(0,3)\setminus\{1,2\}$.

Assume that $\lambda=2$. Then we expand $(1-\exp(-x))^{-2}=\sum_{n\ge0}(n+1)\exp(-nx)$ which converges uniformly on (ϵ,∞) and obtain

$$\begin{aligned}&\beta\int_0^\infty \left(e^{ixz}-1-ixz\right)\frac{e^{-\alpha\beta x}}{(1-e^{-\beta x})^2}dx\\ &=\sum_{n\ge0}\left[\frac{n+1}{n+\alpha-\frac{iy}{\beta}}-\frac{n+1}{n+\alpha}-\frac{iy}{\beta}\frac{n+1}{(n+\alpha)^2}\right]\\ &=\left(1-\alpha+\frac{iy}{\beta}\right)\sum_{n\ge0}\left[\frac{1}{n+\alpha-\frac{iy}{\beta}}-\frac{1}{n+\alpha}\right]-\frac{iy}{\beta}(1-\alpha)\sum_{n\ge0}\frac{1}{(n+\alpha)^2}\end{aligned}$$

and using the series representation for digamma function we obtain formula (37). Formula (36) corresponding to the case $\lambda=1$ is proved similarly. □

The following theorem is the analogue of theorem 5 and it is the main result in this section.

THEOREM 10. *Assume that $q>0$.*

(i) Equation $\Psi(i\zeta)+q=0$ has infinitely many solutions, all of which are real and simple. They are located as follows:

$$\begin{aligned}&\zeta_0^-\in(-\beta_1\alpha_1,0)\\ &\zeta_0^+\in(0,\beta_2\alpha_2)\\ &\zeta_n\in(\beta_2(\alpha_2+n-1),\beta_2(\alpha_2+n)),\quad n\ge1\\ &\zeta_n\in(\beta_1(-\alpha_1+n),\beta_1(-\alpha_1+n+1)),\quad n\le-1\end{aligned}\tag{38}$$

(ii) If $\sigma\neq0$ we have

$$\begin{aligned}\zeta_n &= \beta_2(n+\alpha_2)\\ &+ \frac{2c_2}{\sigma^2\beta_2^2\Gamma(\lambda_2)}(n+\alpha_2)^{\lambda_2-3}+O(n^{\lambda_2-3-\epsilon}),\quad n\to+\infty\\ \zeta_n &= \beta_1(n-\alpha_1)\\ &- \frac{2c_1}{\sigma^2\beta_1^2\Gamma(\lambda_1)}(-n+\alpha_1)^{\lambda_1-3}+O(n^{\lambda_1-3-\epsilon}),\quad n\to-\infty\end{aligned}\tag{39}$$

(iii) If $\sigma = 0$ we have

$$\begin{aligned} \zeta_n &= \beta_2(n+\alpha_2+\omega_0) \\ &+ A(n+\alpha_2+\omega_0)^{\lambda} + O(n^{\lambda-\epsilon}), \quad n \to +\infty \end{aligned} \tag{40}$$

where coefficients are presented in Table 1 and

$$x_0 = \frac{1}{\pi} \arctan\left(\sin(\pi\lambda_2) \left(\frac{c_1 \beta_2^{\lambda_2} \Gamma(1-\lambda_1)}{c_2 \beta_1^{\lambda_1} \Gamma(1-\lambda_2)} - \cos(\pi\lambda_2) \right)^{-1} \right)$$

The corresponding results for $n \to -\infty$ can be obtained by symmetry considerations.

	ω_0	A	λ
$\lambda_1 < 2,\ \lambda_2 < 2$	0	$\frac{c_2}{\rho\beta_2\Gamma(\lambda_2)}$	$\lambda_2 - 2$
$\lambda_1 < 2,\ \lambda_2 > 2$	$2-\lambda_2$	$-\frac{\sin(\pi\lambda_2)\beta_2^3\rho}{\pi c_2 \Gamma(1-\lambda_2)}$	$2-\lambda_2$
$\lambda_1 > 2,\ \lambda_2 < \lambda_1$	0	$\frac{c_2\beta_1^{\lambda_1}}{c_1\beta_2^{\lambda_1-1}\Gamma(1-\lambda_1)\Gamma(\lambda_2)}$	$\lambda_2-\lambda_1$
$\lambda_1 > 2,\ \lambda_2 > \lambda_1$	$2-\lambda_2$	$-\frac{\sin(\pi\lambda_2)}{\pi}\frac{c_1\beta_2^{\lambda_1+1}\Gamma(1-\lambda_1)}{c_2\beta_1^{\lambda_1}\Gamma(1-\lambda_2)}$	$\lambda_1-\lambda_2$
$\lambda_1 > 2,\ \lambda_2 = \lambda_1$	x_0	$-\rho\frac{\sin(\pi x_0)^2}{\pi^2}\frac{\beta_2^3}{c_2}\Gamma(\lambda_2)$	$2-\lambda_2$

TABLE 1
Coefficients for asymptotic expansion of ζ_n when $\sigma = 0$.

(iv) Function $q(\Psi(z)+q)^{-1}$ can be factorized as follows

$$\begin{aligned} \frac{q}{\Psi(z)+q} &= \frac{1}{\left(1+\frac{iz}{\zeta_0^+}\right)\left(1+\frac{iz}{\zeta_0^-}\right)} \\ &\times \prod_{n\ge1} \frac{1+\frac{iz}{\beta_2(n+\alpha_2)}}{1+\frac{iz}{\zeta_n}} \prod_{n\le-1} \frac{1+\frac{iz}{\beta_1(n-\alpha_1)}}{1+\frac{iz}{\zeta_n}} \end{aligned} \tag{41}$$

where the infinite product converges uniformly on the compact subsets of the complex plane excluding zeros/poles of $\Psi(z)+q$.

REMARK 4. When $\sigma = 0$ the remaining cases $\lambda_1 < 2$, $\lambda_2 = 2$ and $\lambda_1 = 2$, $0 < \lambda_2 < 3$ are not covered by theorem 10. The interested reader can derive these asymptotic expansions by using formulas (36), (37) and the following results for the digamma function (see [11], [12]):

$$\psi(1-z) = \psi(z) + \pi\cot(\pi z), \qquad \psi(z) = \ln(z) - \frac{1}{2z} + O(z^{-2}), \quad z \to \infty$$

Proof of theorem 10: The proof of (i) is very similar to the corresponding part of the proof of theorem 5. We separate equation $\Psi(i\zeta) + q = 0$ into jump part and diffusion part, find points where the jump part goes to infinity, and by analyzing the signs we conclude that on every interval between these points there should exist a solution.

The proof of (ii) and (iii) is based on the following two asymptotic formulas as $\zeta \to +\infty$ (see [11],[12]):

$$\begin{aligned} \mathrm{B}(\alpha+\zeta;\gamma) &= \Gamma(\gamma)\zeta^{-\gamma}\left[1-\frac{\gamma(2\alpha+\gamma-1)}{2\zeta}+O(\zeta^{-2})\right] \\ \mathrm{B}(\alpha-\zeta;\gamma) &= \Gamma(\gamma)\frac{\sin(\pi(\zeta-\alpha-\gamma))}{\sin(\pi(\zeta-\alpha))}\zeta^{-\gamma}\left[1+\frac{\gamma(2\alpha+\gamma-1)}{2\zeta}+O(\zeta^{-2})\right] \end{aligned}$$

If $\sigma \neq 0$ and $\zeta \to +\infty$ we use the above formulas and rewrite equation $\Psi(i\zeta)+q=0$ as

$$\frac{\sin\left(\pi\left(\frac{\zeta}{\beta_2}-\alpha_2+\lambda_2\right)\right)}{\sin\left(\pi\left(\frac{\zeta}{\beta_2}-\alpha_2\right)\right)} = \frac{\sigma^2\beta_2^{\lambda_2}}{2c_2\Gamma(1-\lambda_2)}\zeta^{3-\lambda_2}+O(\zeta^{1-\lambda_2})+O(\zeta^{\lambda_1-\lambda_2})$$

and if $\sigma=0$ and $\zeta \to +\infty$ we have

$$\begin{aligned} \frac{\sin\left(\pi\left(\frac{\zeta}{\beta_2}-\alpha_2+\lambda_2\right)\right)}{\sin\left(\pi\left(\frac{\zeta}{\beta_2}-\alpha_2\right)\right)} &= \frac{\beta_2^{\lambda_2}\rho}{c_2\Gamma(1-\lambda_2)}\zeta^{2-\lambda_2} \\ &+\frac{c_1\beta_2^{\lambda_2}\Gamma(1-\lambda_1)}{c_2\beta_1^{\lambda_1}\Gamma(1-\lambda_2)}\zeta^{\lambda_1-\lambda_2}+O(\zeta^{1-\lambda_2})+O(\zeta^{\lambda_1-\lambda_2-1}) \end{aligned}$$

Asymptotic expansions (39) and (40) can be derived from the above formulas using the same method as in the proof of theorem 5.

In order to prove factorization identity (41) and the fact that there are no other roots we use exactly the same approach as in the proof of theorem 5, except that we use different entire functions $P(z)$ and $Q(z)$:

$$\begin{aligned} P(z) &= \left[\Gamma\left(\alpha_1-\frac{iz}{\beta_1}\right)\Gamma\left(\alpha_2+\frac{iz}{\beta_2}\right)\right]^{-1}, \\ Q(z) &= q^{-1}(\Psi(z)+q)P(z). \end{aligned} \tag{42}$$

The factorization of gamma function in infinite product can be found in [12]. Function $P(z)$ and $Q(z)$ have order of growth equal to one (see [14]) and the asymptotics for infinite products is supplied by lemma 6. □

We can also derive a result similar to lemma 7 using the entire function $Q(z)$ defined by equation (42). While there is no closed form expression for derivatives of gamma function, they can be easily computed numerically. Our final result in this section is the analogue of theorem 8, and the proof is identical.

THEOREM 11. *For $q>0$*

$$\phi_q^-(z)=\frac{1}{1+\frac{iz}{\zeta_0^+}}\prod_{n\geq 1}\frac{1+\frac{iz}{\beta_2(n+\alpha_2)}}{1+\frac{iz}{\zeta_n}}, \quad \phi_q^+(z)=\frac{1}{1+\frac{iz}{\zeta_0^-}}\prod_{n\leq -1}\frac{1+\frac{iz}{\beta_1(n-\alpha_1)}}{1+\frac{iz}{\zeta_n}}$$

Infinite products converge uniformly on compact subsets of $\mathbb{C}\setminus i\mathbb{R}$. The density of $M_{\tau(q)}$ is given by

$$p^M(q,x)=\frac{d}{dx}\,\mathbb{P}(M_{\tau(q)}<x)=-c_0^-\zeta_0^-e^{\zeta_0^-x}-\sum_{k\geq 1}c_k^-\zeta_{-k}e^{\zeta_{-k}x}$$

where

$$c_0^-=\prod_{n\leq -1}\frac{1-\frac{\zeta_{-k}}{\beta_1(n-\alpha_1)}}{1-\frac{\zeta_{-k}}{\zeta_n}}, \quad c_k^-=\frac{1-\frac{\zeta_{-k}}{\beta_1(-k-\alpha_1)}}{1-\frac{\zeta_{-k}}{\zeta_0^-}}\prod_{n\leq -1,\, n\neq -k}\frac{1-\frac{\zeta_{-k}}{\beta_1(n-\alpha_1)}}{1-\frac{\zeta_{-k}}{\zeta_n}}.$$

5. Implementation and numerical results. In this section we discuss implementation details for computing the density of $M_{\tau(q)}$ and M_t. In order to illustrate the main ideas we will use the process X_t defined in section 3, however the implementation for a general X_t from the β-family would be quite similar. Our main tools are theorem 8 and asymptotic expansion for ζ_n given in theorem 5.

First let us discuss the computation of density of $M_{\tau(q)}$. The first step would be to compute solutions to equation $\Psi(i\zeta)+q=0$, and for q real this is a simple task: for n large we use Newton's method which is started from the point given by asymptotic expansion (16) or (17). To compute $\zeta_0^{\pm}$ or ζ_n with n small we use localization result (15) and secant (or bisection) method to get the starting point for Newton's iteration. Overall this part of the algorithm is very computationally efficient and can be made even faster if we compute different ζ_n in parallel.

The second step is to compute coefficient c_k^- which are given by (33). Each term in the infinite product is $1+O(n^{-2})$, however as we show in proposition 12 we can considerably improve convergence by using our knowledge of the asymptotic expansion for ζ_n. The final step is to compute the density of $M_{\tau(q)}$ using formula (32). Note that the series converges exponentially for $x>0$. When x is small the convergence is slow, and the asymptotic behavior as $x\to 0^+$ would depend on the decay rate of coefficient c_k^-, however we were unable to prove any results in this direction.

PROPOSITION 12. *Assume that* $\zeta_n = n+\beta+\frac{A_1}{(n+\beta)}+\frac{A_2}{(n+\beta)^2}+O(n^{-3})$ *as* $n\to+\infty$. *Then as* $N\to+\infty$ *we have*

$$\prod_{n\geq N}\frac{1+\frac{z}{n+\alpha}}{1+\frac{z}{\zeta_n}} = \frac{\Gamma(N+\alpha)\Gamma(N+\beta+z)}{\Gamma(N+\beta)\Gamma(N+\alpha+z)} \times \exp\Big[A_1\left(f_{1,1}(\beta,\beta;N)-f_{1,1}(z+\beta,\beta;N)\right) + A_2\left(f_{1,2}(\beta,\beta;N)-f_{1,2}(z+\beta,\beta;N)\right)+O(N^{-3})\Big] \tag{43}$$

where $f_{\alpha_1,\alpha_2}(z_1,z_2;N)$ *can be computed as follows*

$$\begin{aligned} f_{\alpha_1,\alpha_2}(z_1,z_2;N) &= \sum_{k\geq 0}\frac{\binom{-\alpha_2}{k}(z_2-z_1)^k}{\alpha_1+\alpha_2+k-1}(z_1+N)^{1-\alpha_1-\alpha_2-k} \\ &+ (z_1+N)^{-\alpha_1}(z_2+N)^{-\alpha_2}\left[\frac{1}{2}+\frac{\alpha_1}{12(z_1+N)}+\frac{\alpha_2}{12(z_2+N)}\right] \\ &+ O(N^{-\alpha_1-\alpha_2-3}) \end{aligned} \tag{44}$$

PROOF. First we define for $\alpha_1+\alpha_2>1$

$$f_{\alpha_1,\alpha_2}(z_1,z_2;N)=\sum_{n\geq N}(n+z_1)^{-\alpha_1}(n+z_2)^{-\alpha_2}$$

The proof of the asymptotic expansion (44) is based on the Euler-Maclaurin formula

$$\sum_{n\geq N} f(n) = \int_N^{\infty} f(x)dx + \frac{f(N)}{2} - \frac{f'(N)}{12} + O(f^{(3)}(N))$$

with $f(x)=(x+z_1)^{-\alpha_1}(x+z_2)^{-\alpha_2}$. To obtain (44) we compute the integral by changing variables $y=(x+z_1)^{-1}$, expanding the resulting integrand in Taylor series at $y=0$ and integrating term by term.

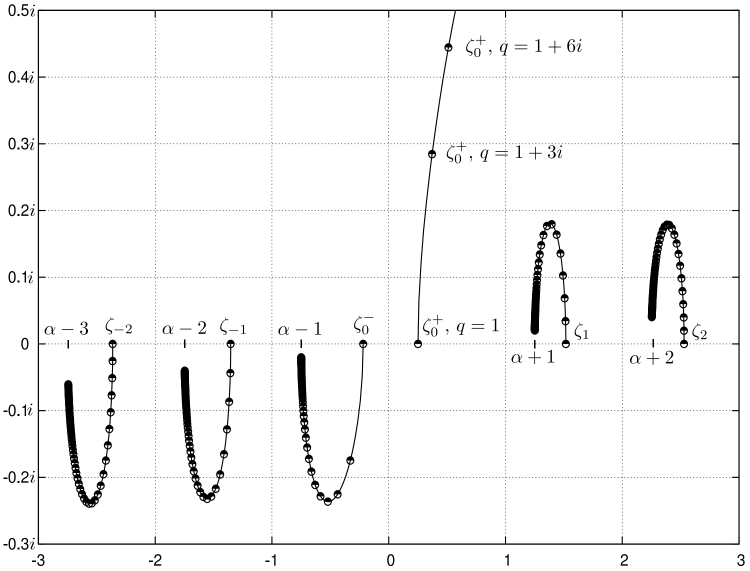

FIG 2. *The values of* $\zeta_0^\pm$ *and* ζ_n *for* $q \in [1, 1+200i]$.

In order to obtain formula (43) we follow the steps of the proof of lemma 6:

$$\begin{aligned}\prod_{n\ge N} \frac{1+\frac{z}{n+\alpha}}{1+\frac{z}{\zeta_n}} &= \prod_{n\ge N} \frac{1+\frac{z}{n+\alpha}}{1+\frac{z}{n+\beta}} \prod_{n\ge N} \frac{1+\frac{z}{n+\beta}}{1+\frac{z}{\zeta_n}} \\ &= \frac{\Gamma(N+\alpha)\Gamma(N+\beta+z)}{\Gamma(N+\beta)\Gamma(N+\alpha+z)} \prod_{n\ge N} \frac{1+\frac{z}{n+\beta}}{1+\frac{z}{\zeta_n}}\end{aligned}$$

Next, we approximate $\ln(1+\omega) = \omega + O(\omega^2)$ as $\omega \to 0$ and obtain

$$\begin{aligned}\sum_{n\ge N} \ln\left(\frac{1+\frac{z}{n+\beta}}{1+\frac{z}{\zeta_n}}\right) &= \sum_{n\ge N} \ln\left(\frac{\zeta_n}{n+\beta}\right) - \sum_{n\ge N} \ln\left(\frac{z+\zeta_n}{z+n+\beta}\right) \\ = \sum_{n\ge N} &\left(\frac{A_1}{(n+\beta)^2} + \frac{A_2}{(n+\beta)^3} + O(n^{-4})\right) \\ - \sum_{n\ge N} &\left(\frac{A_1}{(z+n+\beta)(n+\beta)} + \frac{A_2}{(z+n+\beta)(n+\beta)^2} + O(n^{-4})\right)\end{aligned}$$

which ends the proof. $\square$

Computing the density $p_t(x)$ of M_t at a deterministic time $t>0$ requires more work. Our starting point is the the fact that the density of $M_{\tau(q)}$ (which we denoted by $p^M(q,x)$) is the Laplace transform of $q \times p_t(x)$:

$$p^M(q,x) = \mathbb{E}[p_{\tau(q)}(x)] = q \int_0^\infty e^{-qt} p_t(x) dt.$$

Thus $p_t(x)$ can be recovered as the inverse Laplace transform of $q^{-1} \times p^M(q,x)$, which we rewrite in term of cosine transform:

$$p_t(x) = \frac{1}{2\pi i} \int_{q_0+i\mathbb{R}} e^{qt} p^M(q,x) \frac{dq}{q} = \frac{e^{q_0 t}}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{p^M(q_0+iu,x)}{q_0+iu}\right] \cos(tu) du.$$

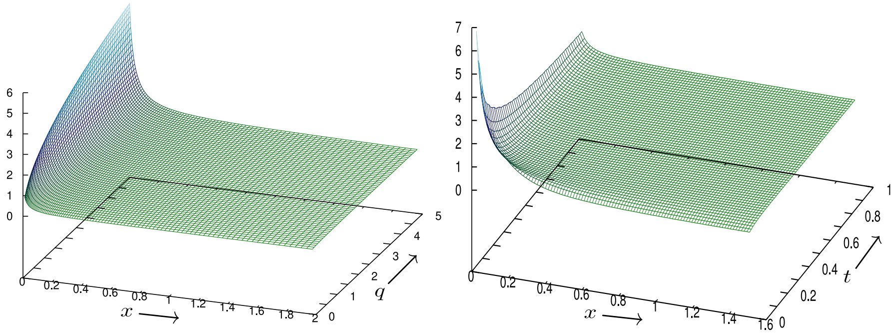

Fig 3. *Surface plot of* $p^M(q,x)$ *(left) and* $p_t(x)$ *(right).*

We see that to compute this Fourier integral numerically we need to be able to compute $p^M(q,x)$ for q lying in some interval $q \in [q_0, q_0 + iu_0]$ in the complex plane. The main problem is that we need to solve many equations (20) with complex q. While the asymptotic expansions for ζ_n presented in theorem 5 are still true, we do not have any localization results in the complex plane. It is certainly possible to compute the roots using argument principle, and originally all the computations were done by the author using this method. However as we will see, there is a much more efficient algorithm.

We need to compute the solutions of equation $\Psi(i\zeta) + q = 0$ for all $q \in [q_0, q_0 + iu_0]$. First we compute the initial values: the roots $\zeta_0^{\pm}$, ζ_n for real value of $q = q_0$ using the method discussed above. Next we consider each root as an implicit function of u: $\zeta_n(u)$ is defined as

$$\Psi(i\zeta_n(u)) + (q_0 + iu) = 0, \qquad \zeta_n(0) = \zeta_n.$$

Using implicit differentiation we obtain a first order differential equation

$$\frac{d\zeta_n(u)}{du} = -\frac{1}{\Psi'(i\zeta_n(u))},$$

with initial condition $\zeta_n(0) = \zeta_n$. We compute the solution to this ODE using a numerical scheme, for example an adaptive Runge-Kutta method, and at each step we correct the solution by applying several iterations of Newton's method. Again, for different n we can compute $\zeta_n(u)$ in parallel.

Figure 2 shows the result of this procedure. We have used the following values of parameters: $\sigma = 1$, $\mu = -0.1$ and $\alpha = 0.25$ and computed zeros $\zeta_0^{\pm}$ and ζ_n for $q \in [1, 1 + 200i]$. The graph shows interesting qualitative behavior: all zeros except ζ_0^+ converge to the closest pole of $\Psi(i\zeta)$ at $\alpha + n$, while ζ_0^+ has no pole nearby (since $\Psi(i\zeta)$ is regular at $\zeta = \alpha$) and it converges to ∞ while always staying in $\mathbb{C}^+$. If $\alpha < 0$ the situation is exactly the same, except that now ζ_0^- escapes to ∞ while always staying in $\mathbb{C}^-$. We have repeated this procedure for many different values of parameters, and from this numerical evidence we can make some observations/conjectures. It appears to be true that the roots never collide - which means that we have no higher order solutions to $\Psi(z) + q = 0$ for all $q \in \mathbb{C}$. It also seems that the roots never cross the real line. All these observations are based on numerical evidence, and we did not pursue this any further to obtain rigorous proofs. However there is one fact that we can prove rigorously: there are not going to appear any new, unaccounted zeros. This could be proved by an argument that we have used in the proof of theorem 5 to show that there are no extra zeros.

The results of our computations are presented in Figure 3. The parameters are $\sigma = 1$, $\mu = -0.1$ and $\alpha = 0.25$, the surface $p^M(q,x)$ is on the left and $p_t(x)$ on the right.

6. Conclusion. In this paper we have introduced a ten-parameter family of Lévy processes, characterized by the fact that the characteristic exponent is a meromorphic function expressed in terms of beta and digamma functions. This family is quite rich, in particular it includes processes with the complete range of behavior of small positive/negative jumps. We have presented results on the Wiener-Hopf factorization for these processes, including semi-explicit formulas for Wiener-Hopf factors and the density of the supremum process $M_{\tau(q)}$.

These Lévy processes might be used for modeling purposes whenever one needs to compute distributions related to such functionals as the first passage time, overshoot, extrema, last time before achieving extrema, etc. Some possible applications in Mathematical Finance include pricing barrier, lookback and perpetual American options, building structural models with jumps in Credit Risk, computing ruin probabilities, etc.

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