VALUATION OF CONTINUOUSLY MONITORED DOUBLE BARRIER OPTIONS AND RELATED SECURITIES

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ABSTRACT. In this article we apply Carr's randomization approximation and the operator form of the Wiener-Hopf method to double barrier options in continuous time. Each step in the resulting backward induction algorithm is solved using a simple iterative procedure that reduces the problem of pricing options with two barriers to pricing a sequence of certain contingent claims with first-touch single barrier features. This procedure admits a clear financial interpretation that can be formulated in the language of embedded options.

Our approach results in a fast and accurate pricing method that can be used in a rather wide class of Lévy-driven models including Variance Gamma processes, Normal Inverse Gaussian processes and KoBoL processes (a.k.a. the CGMY model). At the same time, our work gives new insight into the known explicit formulas obtained by other authors in the setting of the Black-Scholes model. The operator form of the Wiener-Hopf method is generalized for wide classes of processes including the important class of Variance Gamma processes.

Our method can be applied to double barrier options with arbitrary bounded terminal payoff functions, which, in particular, allows us to price knock-out double barrier put/call options as well as double-no-touch options.

Key words and phrases: Option pricing, double barrier options, double-no-touch options, Lévy processes, Variance Gamma processes, KoBoL processes, CGMY model, fast Fourier transform, Carr's randomization, Wiener-Hopf factorization, Laplace transform.

CONTENTS

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Introduction

The problem of finding efficient numerical algorithms for pricing options with single and double barriers has been widely studied in the literature on mathematical finance. In this work we will mainly focus on continuously monitored knock-out double barrier options. Such an option expires worthless if, at any time t prior to (and including) the maturity date $T > 0$, the price, S_t, of the underlying leaves a pre-specified interval (L, U), where $0 < L < U$ are the *lower* and *upper barrier* of the option, respectively. Otherwise, at time T, the option yields payoff equal to $G(S_T)$, where G is the terminal payoff function, which is defined on the interval (L, U).

The method we develop can be applied to an arbitrary (measurable and bounded) terminal payoff function G. However, it may be useful to keep in mind two concrete examples. If G coincides with the terminal payoff function $G(S) = (S - K)_+$ (respectively, $G(S) = (K - S)_+$) of a plain vanilla European call (respectively, put) option with strike price K, we obtain a *knock-out double barrier put* (respectively, *call*) option. If $G(S) = 1$ for all $L < S < U$, we obtain a *double-no-touch* (DNT) option. Options of these types were recently studied by Carr and Crosby [15] in the setting of foreign exchange option markets. In particular, the problem of fast and accurate pricing of DNT options is important due to the fact that they are the most liquid and actively traded exotic options on FX option markets.

In the classical Black-Scholes framework [3], the problem of pricing continuously monitored double barrier options has been studied by many authors. Among the works devoted to this topic, the earliest of which dates back to 1992, let us mention [34, 24, 22, 48, 19, 37, 43, 25, 46, 21, 38, 27] (the list is by no means complete).

For the study of the pricing problem for continuously monitored double barrier options in Kou's model (also known as the double-exponential jump-diffusion model) and the relevant background, we refer the reader to [31, 32, 33, 47, 7]. The same methodology can be extended to hyper-exponential jump-diffusion models (cf. [7, 15]).

From a broad perspective, the works listed above share one general feature. Using certain probabilistic arguments (such as fluctuation theory for Lévy processes), the authors derive a formula for the Laplace transform of the no-arbitrage price of the option with respect to the time to maturity. It is typically expressed as the sum of an infinite series, with the individual terms being given by explicit analytical formulas. In the Black-Scholes setting, it is furthermore possible to perform explicit Laplace inversion for the individual terms of the series. In Kou's model, this is no longer the case, but the series in question converges rapidly, and its terms can be easily computed, so that the calculation of the option value essentially reduces to a single numerical Laplace inversion. This allows one to develop rather fast pricing algorithms for double barrier options in the Black-Scholes model and in Kou's model, a feature which makes these models attractive from the practical viewpoint.

However, empirical evidence shows that other Lévy processes, such as Variance Gamma (VG) processes, Normal Inverse Gaussian (NIG) processes and KoBoL processes (a.k.a. the CGMY model), often provide a much better fit to the observed stock prices. For stocks of this type, the Black-Scholes model and Kou's model may yield inaccurate prices of single and double barrier options when the spot price of the underlying is close to (one of) the barrier(s). The reason is that the value function of the option in the Black-Scholes and Kou's models remains continuously differentiable with respect to the spot price up to and including the barrier(s), while the same property fails for VG, NIG and KoBoL processes (see [28, 6] for more details in the single barrier setting).

The accuracy can be improved, to some extent, by considering a natural generalization of the Kou's model constructed and studied in [35] in the context of American options and applied later under the name hyper-exponential jump-diffusion (HEJD) models in [26] to price barrier options. The computational advantage of the Kou's model and HEJD model stems from the fact that the Wiener-Hopf factorization is almost trivial for these processes (see also [1] for the probabilistic background for similar processes). By increasing the number of different jumps that are allowed in the positive and negative directions, one can approximate the jump densities of VG, NIG and KoBoL processes with the jump densities of hyper-exponential jump-diffusions. The resulting pricing algorithms and numerical examples in the context of options with single barriers can be found in [26]. As we noted in [6], this approach retains the disadvantage that hyper-exponential jump-diffusions have finite jump activity, whereas VG, NIG and KoBoL processes have infinite jump activity. More importantly, we explained in *op. cit.* that as the number of jumps increases, the computational cost of evaluating the Laplace transform of the value function of the option grows very rapidly, which often necessitates a rather undesirable trade-off between the accuracy and the speed of the calculations.

From the computational viewpoint, the present work continues the tradition of a series of articles including [17, 10, 29, 20, 5, 6] that demonstrate the high efficiency of numerical methods based on Fourier transformations with respect to the log-spot price of the underlying asset. We would especially like to single out the important work of Feng and Linetsky [20], which develops a very fast and accurate method of pricing *discretely monitored* single and double barrier options under a wide class of Lévy-driven models (including NIG and KoBoL processes). Our work can be viewed as the counterpart of *op. cit.* in the setting of continuously monitored options.

On the other hand, Laplace transforms are also present in the theoretical portion of the paper, because pricing methods for single and double barrier options that are based on Carr's randomization technique [16, 14] are closely related to the problem of calculating the Laplace transforms of the value functions of these options with respect to the maturity date. As a result, our article gives new insight into the formulas obtained by other authors in the Black-Scholes and Kou's models.

One of the key features of our approach, which contributes to its intrinsic accuracy, is that our method does *not* involve replacing the underlying Lévy process of the model with an approximation thereof. The class of Lévy processes to which our method can be applied (see §1.6 for a list of examples) includes diffusion processes, hyper-exponential jump-diffusions, VG, NIG and KoBoL processes.

At the same time, our algorithm is easy to implement in practice (see §3). The high computational speed is achieved by combining Carr's randomization procedure with the efficient numerical techniques developed in [5, 6].

The rest of this article is organized as follows. Our approach to the valuation of perpetual double barrier options in Lévy-driven models is explained in §1. The key idea is presented in §1.2, where we show how the pricing problem for these options can be reduced to a sequence of pricing problems for perpetual first-touch single barrier contingent claims, by means of constructing certain auxiliary embedded options. As an example, we demonstrate in §1.3 that the latter sequence of problems admits a simple and explicit solution in the framework of the classical Black-Scholes model [3]. The remainder of §1 is devoted to an explanation of how this sequence of problems can be solved in general, as well as to a description of a wide class of Lévy processes to which our method can be applied, along with a list of concrete examples (§1.6).

In §2 we study finite-lived knock-out double barrier options in Lévy-driven models. We use Carr's randomization approximation to compute the no-arbitrage price of such an option, and we reduce each step of the resulting backward induction algorithm to a sequence of calculations involving the expected present value operators of the underlying process, which lends our method to an efficient numerical realization. We also derive a formula for the Laplace transform with respect to the maturity date of the value function of a knock-out double barrier option under an arbitrary regular Lévy process of exponential type, which (to the best of our knowledge) is a new result. The shape of our formula is similar to the shapes of the known formulas in the special case of hyper-exponential jump-diffusion processes.

A detailed algorithm for pricing continuously monitored knock-out double barrier options is presented in §3. The implementation of this algorithm is based on certain computational techniques that were developed in [5, 6], and that are summarized, for convenience, in Appendix C. Some numerical examples, which were obtained using our method, are presented and discussed in §4.

While the material contained in §§1–3 suffices for the practical applications of our method, our exposition would not be complete without a theoretical justification of the validity of the method. To this end, we start "from scratch" in Appendices A and B and present an essentially self-contained exposition of this method, with complete proofs. We tried to make sure (at the expense of a few repetitions) that these two appendices can mostly be read independently from the rest of the article. We believe that they are interesting in their own right, in particular because they include a streamlined exposition of the operator form of the Wiener-Hopf method for pricing *single* barrier options, developed earlier by S.I. Boyarchenko and the second author (see, e.g., [10]), often removing unnecessary assumptions, and simplifying some of the arguments. In particular, the results in [10] are proved for a class of processes, which does not include the important class of VG processes; the present paper fills in this gap. We also hope that the theory is clarified thanks to our consistent use of a single purely probabilistic framework, while avoiding nontrivial analytical tools whenever possible.

1. Perpetual payoffs streams with barriers

The seminal works of Carr and Faguet [16] and Carr [14] introduced a very useful approximation procedure into mathematical finance, which is nowadays known as *Canadization* or *Carr's randomization* (we prefer the latter term). In a number of different situations, it allows one to develop an efficient algorithm for numerically solving a pricing problem for a finite-lived option by replacing it with a sequence of perpetual option pricing problems. In preparation for using this procedure, we will study pricing problems for perpetual options with single and double barrier features.

1.1. **Contingent claims with barriers.** In this section we consider certain types of perpetual contingent claims on a stock whose price process, $\{S_t\}_{t\geq 0}$, has the form $S_t = S_0 e^{X_t}$, where $S_0 > 0$ and $X = \{X_t\}_{t\geq 0}$ is a 1-dimensional Lévy process[1]. (Some examples of Lévy processes used in financial modeling appear in §1.6 below.) We also assume that it is possible to borrow and lend money at a fixed rate of return $q > 0$. Our goal is to derive formulas for the expected present values of the claims we are about to describe, which can be used for practical calculations.

(1) Let $H > 0$ be fixed, let $h = \ln H$, and let $f(x)$ be a bounded measurable[2] function defined on $(-\infty, h)$. These data determine an *up-and-out perpetual stream of payoffs*, whose instantaneous payoff at time $t \geq 0$ (while the stream is active) is equal to $f(\ln S_t)$, and which is abandoned as soon as S_t reaches or exceeds H.

(2) Similarly, if H and h are as above, then a bounded measurable function $f(x)$ defined on $(h, +\infty)$ determines a *down-and-out perpetual stream of payoffs*.

[1]The definition of a Lévy process is recalled in §A.2.

[2]Throughout this article, for a function defined on $\mathbb{R}$, "measurable" will mean "Borel measurable."

(3) Let H and h be as above, and let $G(x)$ be a bounded measurable function defined on $[h, +\infty)$ (respectively, $(-\infty, h]$). Consider a contingent claim that pays its owner $G(\ln S_t)$ at the first moment when S_t reaches or exceeds H (respectively, reaches or falls below H). It is called a *perpetual up-and-in* (respectively, *down-and-in*) *first-touch contingent claim*.

(4) Finally, fix $0 < H_- < H_+$ and put $h_\pm = \ln H_\pm$. Let $g(x)$ be a bounded measurable function defined on (h_-, h_+), let $G_+(x)$ be a bounded measurable function defined on $[h_+, +\infty)$, and let $G_-(x)$ be a bounded measurable function defined on $(-\infty, h_-]$. Then $g(x)$ determines a *perpetual knock-out double barrier stream of payoffs*, whose instantaneous payoff at time $t \geq 0$ (while the stream is active) is equal to $g(\ln S_t)$, and which is abandoned as soon as S_t leaves the open interval (H_-, H_+). The pair of functions $G_\pm$ determines a *perpetual first-touch double barrier contingent claim*, which pays its owner $G_\pm(\ln S_t)$ at the first moment when S_t leaves the open interval (H_-, H_+), where the subscript "+" or "−" is chosen according to whether $S_t \geq H_+$ or $S_t \leq H_-$.

A pricing method for perpetual streams of types (1)–(2) is recalled in §1.4. A pricing method for perpetual contingent claims of type (3) is recalled in §1.7. One of the fundamental contributions of the present work is a pricing method for perpetual streams and contingent claims of type (4), which we now develop. Our approach is similar to, and was motivated by, the approach developed by S.I. Boyarchenko in [7] in the special case of hyper-exponential jump-diffusion models.

1.2. **Iterative procedure for streams with double barriers.** In this subsection we explain how to value perpetual knock-out double barrier streams of payoffs in terms of perpetual first-touch *single barrier* contingent claims. The arguments we give have a clear financial interpretation, and are valid for an arbitrary process that has the *strong Markov property* (see, e.g., [44, p. 247] or [45, p. 278]).

1.2.1. *Definition of the value function.* We remain in the setup of §1.1. In particular, we are given a 1-dimensional Lévy process $X = \{X_t\}_{t\geq 0}$ and a real number $q > 0$, representing the killing rate in our model. We fix $0 < H_- < H_+$, put $h_\pm = \ln H_\pm$, and let $g(x)$ be a bounded measurable function on (h_-, h_+). Let $V_{k.o.}(x; g)$ denote the value function of the associated perpetual knock-out double barrier stream of payoffs. In other words, $V_{k.o.}(x; g)$ is the expected present value of this stream, assuming that the initial spot price of the underlying equals $S_0 = e^x$. By definition,

$$V_{k.o.}(x; g) = \mathbb{E}\left[\int_0^{\tau_{h_- - x, h_+ - x}} e^{-qt} g(x + X_t)\, dt\right], \tag{1.1}$$

where τ_{h_-,h_+} is the first entrance time of the process X into the set $(-\infty, h_-] \cup [h_+, +\infty)$, that is, $\tau_{h_-,h_+}(\omega) = \inf\{t \geq 0 \mid X_t(\omega) \geq h_+ \text{ or } X_t(\omega) \leq h_-\}$.

Remark 1.1. In what follows, various contingent claims that we introduce will be implicitly identified with their value functions. The barriers $H_\pm$ are fixed once and for all, so we suppress them from our notation, for the sake of readability.

1.2.2. *Reduction to a first-touch double barrier claim.* Next, let $G^0(x)$ denote the expected present value of the stream of payoffs $\{g(\ln S_t)\}$ that is never abandoned, let $G^0_+(x)$ denote the restriction of $G^0(x)$ to $[h_+, +\infty)$, and let $G^0_-(x)$ denote the restriction of $G^0(x)$ to $(-\infty, h_-]$. The pair of functions $G^0_\pm(x)$ determines a perpetual first-touch double barrier contingent claim with barriers (H_-, H_+). Let us denote the value function of this claim by $V_{f.t.}(x; G^0_\pm)$; by definition,

$$V_{f.t.}(x; G^0_\pm) = \mathbb{E}\left[e^{-q\tau_{h_- - x, h_+ - x}} G^0(x + X_{\tau_{h_- - x, h_+ - x}})\right]. \tag{1.2}$$

We have $V_{k.o.}(x; g) + V_{f.t.}(x; G^0_\pm) = G^0(x)$ for all $x \in \mathbb{R}$. This follows from Dynkin's formula (cf. §B.2), but is also clear from the financial viewpoint. Namely, due to the strong Markov property of X, we can interpret $V_{f.t.}(x; G^0_\pm)$ as the expected present value of the perpetual stream of payoffs $\{g(\ln S_t)\}$ that becomes *activated* (rather than deactivated) as soon as S_t leaves the open interval (H_-, H_+).

1.2.3. *First approximation to the value function.* Let us try to calculate $V_{f.t.}(x; G^0_\pm)$. Let $G^1_+(x) = V_{d.i.}(x; G^0_-)$ and $G^1_-(x) = V_{u.i.}(x; G^0_+)$ denote[3] the value functions of the perpetual down-and-in and up-and-in first-touch contingent claims, with barriers H_- and H_+, determined by the functions $G^0_-(x)$ and $G^0_+(x)$, respectively. As an initial "approximation" to $V_{f.t.}(x; G^0_\pm)$, we could attempt to use the sum $G^1_+(x) + G^1_-(x)$. However, the portfolio consisting of the contingent claims $G^1_\pm(x)$ is worth more than the contingent claim $V_{f.t.}(x; G^0_\pm)$, because S_t could enter one of the intervals $(0, H_-]$ or $[H_+, +\infty)$, and then enter the other one at a later time.

1.2.4. *Embedded options.* In order to compensate for the amount by which the sum $G^1_+(x) + G^1_-(x)$ exceeds $V_{f.t.}(x; G^0_\pm)$, we introduce embedded options. Let $G^2_+(x)$ denote the value function of the contingent claim that pays its owner one contingent claim $G^1_-(x)$ at the first moment when S_t reaches or falls below H_-. Similarly, let $G^2_-(x)$ denote the value function of the contingent claim that pays its owner one contingent claim $G^1_+(x)$ at the first moment when S_t reaches or exceeds H_+. Thus:

$$G^2_+(x) = V_{d.i.}\big(x; V_{u.i.}(y; G^0_+)\big), \qquad G^2_-(x) = V_{u.i.}\big(x; V_{d.i.}(y; G^0_-)\big).$$

It is clear that the portfolio consisting of the three claims $V_{f.t.}(x; G^0_\pm)$, $G^2_+(x)$ and $G^2_-(x)$ is worth more than the portfolio consisting of the two claims $G^1_\pm(x)$. The discrepancy between the two portfolios can be measured by another first-touch double barrier contingent claim. Namely, consider the perpetual contingent claim that pays its owner one contingent claim $G^2_\pm(x)$ at the first moment when S_t leaves the open interval (H_-, H_+), where the subscript "+" or "−" is chosen depending on whether $S_t \geq H_+$ or $S_t \leq H_-$. Its value function is $V_{f.t.}(x; G^2_\pm)$, and we have

[3] Note that the subscripts "+" and "-" have been interchanged.

Proposition 1.2. $V_{f.t.}(x; G^0_\pm) + G^2_+(x) + G^2_-(x) = G^1_+(x) + G^1_-(x) + V_{f.t.}(x; G^2_\pm)$.

The assertion of the proposition is easily verified via a case-by-case inspection of the following mutually exclusive possibilities for the realization of the uncertainty in the future dynamics of the price process $\{S_t\}$:

(i) S_t never leaves the interval (H_-, H_+);
(ii) S_t reaches $[H_+, +\infty)$, but never reaches $(0, H_-]$;
(iii) S_t reaches $(0, H_-]$, but never reaches $[H_+, +\infty)$;
(iv) S_t reaches $[H_+, +\infty)$ before it reaches $(0, H_-]$, then reaches $(0, H_-]$ at a later time, and never returns to $[H_+, +\infty)$ afterwards;
(v) S_t reaches $(0, H_-]$ before it reaches $[H_+, +\infty)$, then reaches $[H_+, +\infty)$ at a later time, and never returns to $(0, H_-]$ afterwards;
(vi) S_t reaches $[H_+, +\infty)$ before it reaches $(0, H_-]$, then reaches $(0, H_-]$ at a later time, and then reaches $[H_+, +\infty)$ again later;
(vii) S_t reaches $(0, H_-]$ before it reaches $[H_+, +\infty)$, then reaches $[H_+, +\infty)$ at a later time, and then reaches $(0, H_-]$ again later.

The verification is straightforward, albeit somewhat tedious, so we skip the details.

1.2.5. *The valuation formulas.* We remain in the same setup as before. The formula obtained in Proposition 1.2 can be rewritten as follows:

$$V_{f.t.}(x; G^0_\pm) = G^1_+(x) + G^1_-(x) - G^2_+(x) - G^2_-(x) + V_{f.t.}(x; G^2_\pm), \tag{1.3}$$

where $G^j_+(x) = V_{d.i.}(x; G^{j-1}_-)$ and $G^j_-(x) = V_{u.i.}(x; G^{j-1}_+)$ for $j = 1, 2$.

We will see in Theorem A.14(b) that there exists a number $0 < \epsilon < 1$, depending only on the process X, the barriers (H_-, H_+), and the killing rate $q > 0$, that has the following property: $\sup_{x\in\mathbb{R}}|G^n_\pm(x)| \le \epsilon \cdot \sup_{x\in\mathbb{R}}|G^{n-1}_\mp(x)|$ for all $n = 1, 2, \ldots$. This inequality means that the method by which formula (1.3) was obtained can be iterated to yield a *convergent* procedure for calculating the value function $V_{f.t.}(x; G^0_\pm)$. Namely, we replace the functions $G^0_\pm(x)$ with the functions $G^2_\pm(x)$ and apply the procedure described in §§1.2.3–1.2.4. Let us denote the resulting functions by $G^3_\pm(x)$ and $G^4_\pm(x)$. Continuing in the same fashion, we obtain a formula

$$V_{f.t.}(x; G^0_\pm) = G^1_+(x) + G^1_-(x) - G^2_+(x) - G^2_-(x) + G^3_+(x) + G^3_-(x) - \cdots, \tag{1.4}$$

where the series on the right hand side converges absolutely and uniformly on the whole real axis. In §1.3 below, we show that in the Black-Scholes model, once the function $G^0(x)$ is known, the right hand side of (1.4) can be evaluated explicitly. In the general setting, it can be calculated numerically using the methods outlined in the remainder of this section (the computational aspects are discussed in §3).

Finally, formula (1.4) and the discussion in §1.2.2 imply the following identity that expresses the value function $V_{k.o.}(x; g)$ of a perpetual knock-out double barrier stream of payoffs as the sum of an absolutely and uniformly convergent series:

$$
\begin{aligned}
V_{k.o.}(x;g) = G^0(x) - G^1_+(x) - G^1_-(x) + G^2_+(x) + G^2_-(x) \\
- G^3_+(x) - G^3_-(x) + G^4_+(x) + G^4_-(x) - \cdots ,
\end{aligned}
\tag{1.5}
$$

where $G^n_+(x) = V_{d.i.}(x; G^{n-1}_-)$ and $G^n_-(x) = V_{u.i.}(x; G^{n-1}_+)$ for all $n \geq 1$.

1.3. **Example: the Black-Scholes model.** It is well known that the valuation of a first-touch contingent claim with a single barrier in the classical Black-Scholes [3] framework is a simple problem that has an explicit solution. Indeed, let us suppose that $X = \{X_t\}$ is a Brownian motion with volatility $\sigma > 0$ and drift $\mu \in \mathbb{R}$. Let us also fix $h \in \mathbb{R}$, and let τ_h denote the first moment when X reaches the point h:

$$\tau_h(\omega) = \inf\{t \geq 0 \,|\, X_t(\omega) = h\}.$$

The Laplace transform of the distribution of τ_h can be calculated explicitly using, for instance, [44, Eq. (9.1)]:

$$\mathbb{E}\big[e^{-q\tau_h}\big] = \exp\left(\frac{\mu \cdot h - |h| \cdot \sqrt{\mu^2 + 2q\sigma^2}}{\sigma^2}\right) \qquad \forall\, q > 0. \tag{1.6}$$

Next, let $H = e^h$, suppose that $G(x)$ is a bounded measurable function on $[h, +\infty)$, and suppose that the initial spot price of the underlying equals $S_0 = e^x$, where $x < h$. Due to the fact that (almost) all trajectories of $\{S_t\}$ are continuous, S_t can only enter the interval $[H, +\infty)$ at the point H. Hence the perpetual up-and-in first-touch contingent claim defined by the function G (cf. §1.1) is identical to the claim that pays its owner $G(h)$ at the random time τ_{h-x} defined above. In view of (1.6), we obtain the following formula for the value function of this claim:

$$V^{BS}_{up-and-in}(x;G) = \begin{cases} G(x), & x \geq h; \\ G(h) \cdot \exp\Big((h-x) \cdot \big(\mu - \sqrt{\mu^2 + 2q\sigma^2}\big)/\sigma^2\Big), & x < h. \end{cases} \tag{1.7}$$

Similarly, let $G(x)$ be a bounded measurable function on $(-\infty, h]$, and suppose that the initial spot price of the underlying equals $S_0 = e^x$, where $x > h$. Then the perpetual down-and-in first-touch contingent claim defined by the function G is identical to the claim that pays its owner $G(h)$ at the random time τ_{h-x}. In view of (1.6), we obtain the following formula for the value function of this claim:

$$V^{BS}_{down-and-in}(x;G) = \begin{cases} G(x), & x \leq h; \\ G(h) \cdot \exp\Big((h-x) \cdot \big(\mu + \sqrt{\mu^2 + 2q\sigma^2}\big)/\sigma^2\Big), & x > h. \end{cases} \tag{1.8}$$

Using the facts above, one can derive the following identity (see §B.7 for the detailed calculation):

$$
\begin{aligned}
V^{BS}_{f.t.}(x; G^0_\pm) = {} & \frac{G^0_-(h_-) - G^0_+(h_+) \cdot \exp\big(a_-(h_+ - h_-)\big)}{1 - \exp\big((a_- - a_+)(h_+ - h_-)\big)} \cdot \exp\big(-a_+(x - h_-)\big) \\
& + \frac{G^0_+(h_+) - G^0_-(h_-) \cdot \exp\big(-a_+(h_+ - h_-)\big)}{1 - \exp\big((a_- - a_+)(h_+ - h_-)\big)} \cdot \exp\big(a_-(h_+ - x)\big),
\end{aligned}
\tag{1.9}
$$

valid for all $h_- < x < h_+$, where

$$a_\pm = \big(\mu \pm \sqrt{\mu^2 + 2q\sigma^2}\big)/\sigma^2$$

and $V^{BS}_{f.t.}(x; G^0_\pm)$ denotes the value function (1.2), calculated in the Black-Scholes setup.

1.4. **Normalized EPV operators.** We remain in the general framework of §1.1. In particular, $X = \{X_t\}_{t \geq 0}$ is an arbitrary 1-dimensional Lévy process, and $q > 0$ is fixed. The practical implementation of formulas (1.4) and (1.5) is based on the calculation of the action of the normalized expected present value (EPV) operators $\mathcal{E}^\pm_q$, which we now define. The *supremum process* and the *infimum process* of X are the stochastic processes $\overline{X}$ and $\underline{X}$ defined by

$$\overline{X}_t = \sup_{0 \leq s \leq t} X_s, \qquad \underline{X}_t = \inf_{0 \leq s \leq t} X_s.$$

For any bounded measurable function $f(x)$ on the real line, we set

$$(\mathcal{E}_q^+ f)(x) = \mathbb{E}\left[\int_0^\infty qe^{-qt} f(x+\overline{X}_t)\,dt\right]$$

and

$$(\mathcal{E}_q^- f)(x) = \mathbb{E}\left[\int_0^\infty qe^{-qt} f(x+\underline{X}_t)\,dt\right].$$

For more background on the operators $\mathcal{E}_q^\pm$, we refer the reader to Appendix A.

Example 1.3. If $X = \{X_t\}$ is a Brownian motion with volatility σ and drift μ, then $\mathcal{E}_q^+$ (respectively, $\mathcal{E}_q^-$) is a convolution operator with exponentially decaying kernel $\beta_+ e^{-\beta_+ y}\mathbb{1}_{[0,+\infty)}(y)$ (respectively, $-\beta_- e^{-\beta_- y}\mathbb{1}_{(-\infty,0]}(y)$), where $\beta_- < 0 < \beta_+$ are the roots of the quadratic equation $\frac{\sigma^2}{2}\beta^2 + \mu\beta - q = 0$. For Kou's model, or, more generally, for a hyper-exponential jump-diffusion (§1.6(3)), $\mathcal{E}_q^\pm$ are linear combinations of convolution operators of similar form. Thus very accurate and efficient numerical realizations of the operators $\mathcal{E}_q^\pm$ can be designed in this case.

For other examples of Lévy processes listed in §1.6, efficient numerical realizations of the operators $\mathcal{E}_q^\pm$ are developed in [29, 5, 6]; we recall one of them in §3.4 below.

For us, the first application of the operators $\mathcal{E}_q^\pm$ will be to the problem of pricing perpetual knock-out single barrier streams of payoffs. The formulas we give below were obtained earlier by S.I. Boyarchenko and the second author (see, e.g., [10]) under certain additional assumptions, which exclude driftless VG processes. In Appendices A–B, we present a self-contained derivation of these formulas that is valid for arbitrary Lévy processes.

Let $h \in \mathbb{R}$, let $f(x)$ be a bounded measurable function on $(-\infty, h)$, and let us consider a perpetual up-and-out stream of payoffs defined by the function $f(x)$ (cf. §1.1(1)). By Proposition A.6, its value function is given by

$$V_{up-and-out}(x; f) = q^{-1}\cdot\mathcal{E}_q^+\big(\mathbb{1}_{(-\infty,h)}(x)\cdot(\mathcal{E}_q^- f)(x)\big) \tag{1.10}$$

Similarly, a bounded measurable function $f(x)$ on $(h, +\infty)$ determines a perpetual down-and-out stream of payoffs (cf. §1.1(2)). By Proposition A.7, its value function is given by

$$V_{down-and-out}(x; f) = q^{-1}\cdot\mathcal{E}_q^-\big(\mathbb{1}_{(h,+\infty)}(x)\cdot(\mathcal{E}_q^+ f)(x)\big). \tag{1.11}$$

Note that we can either let $h = +\infty$ in (1.10), or let $h = -\infty$ in (1.11), and obtain a formula for the value function of a stream of payoffs that is never abandoned:

$$V_{perpetual}(x; f) = q^{-1}\big(\mathcal{E}_q^+\mathcal{E}_q^- f\big)(x) = q^{-1}\big(\mathcal{E}_q^-\mathcal{E}_q^+ f\big)(x). \tag{1.12}$$

In fact, the compositions $\mathcal{E}_q^+\mathcal{E}_q^-$ and $\mathcal{E}_q^-\mathcal{E}_q^+$ are equal to the normalized expected present value operator $\mathcal{E}_q$, which is defined by

$$(\mathcal{E}_q f)(x) = \mathbb{E}\left[\int_0^\infty qe^{-qt} f(x+X_t)\,dt\right].$$

This result is one of the forms of the celebrated *Wiener-Hopf factorization formula*, which gave rise to the name "Wiener-Hopf method for pricing barrier options." Once again, we refer to Appendix A for all the details and additional information.

1.5. **How to calculate the action of $\mathcal{E}_q^\pm$.** In this subsection we explain how the action of the normalized EPV operators $\mathcal{E}_q^\pm$ can be calculated efficiently in practice for a rather wide class of Lévy processes. We also take this opportunity to discuss the conditions under which the numerical methods developed in the present paper can be applied. Several examples that satisfy these conditions and appear frequently in financial modeling are given in §1.6 below.

Let $T_q \sim \operatorname{Exp} q$ be an exponentially distributed random variable with mean q^{-1}, which is independent of the process X. Since the distribution of T_q is $qe^{-qt}\mathbb{1}_{[0,+\infty)}(t)$, we see that $(\mathcal{E}_q^+ f)(x) = \mathbb{E}\big[f(x+\overline{X}_{T_q}\big]$ and $(\mathcal{E}_q^- f)(x) = \mathbb{E}\big[f(x+\underline{X}_{T_q}\big]$ for every bounded measurable function $f(x)$ on $\mathbb{R}$. To rewrite these formulas in a slightly different way, let $p_q^+(dx)$ (respectively, $p_q^-(dx)$) denote the probability distribution of the random variable $\overline{X}_{T_q}$ (respectively, $\underline{X}_{T_q}$). From the definition of $\overline{X}$ and $\underline{X}$, we see that $p_q^+(dx)$ is supported on $[0,+\infty)$ and $p_q^-(dx)$ is supported on $(-\infty,0]$. We then obtain the following representations of $\mathcal{E}_q^\pm$ as convolution operators:

$$(\mathcal{E}_q^+ f)(x) = \int_0^{+\infty} f(x+y)\,p_q^+(dy), \quad (\mathcal{E}_q^- f)(x) = \int_{-\infty}^0 f(x+y)\,p_q^-(dy). \tag{1.13}$$

In order to calculate the action of $\mathcal{E}_q^\pm$ in practice, it is natural to consider the Fourier transforms of the measures $p_q^\pm(dx)$. Using the normalization of the Fourier transform that is common in probability theory, we define functions $\phi_q^\pm(\xi)$ (the Wiener-Hopf factors) by

$$\phi_q^+(\xi) = \widehat{p_q^+}(\xi) \stackrel{\text{def}}{=} \mathbb{E}\big[e^{i\xi\overline{X}_{T_q}}\big], \qquad \phi_q^-(\xi) = \widehat{p_q^-}(\xi) \stackrel{\text{def}}{=} \mathbb{E}\big[e^{i\xi\underline{X}_{T_q}}\big]. \tag{1.14}$$

If the functions $\phi_q^\pm(\xi)$ are known, we can (approximately) compute the probability measures $p_q^\pm(dx)$ by means of Fourier inversion, which allows us to calculate the action of the operators $\mathcal{E}_q^\pm$ using (1.13). The numerical realization of this approach is explained in §3.4. In certain cases, such as the Black-Scholes model or Kou's model, $\phi_q^\pm(\xi)$ are rational functions that are given by explicit formulas. In general, however, they must be calculated numerically.

To this end, let us recall that every Lévy process $X = \{X_t\}_{t\geq 0}$ has a *characteristic exponent*, which is a continuous function $\psi : \mathbb{R} \longrightarrow \mathbb{C}$ satisfying $\psi(0) = 0$ and

$$\mathbb{E}\big[e^{i\xi X_t}\big] = e^{-t\psi(\xi)} \qquad \forall\, \xi \in \mathbb{R},\ t \geq 0;$$

and, conversely, the law of a Lévy process is uniquely determined by its characteristic exponent [45, Thm. 7.10]. In this paper, we always work with concrete examples of Lévy processes in terms of their characteristic exponents.

The numerical methods developed in the paper are justified under the following conditions on the characteristic exponent $\psi(\xi)$ and the Wiener-Hopf factors $\phi_q^\pm(\xi)$:

(i) There exist $-\infty \leq \lambda_- < 0 < \lambda_+ \leq +\infty$ such that $\psi(\xi)$ admits analytic extension into the open strip $\lambda_- < \operatorname{Im}\xi < \lambda_+$ in the complex plane, and grows at most polynomially within every closed sub-strip $\operatorname{Im}\xi \in [\omega_-, \omega_+] \subset (\lambda_-, \lambda_+)$.

(ii) The following integral formulas for the functions $\phi_q^\pm(\xi)$ are valid[4]:

$$\phi_q^\pm(\xi) = \exp\left[\pm\frac{1}{2\pi i}\int_{\operatorname{Im}\eta=\omega_\mp} \frac{\xi\cdot\ln(1+q^{-1}\psi(\eta))}{\eta(\xi-\eta)}\,d\eta\right], \tag{1.15}$$

where $\lambda_- < \omega_- < 0 < \omega_+ < \lambda_+$ are chosen subject to the condition that $q + \operatorname{Re}\psi(\xi)$ remains bounded away from 0 within the strip $\operatorname{Im}\xi \in [\omega_-, \omega_+]$;

(iii) On each line $\operatorname{Im}\xi = \omega \in (\lambda_-, \lambda_+)$, the derivatives of the characteristic exponent admit estimates

$$|\psi^{(s)}(\xi)| \leq C_s(1+|\xi|)^{\bar{\nu}-s}, \tag{1.16}$$

where $\bar{\nu} \in [0, 2]$ is independent of $s = 1, 2, \ldots$, and the constants C_s are independent of ξ (but may depend on ω).

We call a Lévy process *tame*, if conditions (i)–(iii) are satisfied. A numerical method of calculating $\phi_q^\pm(\xi)$ based on (1.15) is presented in §3.3.

1.6. **Model classes of Lévy processes.** The following classes of Lévy processes are tame

(1) A Brownian motion (used in the classical Black-Scholes model [3]) is a tame Lévy process with $\lambda_\pm = \pm\infty$. Its characteristic exponent is given by $\frac{\sigma^2}{2}\xi^2 - i\mu\xi$, where $\sigma > 0$ is the volatility and $\mu \in \mathbb{R}$ is the drift of the process.

(2) In Merton's model [42], the underlying log-price process is a Lévy process with characteristic exponent $\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \lambda\cdot\big(1 - e^{im\xi - \frac{s^2}{2}\xi^2}\big)$, where $\sigma, s, \lambda > 0$ and $\mu, m \in \mathbb{R}$. A process of this kind is also tame with $\lambda_\pm = \pm\infty$.

[4] Because of the first property, it is clear that the integral on the right hand side of (1.15) converges.

(3) A hyper-exponential jump-diffusion process [1, 35, 26] has characteristic exponent

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \lambda^+ \cdot \sum_{j=1}^{n^+} \frac{ip_j^+\xi}{i\xi - \alpha_j^+} + \lambda^- \cdot \sum_{k=1}^{n^-} \frac{ip_k^-\xi}{i\xi + \alpha_k^-}, \tag{1.17}$$

where $n^\pm$ are positive integers and $\alpha_j^\pm, \lambda^\pm, p_j^\pm > 0$ satisfy $\sum_{j=1}^{n^\pm} p_j^\pm = 1$. Kou's model [31], which was discovered earlier, can be obtained as a special case of hyper-exponential jump-diffusion models by taking $n^+ = n^- = 1$.

(4) Lévy processes of the *extended Koponen family* (generalizing the class of processes introduced by Koponen [30]) were defined by S.I. Boyarchenko and the second author in [8]. Later the same family of Lévy processes was used in [18] under the name "CGMY-model," and in [10] under the name "KoBoL processes." We adopt the latter terminology. The characteristic exponent of a KoBoL process of order $\nu \in (0,2)$, $\nu \neq 1$, has the form[5]

$$\psi(\xi) = -i\mu\xi + c \cdot \Gamma(-\nu) \cdot \left[(-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu + \lambda_+^\nu - (\lambda_+ + i\xi)^\nu\right], \tag{1.18}$$

where $\lambda_- < 0 < \lambda_+$ are called the *steepness parameters* of the process, $c > 0$ is its *intensity*, and $\mu \in \mathbb{R}$.

(5) Variance Gamma (V.G.) processes were first used in empirical studies of financial markets by Madan and collaborators [41, 40, 39]. The characteristic exponent of a V.G. process has the form[6]:

$$\psi(\xi) = -i\mu\xi + c \cdot \left[\ln(-\lambda_- - i\xi) - \ln(-\lambda_-) + \ln(\lambda_+ + i\xi) - \ln(\lambda_+)\right], \tag{1.19}$$

where $\lambda_- < 0 < \lambda_+$, $c > 0$ and $\mu \in \mathbb{R}$.

(6) Normal Inverse Gaussian (NIG) processes were constructed by Barndorff-Nielsen, and were applied to empirical studies of financial markets in [2]. The characteristic exponent of a NIG process has the form

$$\psi(\xi) = -i\mu\xi + \delta \cdot \left[\left(\alpha^2 - (\beta + i\xi)^2\right)^{1/2} - (\alpha^2 - \beta^2)^{1/2}\right], \tag{1.20}$$

where $\alpha > |\beta| > 0$, $\delta > 0$ and $\mu \in \mathbb{R}$.

Remark 1.4. All the Lévy processes listed above are tame. In examples (4) and (5), the conditions (i) and (iii) hold with the same parameters $\lambda_\pm$, so there is no conflict of notations. In example (6), one can take $(\lambda_-, \lambda_+) = (\beta - \alpha, \beta + \alpha)$. In example (3), one can take $(\lambda_-, \lambda_+) = \left(\max\{-\alpha_k^-\}, \min\{\alpha_j^+\}\right)$. Condition (ii) is verified (that is, (1.15)) is proved for wide classes of Lévy processes of exponential type in [10],

[5]In the formulas below, and elsewhere in the text, we use the standard convention that $z^\nu = e^{\nu \cdot \ln z}$ for any $\nu \in \mathbb{C}$ and any $z \in \mathbb{C}$ such that $z \notin (-\infty, 0]$. In turn, $\ln z$ denotes the unique branch of the natural logarithm function defined on the complex plane with the negative real axis $(-\infty, 0]$ removed, determined by the requirement that $\ln(1) = 0$.

[6]What we present is not the most common way of writing the formula. Rather, we chose an expression that is equivalent to the standard one and makes the analogy with (1.18) transparent.

Theorem 3.2. This class contains all the model classes listed above but VG processes. The proof of (1.15) for VG processes will be given in Subsection A.5.

Other examples of tame Lévy processes can be found in [10, Ch. 3].

1.7. **First-touch single barrier claims.** We now return to the question of how to calculate the right hand side of formulas (1.4) and (1.5) in practice. It is clear that the problem of pricing perpetual first-touch contingent claims described in §1.1(3) must be solved as an intermediate step. To this end, let us fix $h \in \mathbb{R}$, and let $G_+(x)$ be a bounded measurable function defined on $[h, +\infty)$. Then $G_+(x)$ determines a perpetual up-and-in first-touch contingent claim on the underlying $\{S_t = S_0 e^{X_t}\}_{t\geq 0}$. As before, its value function is denoted by $V_{u.i}(x; G_+)$.

In order to calculate $V_{u.i.}(x; G_+)$, let us first suppose that $G_+(x)$ can itself be written as the expected present value of a perpetual payoff stream $f(\ln S_t)$ that is never abandoned, where $f(x)$ is a bounded measurable function on the real line. Then, due to the strong Markov property of the process $X = \{X_t\}$, we can interpret $V_{u.i.}(x; G_+)$ as the expected present value of the stream of payoffs $f(\ln S_t)$ that is *activated* as soon as S_t reaches or exceeds $H = e^h$. If $V_{u.o.}(x; f)$ denotes the value function of the corresponding perpetual *up-and-out* stream of payoffs, it follows that the sum $V_{u.i.}(x; G_+) + V_{u.o.}(x; f)$ is equal to the expected present value of the perpetual stream $f(\ln S_t)$ that is never abandoned.

By construction, the value function of the latter stream is $G_+(x)$. Furthermore, the function $V_{u.o.}(x; f)$ can be calculated using formula (1.10). It also follows from §1.4 that $G_+ = q^{-1}\mathcal{E}_q^+\mathcal{E}_q^- f$. Combining these facts, we deduce that

$$V_{u.i.}(x; G_+) = q^{-1} \cdot \mathcal{E}_q^+\big(\mathbb{1}_{[h,+\infty)}(x) \cdot (\mathcal{E}_q^- f)(x)\big).$$

Finally, in order to remove the notational dependence on the function f, we note that, thanks to the identity $G_+ = q^{-1}\mathcal{E}_q^+\mathcal{E}_q^- f$, we may (formally) write $q^{-1}\mathcal{E}_q^- f = (\mathcal{E}_q^+)^{-1}G_+$, which leads to the formula

$$V_{u.i.}(x; G_+) = \mathcal{E}_q^+\Big(\mathbb{1}_{[h,+\infty)}(x) \cdot \big((\mathcal{E}_q^+)^{-1}G_+\big)(x)\Big). \tag{1.21}$$

In §A.6 we explain the interpretation of the right hand side of (1.21) more carefully and discuss the conditions under which the formula is valid; the rigorous proof appears in §B.3. The informal derivation of (1.21) that we gave suffices for practical purposes.

The treatment of perpetual down-and-in first-touch contingent claim is entirely analogous. In the situation above, if $G_-(x)$ is a bounded measurable function on $(-\infty, h]$, then, under suitable assumptions (explained in §A.6), we have the following formula for the value function of the corresponding contingent claim:

$$V_{d.i.}(x; G_-) = \mathcal{E}_q^-\Big(\mathbb{1}_{(-\infty,h]}(x) \cdot \big((\mathcal{E}_q^-)^{-1}G_-\big)(x)\Big). \tag{1.22}$$

1.8. **Perpetual double barrier streams and EPV operators.** Finally, we apply formulas (1.21)–(1.22) to calculate the right hand side of (1.5). As we explain in §A.7, the formulas below are valid for an arbitrary tame Lévy process, and, in particular, for all the examples of Lévy-driven models listed in §1.6.

We fix a tame 1-dimensional Lévy process $X = \{X_t\}_{t\geq 0}$, a real number $q > 0$, barriers $0 < H_- < H_+$, and a bounded measurable function $g(x)$ defined on the interval (h_-, h_+), where $h_\pm = \ln H_\pm$. Let $V_{k.o.}(x; g)$ denote the expected present value of the perpetual stream of payoffs $g(\ln S_t)$ that is abandoned as soon as $S_t = S_0 e^{X_t}$ leaves the interval (H_-, H_+), assuming that the killing rate equals q and that the initial spot price of the underlying equals $S_0 = e^x$, where $h_- < x < h_+$ (see (1.1)).

We proved in §1.2 that the function $V_{k.o.}(x; g)$ can be calculated using formula (1.5), where the series on the right hand side converges absolutely and uniformly for all $h_- < x < h_+$. The individual terms of the series can be calculated inductively using the following prescription. First, extend the function $g(x)$ to a bounded measurable function on the whole real line, and calculate[7]

$$G^0(x) = q^{-1}(\mathcal{E}_q^+ \mathcal{E}_q^- g)(x) = q^{-1}(\mathcal{E}_q^- \mathcal{E}_q^+ g)(x). \tag{1.23}$$

Next, set

$$G^0_+(x) = G^0(x)\big|_{[h_+,+\infty)}, \qquad G^0_-(x) = G^0(x)\big|_{(-\infty,h_-]}, \tag{1.24}$$

and calculate $G^n_\pm$ for $n = 1, 2, 3, \ldots$ using the formulas

$$G^n_+(x) = \mathcal{E}_q^-\Big(\mathbb{1}_{(-\infty,h_-]}(x)\cdot\big((\mathcal{E}_q^-)^{-1}G^{n-1}_-\big)(x)\Big) \qquad \forall\, n \geq 1 \tag{1.25}$$

and

$$G^n_-(x) = \mathcal{E}_q^+\Big(\mathbb{1}_{[h_+,+\infty)}(x)\cdot\big((\mathcal{E}_q^+)^{-1}G^{n-1}_+\big)(x)\Big) \qquad \forall\, n \geq 1. \tag{1.26}$$

Numerical realizations of the formulas above are discussed in §§3.4–3.5.

2. Finite-lived double barrier options

2.1. **Market specifications.** In this section we consider a model frictionless market consisting of a riskless bond and a risky asset (for instance, a stock). We assume that the riskless rate, $r > 0$, is constant, and let S_t denote the spot price of the underlying at time t. We also assume that $S_t = S_0 e^{X_t}$, where $X = \{X_t\}_{t\geq 0}$ is a Lévy process under a chosen equivalent martingale measure (EMM). The *EMM-condition* means that the discounted price process of the stock, $\big\{e^{-rt}S_t\big\}_{t\geq 0}$, is a martingale:

$$\mathbb{E}\big[e^{-rt_2}S_{t_2}\,\big|S_{t_1} = S\big] = e^{-rt_1}S \qquad \forall\, t_2 > t_1 \geq 0.$$

[7]As we note in Remark 3.4(2), it is often computationally more efficient to calculate the action of the composition $\mathcal{E}_q^+\mathcal{E}_q^-$ at this step, rather than the action of $\mathcal{E}_q$.

We remark that in general, an EMM (also called a "risk-neutral measure") is not unique. We assume that an EMM has been fixed once and for all, and all expectation operators appearing in this section will be with respect to this measure.

Our goal is to study pricing problems for finite-lived knock-out double barrier options on the stock $\{S_t\}$. The types of options we consider are described in §2.2, where we also give formulas for their no-arbitrage value functions in terms of certain stochastic expressions. These formulas are unsuitable for practical calculations, and the rest of the section will be devoted to developing more computationally efficient approaches to the valuation of finite-lived double barrier options.

As an intermediate step, we derive in §2.3 a formula for the Laplace transform with respect to the maturity date of a knock-out double barrier option in a tame Lévy-driven model. This formula, while still being computationally inefficient, is then used in §2.4 to design a fast and accurate procedure for the valuation of finite-lived double barrier options based on Carr's randomization [16, 14]. For justification of Carr's randomization procedure for single barrier options, see [4]. The case of double barrier options can be treated similarly. In both cases, the process must satisfy certain conditions, which hold for all model classes of Lévy processes[8]

2.2. **Knock-out double barrier options.** We remain in the setup of §2.1. Let us fix two *barriers*, $0 < H_- < H_+$, and define $h_\pm = \ln H_\pm$. We also let $g(x)$ denote a bounded nonnegative measurable function on the interval (h_-, h_+). The *knock-out double barrier option with maturity date $T > 0$, barriers (H_-, H_+) and terminal payoff $g(x)$* is defined as the contingent claim that expires worthless if the price, S_t, of the underlying leaves the open interval (H_-, H_+) at any time $0 \le t \le T$, and pays its owner $g(\ln S_T)$ at time T otherwise.

Examples 2.1. (1) If $K > 0$ is fixed and $g(x) = (e^x - K)_+$ (or $g(x) = (K - e^x)_+$), we obtain a knock-out double barrier call (put) option with strike price K.
(2) If $g(x) = 1$ for all x, we obtain a double-no-touch (DNT) option (cf. [15]).

In general, the no-arbitrage value of a knock-out double barrier option with maturity date $T > 0$, barriers (H_-, H_+) and terminal payoff $g(x)$ is given by the formula

$$V_{k.o.}(x, T; g) = \mathbb{E}\big[e^{-rT}\mathbb{1}_{\{\tau_{h_- - x, h_+ - x} > T\}} g(x + X_T)\big], \tag{2.1}$$

where $x = \ln S_0$ is the initial log-spot price of the underlying and τ_{h_-,h_+} denotes the first entrance time of the process X into the set $(-\infty, h_-] \cup [h_+, +\infty)$ (cf. §1.2.1).

2.3. **Laplace transform of the value function.** In this subsection we elucidate the relationship between the value function of a perpetual knock-out option on the one hand, and the Laplace transform with respect to the maturity date of a finite-lived option of the same type on the other hand.

[8] In the case of the Kou's models and, more generally, HEJD model, the diffusion component must be non-trivial.

Let us consider the value function defined by (2.1). Since $g(x)$ is bounded by assumption, it is clear that the right hand side of (2.1) is bounded as a function of (x, T) as well, and hence its Laplace transform,

$$(\mathcal{L}V_{k.o.})(x, q; g) = \int_0^{\infty} e^{-qT} V_{k.o.}(x, T; g)\, dT, \tag{2.2}$$

is well defined for $q > 0$. Let us substitute (2.1) into the last expression, replace T with t for notational convenience, and use Fubini's theorem (which is justified because we are working with nonnegative functions). We obtain

$$(\mathcal{L}V_{k.o.})(x, q; g) = \mathbb{E}\left[\int_0^{\tau_{h_- - x, h_+ - x}} e^{-(r+q)t} g(x + X_t)\, dt\right]. \tag{2.3}$$

Comparing this formula with (1.1), we see that: *for every $q > 0$, the value of the Laplace transform $(\mathcal{L}V_{k.o.})(x, q; g)$ equals the expected present value of the perpetual knock-out double barrier stream of payoffs $g(S_t)$, when the latter is computed in the framework of §1.1 under the assumption that the killing rate is equal to $r + q$.*

Let us now assume that the Lévy process X is tame (cf. §1.5). Combining the last observation with the formulas of §1.8, we obtain a formula for the Laplace transform $(\mathcal{L}V_{k.o.})(x, q; g)$ as a convergent infinite series:

$$\begin{aligned}(\mathcal{L}V_{k.o.})(x, q; g) = G^0(x) - \ & G^1_+(x) - G^1_-(x) + G^2_+(x) + G^2_-(x) \\ & - G^3_+(x) - G^3_-(x) + G^4_+(x) + G^4_-(x) - \cdots,\end{aligned} \tag{2.4}$$

where the individual terms can be calculated inductively as follows:

$$G^0(x) = (r+q)^{-1}(\mathcal{E}^+_{r+q}\mathcal{E}^-_{r+q} g)(x) = (r+q)^{-1}(\mathcal{E}^-_{r+q}\mathcal{E}^+_{r+q} g)(x),$$

$$G^0_+(x) = G^0(x)\big|_{[h_+, +\infty)}, \qquad G^0_-(x) = G^0(x)\big|_{(-\infty, h_-]},$$

$$G^n_+(x) = \mathcal{E}^-_{r+q}\Big(\mathbb{1}_{(-\infty, h_-]}(x) \cdot \big((\mathcal{E}^-_{r+q})^{-1} G^{n-1}_-\big)(x)\Big) \qquad \forall\, n \geq 1$$

and

$$G^n_-(x) = \mathcal{E}^+_{r+q}\Big(\mathbb{1}_{[h_+, +\infty)}(x) \cdot \big((\mathcal{E}^+_{r+q})^{-1} G^{n-1}_+\big)(x)\Big) \qquad \forall\, n \geq 1.$$

Remark 2.2. One can specialize this discussion to the case of knock-out single barrier options by taking either $h_- = -\infty$ or $h_+ = +\infty$. For instance, in the first case, we obtain an up-and-out option with barrier H_+, and only the first and the third term on the right hand side of (2.4) survive, which leads to the formula

$$(\mathcal{L}V)(x, q; g) = (r+q)^{-1} \cdot \mathcal{E}^+_{r+q}\big(\mathbb{1}_{(-\infty, h_+)}(x) \cdot (\mathcal{E}^-_{r+q} g)(x)\big).$$

2.4. **Carr's randomization for double barrier options.** In practice, one could attempt to use formula (2.4) to calculate the values of the function $V_{k.o.}(x,T;g)$ by means of a numerical Laplace inversion. However, apart from a handful of cases (such as the Black-Scholes model or Kou's model) where explicit analytic formulas for the Wiener-Hopf factors $\phi_q^{\pm}(\xi)$ (see §1.5) are available, such an approach is not very efficient, because numerical Laplace inversion would require calculating the action of $\mathcal{E}^{\pm}_{q+r}$ for many different values of q, which could be rather time-consuming.

Instead, we choose a different approach, based on Carr's randomization [16, 14]. Let us consider the right hand side of (2.1). Following [14], we first replace the deterministic maturity period $[0,T]$ with a random maturity period $[0,T']$, where T' is an exponentially distributed random variable that has mean T and is independent of the process X. By a slight abuse of notation, let us denote the resulting expression by $V_{k.o.}(x,T';g)$ (even though T' is a random variable, $V_{k.o.}(x,T';g)$ is still a deterministic quantity). Using the fact that the distribution of T' equals $\frac{1}{T}\cdot e^{-t/T}\mathbb{1}_{[0,+\infty)}(t)\,dt$ and that T' is independent of X, we obtain

$$\begin{aligned}
V_{k.o.}(x,T';g) &= \mathbb{E}\big[e^{-rT'}\mathbb{1}_{\{\tau_{h_--x,h_+-x}>T'\}}g(x+X_{T'})\big]\\
&= \frac{1}{T}\cdot\mathbb{E}\left[\int_0^\infty e^{-t/T}e^{-rt}\mathbb{1}_{\{\tau_{h_--x,h_+-x}>t\}}g(x+X_t)\,dt\right]\\
&= (1+rT)^{-1}\cdot(r+T^{-1})\cdot\mathbb{E}\left[\int_0^{\tau_{h_--x,h_+-x}} e^{-(r+T^{-1})t}g(x+X_t)\,dt\right]\\
&= (1+rT)^{-1}\cdot(r+T^{-1})\cdot V_{k.o.}(x;g),
\end{aligned}$$

where $V_{k.o.}(x;g)$ is defined by formula (1.1) with q replaced by $r+T^{-1}$.

In general, of course, one cannot expect $V_{k.o.}(x,T';g)$ to be a good approximation to $V_{k.o.}(x,T;g)$. Following Carr [14], we next divide the maturity period of the option into N subintervals, using points $0=t_0<t_1<\cdots<t_N=T$, and we replace each sub-period $[t_s,t_{s+1}]$ with an exponentially distributed random maturity period with mean $\Delta_s=t_{s+1}-t_s$. Moreover, these N random maturity sub-periods are assumed to be independent of each other and of the process X. (In [14], it is assumed that $\Delta_s=T/N$ for all s, but, in principle, we do not have to impose this requirement.)

As in [14], we can calculate the value function of the claim with this new maturity period using backward induction. Namely, let $V^s(x)$ denote the value function of the option after the first s maturity sub-periods. Then, by definition, $V^N(x)=g(x)$, the terminal payoff function. Moreover, for all $0\le s\le N-1$, the function $V^s(x)$ can be interpreted as the value function of a knock-out double barrier option with barriers (H_-,H_+), terminal payoff function $V^{s+1}(x)$, and exponentially distributed maturity date with mean Δ_s. Therefore $V^s(x)$ can be calculated using the method we just explained. The resulting algorithm that computes $V^N(x)$, $V^{N-1}(x)$, ..., $V^0(x)$ will be referred to as "Carr's randomization for double barrier options."

Let us make this procedure a little more explicit. Further details, as well as the computational aspects of the algorithm sketched below, will be discussed in §3.

Carr's randomization algorithm for calculating $V_{k.o.}(x, T; g)$.

1. Choose points $0 = t_0 < t_1 < \cdots < t_N = T$. For each $0 \le s \le N-1$, set $\Delta_s = t_{s+1} - t_s$ and $q_s = r + \Delta_s^{-1}$.
2. Define $V^N(x) = g(x)$, the terminal payoff function of the option.
3. In a cycle with respect to $s = N-1, N-2, \ldots, 0$, calculate

$$G^{0,s}(x) = (\mathcal{E}^+_{q_s}\mathcal{E}^-_{q_s}V^{s+1})(x), \quad G^{0,s}_+(x) = G^{0,s}(x)\big|_{[h_+,+\infty)}, \quad G^{0,s}_-(x) = G^{0,s}(x)\big|_{(-\infty,h_-]},$$

$$G^{n,s}_+(x) = \mathcal{E}^-_{q_s}\Big(\mathbb{1}_{(-\infty,h_-]}(x) \cdot \big((\mathcal{E}^-_{q_s})^{-1}G^{n-1,s}_-\big)(x)\Big) \qquad \forall\, n \ge 1 \tag{2.5}$$

and

$$G^{n,s}_-(x) = \mathcal{E}^+_{q_s}\Big(\mathbb{1}_{[h_+,+\infty)}(x) \cdot \big((\mathcal{E}^+_{q_s})^{-1}G^{n-1,s}_+\big)(x)\Big) \qquad \forall\, n \ge 1; \tag{2.6}$$

then set

$$\begin{aligned} V^s(x) = (1 + r\Delta_s)^{-1} \quad \cdot \quad \big(\, & G^{0,s}(x) - \ G^{1,s}_+(x) - G^{1,s}_-(x) + G^{2,s}_+(x) + G^{2,s}_-(x) \\ & -G^{3,s}_+(x) - G^{3,s}_-(x) + G^{4,s}_+(x) + G^{4,s}_-(x) - \cdots \big), \end{aligned}$$

The function $V^0(x)$ obtained at the last step of this algorithm is the desired *Carr's randomization approximation* to the value function $V_{k.o.}(x, T; g)$ of the original double barrier option with deterministic maturity date T. As the mesh, $\max_s \Delta_s$, of the partition of the maturity period of the option approaches 0, we expect that $V^0(x)$ converges to $V_{k.o.}(x, T; g)$.

Remark 2.3. In practice, we usually apply Carr's randomization to partitions where $\Delta_s = T/N$ for all $0 \le s \le N-1$, i.e., $t_s = sT/N$ for all s. The reason is that q_s then becomes independent of s, and usually it is computationally much more efficient to calculate the action of the operators $\mathcal{E}^{\pm}_q$ (and their inverses) on many different functions for a fixed value of q, rather than for varying values of q. This feature is responsible for the computational superiority of Carr's randomization over methods based on numerical Laplace inversion.

Remark 2.4. At each step of the backward induction procedure sketched above, the function $V^s(x)$ is represented as a sum of an infinite series whose terms are expressed in terms of the function $V^{s+1}(x)$ obtained at the previous step. In practice, one must truncate this infinite series to arrive at a finite sum (the details are explained in §3.7). Numerical examples show that, typically, one only has to keep very few (5–9) terms of the series to force the error of this truncation to become negligible.

3. Explicit pricing algorithm

3.1. **The option and the underlying process.** This section is devoted to the computational aspects of the pricing method for continuously monitored knock-out double barrier options, described in §2.4. As in §2.1, we consider a frictionless market consisting of a riskless bond $B_t = e^{rt}$ (where $r > 0$ is constant) and a risky asset $S_t = S_0 e^{X_t}$, where $X = \{X_t\}_{t\geq 0}$ is a 1-dimensional Lévy process under a chosen equivalent martingale measure; in particular, the discounted price process $\{e^{-rt}S_t\}_{t\geq 0}$ of the underlying is a martingale. We also assume that the Lévy process X is tame, i.e., satisfies the conditions described in §1.5 (see §1.6 for a list of examples).

Next, let us fix a maturity date $T > 0$, barriers $H_+ > H_- > 0$, put $h_\pm = \ln H_\pm$, and let $g(x)$ denote a bounded nonnegative measurable function on (h_-, h_+). Our goal is to describe a numerical procedure for calculating the value function $V_{k.o.}(x, T; g)$ of a knock-out double barrier option with maturity date T, barriers $H_\pm$ and terminal payoff function $g(x)$ (see (2.1)). As special cases, we will obtain pricing algorithm for knock-out double barrier call/put options with strike price $K > 0$ (by taking $g(x) = (\pm e^x \mp K)_+$) and for double-no-touch options (by taking $g(x) = 1$).

The notation and assumptions introduced above will remain in force throughout the section. The pricing algorithm will be given in §§3.6–3.7 after some preliminaries.

3.2. **General remarks.** The method we will use to implement the pricing algorithm outlined in §2.4 uses the same technical tools as the methods we employed in [5, 6]. In particular, in §3.3 we use the integral formulas (1.15) to calculate (approximately) the values of the Wiener-Hopf factors $\phi_q^\pm(\xi)$ (defined by (1.14)) on a suitable grid of points in $\mathbb{R}$, and in §3.4 we use these values to realize the normalized EPV operators $\mathcal{E}_q^\pm$ as certain (discretized) convolution operators. The inverse operators $(\mathcal{E}_q^\pm)^{-1}$ that appear in the formulas of §2.4 will be realized numerically by explicitly calculating the inverses of the discretized forms of $\mathcal{E}_q^\pm$ in §3.5.

However, there exists a difference between the method of the present article and the methods of [5, 6] that is important from both the conceptual and the technical viewpoints. Namely, when one is dealing with an option that has a single barrier H, it suffices to work with a single uniformly spaced grid of points on the real line (it was referred to in *op. cit.* as "*the* x-grid"). For example, if the option is a down-and-out barrier option, the x-grid has the form $h, h + \Delta, h + 2\Delta, \dots, h + (M - 1)\Delta$, where $h = \ln H$ and $\Delta > 0$. At each step in the backward induction procedure based on Carr's randomization for single barrier options, one only works with the arrays of values of various auxiliary functions at the points of this fixed x-grid.

On the other hand, a glance at the formulas of §2.4 will convince the reader that a single x-grid will not suffice for a numerical implementation of our pricing algorithm for *double barrier* options. Instead, we must work with five x-grids: the "main" one, which begins at $h_- = \ln H_-$ and ends at $h_+ = \ln H_+$ (this is the grid of points at which the values of the function $V_{k.o.}(x,T;g)$ will be calculated), and four longer "auxiliary" ones, which extend to the left and to the right of the points h_+ and h_-.

3.3. **Calculation of the Wiener-Hopf factors.** In order to be able to implement the numerical realization of the operators $\mathcal{E}_q^{\pm}$ described in §3.4 below, we must first know how to calculate the values of the functions $\phi_q^{\pm}(\xi)$ that appear in (1.14). Apart from a few special cases (such as the hyper-exponential jump-diffusions [1, 26, 7, 15]), no explicit formulas for $\phi_q^{\pm}(\xi)$ are known. Instead, one must use the integral formulas recalled in §1.5. We will not repeat these formulas here, but will only give the discretized versions thereof, after introducing some auxiliary notation.

Consider a uniformly spaced grid of points $\vec{\xi}=(\xi_k)_{k=1}^M$ in $\mathbb{R}$, where $\xi_k=\xi_1+(k-1)\zeta$ for all $1\le k\le M$ and $\zeta>0$ is fixed. We let $\psi(\xi)$ denote the characteristic exponent of the underlying Lévy process X (cf. §1.5), and we recall that X is assumed to be tame; in particular, the properties stated at the end of §1.5 hold.

We will obtain approximate formulas for $\phi_q^{\pm}(\xi_k)$ by using a simplified version of the trapezoidal rule to discretize (1.15). Due to the fact that the integrand in (1.15) decays somewhat slower than $|\eta|^{-2}$ as $\mathrm{Re}\,\eta\to\pm\infty$, it is sometimes necessary to use an η-grid that is longer than the ξ-grid for this discretization, in order to guarantee the desired precision of the calculation of $\phi_q^{\pm}(\xi_k)$.

With these remarks in mind, and with the notation above, we present an algorithm for the approximate calculation of the values $(\phi_q^{\pm}(\xi_k))_{k=1}^M$.

- Select a positive integer m that controls the length of the η-grid.
- Choose $\omega_-\in(\lambda_-,0)$ and $\omega_+\in(0,\lambda_+)$ such that there exists $\delta>0$ with $\mathrm{Re}(q+\psi(\eta))\ge\delta$ whenever $\mathrm{Im}\,\eta\in[\omega_-,\omega_+]$. As a rule of thumb, we recommend taking $\omega_\pm=\lambda_\pm/3$ in the algorithms that are based on Carr's randomization method (§2.4), since in these examples, q is rather large[9].
- Define the η-grid as follows:
$$\vec{\eta}^{\pm}=(\eta_\ell^{\pm})_{\ell=1}^{mM},\qquad \eta_\ell^{\pm}=-mM\zeta/2+(\ell-1)\zeta+i\omega_{\mp},$$
where $\vec{\eta}^{+}$ (resp., $\vec{\eta}^{-}$) is used for the calculation of ϕ_q^{+} (resp., ϕ_q^{-}).
- Using the simplified trapezoid rule to discretize (1.15) leads to the following approximation:
$$\phi_q^{\pm}(\xi_k)\approx\exp\left[\pm\frac{\zeta\cdot\xi_k}{2\pi i}\sum_{\ell=1}^{mM}\frac{\ln(1+q^{-1}\psi(\eta_\ell^{\pm}))}{\eta_\ell^{\pm}(\xi_k-\eta_\ell^{\pm})}\right].$$

[9]When $q>0$ is small, one has to use a somewhat different approach to the calculation of the factors $\phi_q^{\pm}$. A discussion of this approach is beyond the scope of the present article.

- The last formula can be rewritten as follows:

$$\phi_q^{\pm}(\xi_k) \approx \exp\left[\pm\frac{\zeta\cdot\xi_k}{2\pi i}\sum_{j=1}^{m} I_j(\xi_k)\right], \tag{3.1}$$

  where

$$I_j(\xi_k) = \sum_{\ell=1+(j-1)M}^{jM} \frac{\ln(1+q^{-1}\psi(\eta_\ell^{\pm}))}{\eta_\ell^{\pm}(\xi_k-\eta_\ell^{\pm})}, \qquad 1\le j\le m.$$

- Noting that $\xi_k - \eta_\ell^{\pm} = \big(\xi_1 + mM\zeta/2 - i\omega_{\mp}\big) - (k-\ell)\zeta$ depends only on the difference $k-\ell$, calculate each of the arrays $(I_j(\xi_k))_{k=1}^{M}$ for $j=1,2,\dots,m$ using the "fast convolution" algorithm of §C.5.
- Using the results of the previous step, calculate the right hand side of (3.1).

Remark 3.1. In practice, it is clearly computationally more efficient to calculate *either* the values $(\phi_q^+(\xi_k))$ *or* the values $(\phi_q^-(\xi_k))$, and then use the analytic form (A.18) of the Wiener-Hopf factorization formula, namely, $\phi_q^+(\xi)\phi_q^-(\xi) = q(q+\psi(\xi))^{-1}$, to calculate the values $(\phi_q^-(\xi_k))$ (respectively, $(\phi_q^+(\xi_k))$).

3.4. **Two numerical realizations of the operators $\mathcal{E}_q^{\pm}$.**

3.4.1. *Preliminary comments.* We begin by recalling the numerical realizations of the operators $\mathcal{E}_q^{\pm}$ that were employed in [5, 6]. They are based on formulas (1.13), which represent $\mathcal{E}_q^{\pm}$ as operators of convolution with certain probability measures $p_q^{\pm}(dx)$ that are supported on the half-axes $[0,+\infty)$ and $(-\infty,0]$, respectively. In order to understand these probability measures in more concrete terms, we use Proposition B.4, which states that away from 0, each of the two measures $p_q^{\pm}(dx)$ is represented by an (infinitely differentiable and exponentially decaying) function $g_q^{\pm}(x)$, which, moreover, can be found via Fourier inversion:

$$g_q^{\pm}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\phi_q^{\pm}(\xi)\,d\xi, \qquad \pm x>0. \tag{3.2}$$

It follows that the action of the operators $\mathcal{E}_q^+$ and $\mathcal{E}_q^-$ on a bounded measurable function $f(x)$ can be written in the following way:

$$(\mathcal{E}_q^+ f)(x) = p_q^+(\{0\})\cdot f(x) + \int_0^{+\infty} f(x+y)g_q^+(y)\,dy \tag{3.3}$$

and

$$(\mathcal{E}_q^- f)(x) = p_q^-(\{0\})\cdot f(x) + \int_{-\infty}^{0} f(x+y)g_q^-(y)\,dy. \tag{3.4}$$

Here, $p_q^{\pm}(\{0\})$ are constants responsible for the *scalar components* of the operators $\mathcal{E}_q^{\pm}$ (the remaining summands in (3.3)–(3.4) are their *integral components*); with the notation of §1.5, we have $p_q^+(\{0\}) = \mathbb{P}[\overline{X}_{T_q}=0]$ and $p_q^-(\{0\}) = \mathbb{P}[\underline{X}_{T_q}=0]$.

Remark 3.2. It can happen in practice that $p_q^+(\{0\})$ or $p_q^-(\{0\})$ is nonzero[10]. For instance, suppose X is a V.G. process, or a KoBoL process of order $\nu < 1$, with nonzero drift, $\mu \neq 0$. If $\mu > 0$, then $p_q^-(\{0\}) \neq 0$, and if $\mu < 0$, then $p_q^+(\{0\}) \neq 0$.

3.4.2. *First numerical realization of* $\mathcal{E}_q^+$. Let us consider a uniformly spaced grid of points $\vec{x} = (x_j)_{j=1}^M$ on the real line, where $x_j = x_1 + (j-1)\Delta$ for all $1 \leq j \leq M$, and $\Delta > 0$ is fixed. Given a function $f(x)$ whose values at the points of $\vec{x}$ are known, we would like to calculate approximately the values of $(\mathcal{E}_q^+ f)(x)$ at the same points.

To this end, we use the *enhanced* realization of convolution operators, following the methods developed in [5, 6]. Specifically, we approximate $f(x)$ with a piecewise linear function on the interval $[x_1, x_M]$ using the approximations

$$f(x) \approx f_j + \Delta^{-1} \cdot (f_{j+1} - f_j) \cdot (x - x_j), \qquad x_j \leq x \leq x_{j+1}, \tag{3.5}$$

and we approximate $f(x)$ by 0 outside of $[x_1, x_M]$. As we saw in [6, §3.6.2], this leads to the following approximation of the values of the function $(\mathcal{E}_q^+ f)(x)$:

$$(\mathcal{E}_q^+ f)(x_k) \approx -d_k^+ \cdot f_M + \sum_{j=k}^{M} c_{k-j}^+ \cdot f_j \qquad (1 \leq k \leq M), \tag{3.6}$$

where $f_j = f(x_j)$ for $1 \leq j \leq M$,

$$d_k^+ = \frac{\Delta}{2\pi} \int_{-\infty}^{\infty} e^{i(k-M)\Delta\xi} \cdot \phi_q^+(\xi) \cdot \frac{e^{-i\xi\Delta} + i\xi\Delta - 1}{(i\xi\Delta)^2} \, d\xi$$

for $1 \leq k \leq M$,

$$c_\ell^+ = \frac{\Delta}{2\pi} \int_{-\infty}^{\infty} e^{i\ell\Delta\xi} \cdot \phi_q^+(\xi) \cdot \frac{e^{i\xi\Delta} + e^{-i\xi\Delta} - 2}{(i\xi\Delta)^2} \, d\xi$$

for $1 - M \leq \ell \leq -1$, and c_0^+ is equal to the constant $p_q^+(\{0\})$ appearing in (3.3). Probabilistic considerations suggest that in order to improve the accuracy of the approximation (3.6), the discretized form of the operator $\mathcal{E}_q^+$ should also act as an "expectation-type" operator. This means that once the coefficients c_ℓ^+ were found for $\ell \neq 0$ using the formulas above, one should set

$$c_0^+ = 1 - \sum_{1-M \leq \ell \leq -1} c_\ell^+. \tag{3.7}$$

This observation, which is related to the possibility that the scalar component of $\mathcal{E}_q^+$ may be nonzero (cf. Remark 3.2), becomes especially relevant when one must apply the operator $\mathcal{E}_q^+$ multiple times in the course of a given calculation (as in §3.7). In such a situation, if (3.7) does not hold, the errors of the approximation (3.6) will necessarily accumulate over the course of the computation, which may lead to significant errors of the final result produced by the algorithm.

[10]Of course, $p_q^\pm$ can still be uniquely reconstructed from $g_q^\pm(x)$, since $p_q^\pm(\{0\}) = 1 - p_q^\pm((0, \pm\infty))$.

3.4.3. *First numerical realization of $\mathcal{E}_q^-$.* The enhanced numerical realization of $\mathcal{E}_q^-$ is obtained similarly [6, §3.6.3]. We let $\vec{x}$ and $f(x)$ be as above, and consider the same piecewise linear approximation to $f(x)$ as in §3.4.2. It leads to the following approximation of the values of the function $(\mathcal{E}_q^- f)(x)$:

$$(\mathcal{E}_q^- f)(x_k) \approx -d_k^- \cdot f_1 + \sum_{j=1}^{k} c_{k-j}^- \cdot f_j \qquad (1 \le k \le M), \tag{3.8}$$

where $f_j = f(x_j)$,

$$d_k^- = \frac{\Delta}{2\pi} \int_{-\infty}^{\infty} e^{i(k-1)\Delta\xi} \cdot \phi_q^-(\xi) \cdot \frac{e^{i\xi\Delta} - i\xi\Delta - 1}{(i\xi\Delta)^2}\, d\xi,$$

for $1 \le k \le M$,

$$c_\ell^- = \frac{\Delta}{2\pi} \int_{-\infty}^{\infty} e^{i\ell\Delta\xi} \cdot \phi_q^-(\xi) \cdot \frac{e^{i\xi\Delta} + e^{-i\xi\Delta} - 2}{(i\xi\Delta)^2}\, d\xi$$

for $1 \le \ell \le M-1$, and c_0^- is the coefficient $p_q^-(\{0\})$ appearing in (3.4). As in §3.4.2, once the coefficients c_ℓ^- for $\ell \neq 0$ are found, we set

$$c_0^- = 1 - \sum_{1 \le \ell \le M-1} c_\ell^-. \tag{3.9}$$

3.4.4. *Second numerical realization of $\mathcal{E}_q^+$.* The numerical realization of the operator $\mathcal{E}_q^+$ described in §3.4.2 alone does not suffice for implementing the backward induction algorithm of §2.4. For example, the auxiliary function

$$f_+^{n-1,s}(x) = \mathbb{1}_{[h_+,+\infty)}(x) \cdot \big((\mathcal{E}_{q_s}^+)^{-1} G_+^{n-1,s}\big)(x) \tag{3.10}$$

is supported on $[h_+, +\infty)$, whereas we would like to calculate the values of the function $G_-^{n,s}(x) = (\mathcal{E}_{q_s}^+ f)(x)$ to the left of the point h_+.

This leads us to consider the following problem: in the situation of §3.4.2, let us suppose that we know the values of $f(x)$ on the grid $\vec{x} = (x_j)_{j=1}^M$, and that $f(x)$ vanishes to the left of x_1, but we would like to calculate the values of $(\mathcal{E}_q^+ f)(x)$ on the new[11] grid $(x_{j+1-M})_{j=1}^M$, where we extend the meaning of the notation x_j by setting $x_j = x_1 + (j-1)\Delta$ for all integers j.

The approximate formulas for the values of $(\mathcal{E}_q^+ f)(x)$ on the new grid can be easily obtained from the formulas of §3.4.2, so we simply present the final result:

$$(\mathcal{E}_q^+ f)(x_{k+1-M}) \approx \sum_{j=1}^{k} c_{k-j+1-M}^+ \cdot f_j \qquad (1 \le k \le M), \tag{3.11}$$

where $f_j = f(x_j)$ and the coefficients c_ℓ^+ are calculated as in §3.4.2.

[11] In other words, the new grid was obtained by shifting the old grid to the left, so that the endpoint of the new grid coincides with the initial point of the old grid.

3.4.5. *Second numerical realization of $\mathcal{E}_q^-$.* Similarly to §3.4.4, we now consider the following problem. In the situation of §3.4.3, let us suppose that we know the values of $f(x)$ on the grid $\vec{x} = (x_j)_{j=1}^M$, and that the function $f(x)$ vanishes to the right of x_M, but we would like to calculate the values of $(\mathcal{E}_q^- f)(x)$ on the new[12] grid $(x_{j+M-1})_{j=1}^M$. This leads to the following approximation:

$$(\mathcal{E}_q^- f)(x_{k+M-1}) \approx \sum_{\ell=k}^{M} c_\ell^- \cdot f_{M+k-\ell} \qquad (1 \le k \le M), \tag{3.12}$$

where $f_j = f(x_j)$ and the coefficients c_ℓ^- are calculated as in §3.4.2.

3.5. **Calculation of the inverses of the discretized versions of $\mathcal{E}_q^\pm$.** As the last ingredient in the algorithm presented below, we consider the problem of calculating numerically the auxiliary functions of the form (3.10), which entails inverting the operators $\mathcal{E}_q^\pm$ in a suitable sense. We discovered that this problem has a solution that is both convenient and computationally efficient. Namely, we invert the *discretized* forms of the operators $\mathcal{E}_q^\pm$ that were described in §§3.4.2–3.4.3 above.

3.5.1. *The inverse of the discretized form of $\mathcal{E}_q^+$.* The right hand side of (3.6) can be viewed as an operator acting on the M-dimensional space of vectors $\vec{f} = (f_j)_{j=1}^M$ via

$$\mathcal{E}_{q,disc}^+ \vec{f} = c_0^+ \cdot \vec{f} + \mathcal{E}_{q,sub}^+ \vec{f}$$

where

$$(\mathcal{E}_{q,sub}^+ \vec{f})_k \stackrel{\text{def}}{=} -d_k^+ \cdot f_M + \sum_{j=k+1}^{M} c_{k-j}^+ \cdot f_j \qquad (1 \le k \le M).$$

Numerical experiments show that in the situations that arise in practice, the scalar component of $\mathcal{E}_{q,disc}^+$, defined by the coefficient c_0^+, dominates the remaining term $\mathcal{E}_{q,sub}^+$, which implies that the inverse of $\mathcal{E}_{q,disc}^+$ can be accurately computed. A direct calculation shows that the inverse is an operator of a similar type:

$$(\mathcal{E}_{q,disc}^+)^{-1}(\vec{f})_k = \sum_{j=k}^{M} a_{k-j}^+ \cdot (f_j + b_j^+ f_M) \qquad (1 \le k \le M), \tag{3.13}$$

where

$$b_j^+ = d_j^+ / (c_0^+ - d_1^+) \qquad (1 \le j \le M) \tag{3.14}$$

and the coefficients $a_{-\ell}^+$, for $0 \le \ell \le M-1$, can be found inductively using

$$a_0^+ = (c_0^+)^{-1}, \qquad a_{-\ell}^+ = -(c_0^+)^{-1} \cdot \sum_{j=1}^{\ell} c_{-j}^+ a_{j-\ell}^+ \quad (1 \le \ell \le M-1). \tag{3.15}$$

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[12] In other words, the new grid was obtained by shifting the old grid to the right, so that the initial point of the new grid coincides with the endpoint of the old grid.

Remark 3.3. We do not know of a way of calculating the coefficients $a^+_{-\ell}$ that is more efficient than doing it one step at a time, using (3.15). However, for the values of M that occur in practice, this calculation only takes a small fraction of a second.

3.5.2. *The inverse of the discretized form of* $\mathcal{E}^-_q$. The right hand side of (3.8) can be viewed as an operator acting on the M-dimensional space of vectors $\vec{f} = (f_j)_{j=1}^M$ via

$$(\mathcal{E}^-_{q,disc}\vec{f})_k = -d^-_k \cdot f_1 + \sum_{j=1}^{k} c^-_{k-j} \cdot f_j \qquad (1 \le k \le M).$$

The obvious analogue of the comment appearing before formula (3.13) applies here as well, so the inverse of $\mathcal{E}^-_{q,disc}$ can be accurately calculated by means of the formula

$$(\mathcal{E}^-_{q,disc})^{-1}(\vec{f})_k = \sum_{j=1}^{k} a^-_{k-j} \cdot (f_j + b^-_j f_1) \qquad (1 \le k \le M), \tag{3.16}$$

where

$$b^-_j = c^-_j / (c^-_0 - d^-_1) \qquad (1 \le j \le M) \tag{3.17}$$

and the coefficients a^-_ℓ, for $0 \le \ell \le M-1$, can be found inductively using

$$a^-_0 = (c^-_0)^{-1}, \qquad a^-_\ell = -(c^-_0)^{-1} \cdot \sum_{j=1}^{\ell} c^-_j a^-_{\ell-j} \quad (1 \le \ell \le M-1). \tag{3.18}$$

3.6. **Algorithm setup.** We now commence a step-by-step description of the pricing algorithm for the knock-out double barrier option described in §3.1.

I. One must describe the market by giving the riskless rate $r > 0$ and a formula for the characteristic exponent, $\psi(\xi)$, of the process X. The EMM condition

$$r + \psi(-i) = 0 \tag{3.19}$$

must be satisfied.

II. One must specify the maturity date, $T > 0$, and the barriers, $H_+ > H_- > 0$, of the option. Furthermore, one must specify the terminal payoff function $g(x)$, either by giving an explicit formula for it, or by providing the array of values of g on the "main" x-grid, introduced at the next step.

III. Since $V_{k.o.}(x, T; g)$, the value function of the option, vanishes for $x \le h_-$ and $x \ge h_+$ (where $h_\pm = \ln H_\pm$), we are only interested in calculating its values for $x \in (h_-, h_+)$. Thus we specify a uniformly spaced grid of points $\vec{x} = (x_j)_{j=1}^n$, where $x_1 = h_-$, $x_n = h_+$, and $x_j = x_1 + (j-1)\Delta$ for all $1 \le j \le n$, which means that $\Delta = (h_+ - h_-)/(n-1)$. We refer to it as the "main" x-grid (four "auxiliary" x-grids will be introduced below). The final output of the algorithm will be the array of (approximate) values $\big(V_{k.o.}(x_j, T; g)\big)_{j=1}^n$.

IV. We will use Carr's randomization in the situation where the maturity period, $[0, T]$, of the option is divided into sub-periods of equal lengths. Thus we only need to specify the number, N, of time steps. Set $\Delta_t = T/N$ and $q = r + \Delta_t^{-1}$. In what follows, Carr's randomization approximation to the value function of the option after the first s sub-periods have elapsed will be denoted by $V^s(x)$.

V. Finally, one must choose the error tolerance, $\epsilon > 0$, for the iterative procedure that will be used at each step of the backward induction algorithm. The iterative procedure constructs a sequence of approximations to the function $V^s(x)$ using the method of §1.2. The procedure is terminated when the maximum absolute difference between two successive approximations drops below ϵ/N.

These steps conclude the input of the initial data for the algorithms. The next portion deals with the control parameters and the auxiliary calculations.

VI. Choose a positive integer M, which will be equal to the number of points on each of the following four auxiliary x-grids[13]:

$$\vec{x}^{++} = (x_j^{++})_{j=1}^M, \qquad x_j^{++} = h_+ + (j-1)\Delta \quad \forall\, 1 \le j \le M;$$
$$\vec{x}^{+-} = (x_j^{+-})_{j=1}^M, \qquad x_j^{+-} = h_+ + (j-M)\Delta \quad \forall\, 1 \le j \le M;$$
$$\vec{x}^{-+} = (x_j^{-+})_{j=1}^M, \qquad x_j^{-+} = h_- + (j-1)\Delta \quad \forall\, 1 \le j \le M;$$
$$\vec{x}^{--} = (x_j^{--})_{j=1}^M, \qquad x_j^{--} = h_- + (j-M)\Delta \quad \forall\, 1 \le j \le M.$$

The auxiliary x-grids should be significantly longer than the main x-grid; we recommend taking M to be at least $5 \cdot n$.

VII. Set $\zeta = 2\pi/(M\Delta)$ and choose positive integers M_2, M_3 so that the dual grid

$$\vec{\xi} = (\xi_k)_{k=1}^{M_1}, \qquad \xi_k = -M_1\zeta_1/2 + (k-1)\zeta_1,$$

where $M_1 = M \cdot M_2 \cdot M_3$ and $\zeta_1 = \zeta/M_2$, is sufficiently long and sufficiently fine. Since one of the subsequent steps uses FFT for arrays of length $2M_1$, we recommend making the choices so that M, M_2 and M_3 are powers of 2.

VIII. Calculate the values of $\phi_q^+(\xi)$ on the grid $\vec{\xi}$ using the algorithm of §3.3. Then find the values of $\phi_q^-(\xi)$ on $\vec{\xi}$ using the identity $\phi_q^+(\xi)\phi_q^-(\xi) = q(q + \psi(\xi))^{-1}$.

IX. Use the refined inverse FFT technique recalled in §C.4, together with the formulas of §§3.4.2–3.4.3, to calculate the convolution coefficients

$$(c_\ell^+)_{\ell=1-M}^{-1}, \quad (d_k^+)_{k=1}^M, \quad (c_\ell^-)_{\ell=1}^{M-1}, \quad (d_k^-)_{k=1}^M.$$

Then compute $c_0^\pm$ using formulas (3.7) and (3.9).

X. Calculate the coefficients $a_\ell^\pm$ and $b_\ell^\pm$ appearing in the formulas of §3.5.

[13]The mesh Δ is determined in Step III. The need for auxiliary x-grids was explained in §3.2.

3.7. **Implementation of Carr's randomization.** After the preparations of §3.6, we are ready to explain the implementation of a typical step of the backward induction algorithm appearing in §2.4. The procedure explained in this subsection takes as input the values of the function $V^{s+1}(x)$ on the "main" x-grid $\vec{x} = (x_j)_{j=1}^n$ and produces as output the values of the function $V^s(x)$ on the same grid. Thus the full algorithm can be easily obtained by starting with $V^N(x) = g(x)$, the terminal payoff function, and iterating the procedure described in this subsection N times.

Description of the iterative procedure.

1. Input: the array $\vec{V}^{s+1} = \left(V^{s+1}(x_j)\right)_{j=1}^n$.
2. Observe that the main x-grid can be viewed as a subgrid of $\vec{x}^{-+}$. Pad the array $\vec{V}^{s+1}$ with $M - n$ zeros[14] on the right to get the array of values of V^{s+1} on $\vec{x}^{-+}$. Using this array, calculate the values of $G_+^{0,s}(x) = (\mathcal{E}_q^+\mathcal{E}_q^- V^{s+1})(x)$ on $\vec{x}^{-+}$. Next, discard the first n elements of the array $(G_+^{0,s}(x_j^{-+}))_{j=1}^M$ and pad the resulting array with n zeros[15] on the right to obtain the values of $G_+^{0,s}(x)$ on the grid $\vec{x}^{++}$.
3. Similarly, the main x-grid can be viewed as a subgrid of $\vec{x}^{+-}$. Pad the array $\vec{V}^{s+1}$ with $M - n$ zeros on the left to get the array of values of V^{s+1} on the grid $\vec{x}^{+-}$. Use this array to calculate the values of $G_-^{0,s}(x) = (\mathcal{E}_q^-\mathcal{E}_q^+ V^{s+1})(x)$ on the grid $\vec{x}^{+-}$. Define the values of the function $V^{s,0}(x)$, the first approximation to the function $(1 + r\Delta_t)V^s(x)$, by restricting $G_-^{0,s}$ to the main grid $\vec{x}$. Next, discard the last n elements of the array $(G_-^{0,s}(x_j^{+-}))_{j=1}^M$ and pad the resulting array with n zeros on the left to obtain the values of $G_-^{0,s}(x)$ on the grid $\vec{x}^{--}$.
4. Set $k = 1$ (the counter of the number of iterations).
5. At this point, we have the values of $G_+^{2k-2,s}(x)$ on the grid $\vec{x}^{++}$. Using the formulas of §3.5.1, calculate the values of $(\mathcal{E}_q^+)^{-1}G_+^{2k-2,s}$ on $\vec{x}^{++}$. Next, use the identity (2.6) and the formulas of §3.4.4 to calculate the values of $G_-^{2k-1,s}(x)$ on $\vec{x}^{+-}$. Extract the last n elements of the array $(G_-^{2k-1,s}(x_j^{+-}))_{j=1}^M$ to obtain the values of $G^{2k-1,s}(x)$ on the main x-grid $\vec{x} = (x_j)_{j=1}^n$, to be used in Step 8 below. Next, discard the last n elements of $(G_-^{2k-1,s}(x_j^{+-}))_{j=1}^M$ and pad the resulting array with n zeros on the left to obtain the values of $G_-^{2k-1,s}(x)$ on the grid $\vec{x}^{--}$.
6. Similarly, we also have the values of $G_-^{2k-2,s}(x)$ on the grid $\vec{x}^{--}$. Using the obvious analogue of Step 5 and the identity (2.5), calculate the values of $G_+^{2k-1,s}(x)$ on the main x-grid (to be used in Step 8), as well as the values of $G_+^{2k-1,s}(x)$ on $\vec{x}^{++}$.

[14] Recall that, by construction, V^{s+1} vanishes to the right of $x_n = h_+$.

[15] Of course, this is merely an approximation to the true values of $G_+^{0,s}(x)$ at the last n points of the grid $\vec{x}^{++}$. In practice, however, the grid $\vec{x}^{++}$ is sufficiently long, and the function $G_+^{0,s}$ decays sufficiently quickly, so that the error of this approximation is negligible.

7. Repeating Steps 5–6 with the functions $G_{\pm}^{2k-1,s}$ in place of $G_{\pm}^{2k-2,s}$, calculate the values of $G_{\pm}^{2k,s}$ on the main x-grid (to be used in Step 8), as well as the values of $G_{+}^{2k,s}(x)$ on the grid $\vec{x}^{++}$ and the values of $G_{-}^{2k,s}(x)$ on the grid $\vec{x}^{--}$.
8. Define $V^{s,k}(x_j) = V^{s,k-1}(x_j) - G_{+}^{2k-1,s}(x_j) - G_{-}^{2k-1,s}(x_j) + G_{+}^{2k,s}(x_j) + G_{-}^{2k,s}(x_j)$, the new approximation to $(1 + r\Delta_t)V^s(x)$. Calculate
$$\epsilon_k = \max_{1\le j\le n} |V^{s,k}(x_j) - V^{s,k-1}(x_j)|.$$
If $\epsilon_k > \epsilon/N$, replace k with $k + 1$ and return to Step 5.
9. Otherwise, set $V^s(x) = (1 + r\Delta_t)^{-1} \cdot V^{s,k}(x)$. The output of the algorithm is the array $\vec{V}^s = \big(V^s(x_j)\big)_{j=1}^{n}$.

Remarks 3.4. (1) The algorithm given above involves multiple calculations of the expressions appearing in formulas (3.6), (3.11), (3.13), (3.8), (3.12) and (3.16). All these expressions can be calculated quickly using the fast convolution method explained in §C.5.

(2) The reason that the compositions $\mathcal{E}_q^+\mathcal{E}_q^-$ and $\mathcal{E}_q^-\mathcal{E}_q^+$ are used in place of $\mathcal{E}_q$ in Steps 2–3 of the algorithm above is that, in most cases, this increases the overall speed of the computation. Indeed, in order to realize $\mathcal{E}_q$ as a convolution operator, we would have to calculate certain additional "convolution coefficients" (cf., e.g., the formulas in [6, §3.6.1]), which is more time consuming than calculating the compositions $\mathcal{E}_q^+\mathcal{E}_q^-$ and $\mathcal{E}_q^-\mathcal{E}_q^+$ at each step of Carr's randomization procedure.

4. Numerical examples

The calculations based on the algorithm of §3, the results of which are presented below, were performed in MATLAB© 7.3.0 (R2006b), on a PC with characteristics Intel® Core™ 2 Duo T7200 (2.00GHz, 4MB L2 Cache, 667MHz FSB), under the Genuine Windows® XP Professional operating system.

We assume that under a chosen EMM, the log-spot price, $X_t = \ln S_t$, of the underlying follows a KoBoL process (see §1.6(4)) with parameters $\nu = 0.5$, $c = 1$, $\lambda_+ = 9$, $\lambda_- = -8$. (These parameters are taken from the examples that appear in [29, 5, 6].) As in [6], we assume that the riskless rate is $r = 0.03$, which allows us to find the remaining parameter, $\mu \approx -0.0423$, from the EMM condition (3.19).

For this market, we used the algorithm of §3 to compute the prices of a knock-out double barrier put option on the stock $S_t = e^{X_t}$, with strike price $K = 3500$, lower barrier $H_- = 2800$, upper barrier $H_+ = 4200$, and maturity date $T = 0.1$ years. We also computed the prices of a double-no-touch option with the same parameters. We then compared our results with the results obtained by the Monte-Carlo method. The results of our calculations are represented graphically in Figure 1, and are also recorded in Table 1 that appears after the list of references.

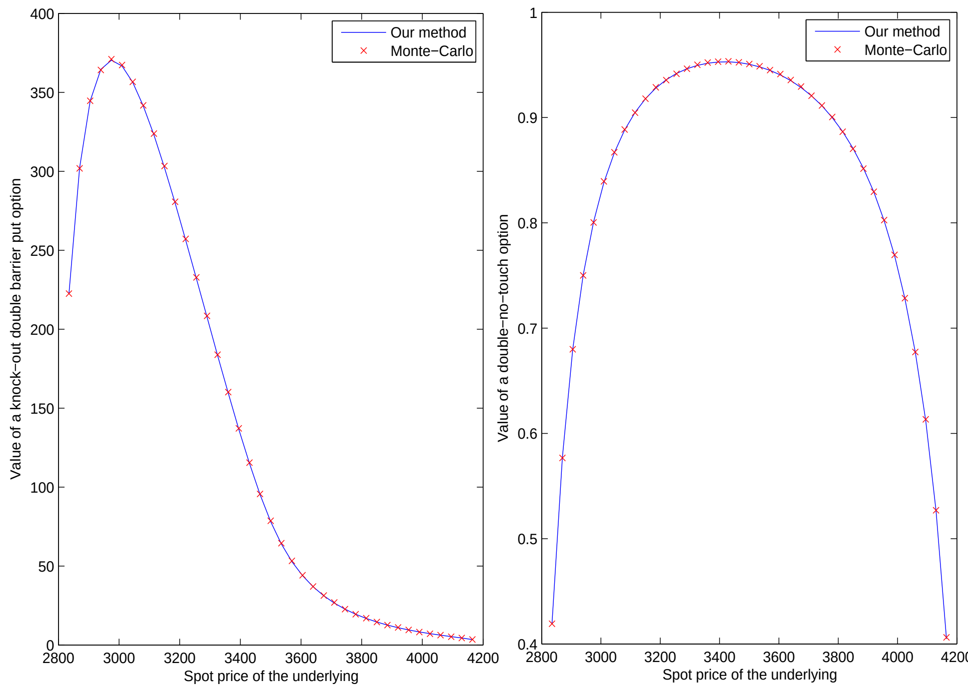

FIGURE 1. Prices of a knock-out double barrier put option (left panel) and of a double-no-touch option (right panel) in the KoBoL model. Solid lines represent the results obtained using the algorithm of §3. Crosses represent the results obtained using Monte-Carlo simulations. *KoBoL parameters:* $\nu = 0.5$, $c = 1$, $\lambda_+ = 9$, $\lambda_- = -8$, $\mu \approx -0.0423$. *Option parameters:* $K = 3500$, $H_- = 2800$, $H_+ = 4200$, $r = 0.03$, $T = 0.1$. *Algorithm parameters:* $n = 812$ (number of points on the "main" x-grid), $\Delta = \frac{\ln H_+ - \ln H_-}{n-1} \approx 0.005$, $M = 4096$, $M_2 = 4$, $M_3 = 16$, $\zeta_1 \approx 0.767$, $m = 8$ (for the calculation of the Wiener-Hopf factors), $N = 80$ (number of time steps), $\epsilon = 10^{-7}$ (error tolerance for the iterative procedure).

The auxiliary parameters of our algorithm are specified in the captions to the figure and to the table. The calculation of all the prices took a total of 14 seconds for each of the two types of options. The iterative procedure used at each step of Carr's randomization converged after just two iterations. For the calculations based on Monte-Carlo simulations, we used 500000 trajectories, with 20000 time steps per year, i.e., 2000 steps along each trajectory.

We observe that for a knock-out double barrier put option, the agreement between the price calculated using our algorithm and the Monte-Carlo price is quite good. The discrepancy does not exceed 0.5% in the out-of-the-money region for the option, and mostly remains under 1.5% (with a single exception of 2.5%) in the in-the-money region. For a double-no-touch option, the agreement is even better: the discrepancy never exceeds 0.4%, and mostly remains under 0.2%, with only a couple of exceptions.

Appendix A. The Wiener-Hopf method

The goal of this appendix and the one that follows is to provide firm theoretical foundations for the method of pricing perpetual knock-out double barrier streams of payoffs that we developed in §1. For the most part, the appendices are rather self contained, and can be read independently from the rest of the article.

A.1. **Overview.** An approach to the valuation of path-dependent options in Lévy-driven models using expected present value operators and the operator form of the Wiener-Hopf factorization formula was pioneered by S.I. Boyarchenko and the second author [9, 10, 11, 35, 36, 12, 13]. In this appendix, the same approach will be applied to the valuation of perpetual knock-out streams of payoffs with two barriers (see §A.7). By definition, if $H_- < H_+$ are the lower barrier and the upper barrier, then such a stream is abandoned as soon as the price, S_t, of the underlying leaves the open interval (H_-, H_+). Prior to the moment when this happens, the instantaneous payoff of the stream at time t is equal to $f(S_t)$, where f is a bounded measurable function on the interval (H_-, H_+). Our main goal is to find a formula for the expected present value of this stream of payoffs that does not involve any stochastic expressions. Such a formula, (A.40), will be derived in §A.7, in the setting of Lévy-driven models of asset pricing (§A.2), under a mild regularity assumption on the underlying price process $\{S_t\}_{t\geq 0}$.

Our approach to the valuation of perpetual streams with two barriers entails the study of certain simpler types of securities as an intermediate step. Thus, in §A.4 we study valuation problems for perpetual knock-out streams with a single barrier, and in §A.6 we study valuation problems for perpetual first-touch contingent claims. Some of the formulas derived there are not new. In particular, valuation formulas for perpetual knock-out streams with a single barrier, and for perpetual first-touch digital options, were obtained earlier by S.I. Boyarchenko and the second author under certain additional assumptions. The novelty of the presentation we give here lies in the fact that we derive the same formulas in the full generality of arbitrary Lévy-driven models, and in the fact that the setup and all our arguments are formulated entirely in the language of probability theory, which is closer in spirit to the financial motivation behind the problems that we consider. Furthermore, we clarify the theory by singling out one important result in the theory of Lévy processes (Lemma A.5) that easily implies all of the main formulas appearing in our work.

Due to the foundational nature of the material presented in this appendix, we decided to carefully formulate all the basic definitions on which this material is based. For every result that we state, we either give a full proof in Appendix B, or give a precise reference to a text where the proof can be found. The definitions of the (normalized) expected present value (EPV) operators of a Lévy process, as well as two forms of the Wiener-Hopf factorization formula, are reviewed in §A.3. Computational aspects of option pricing methods based on EPV operators are discussed in §§3.3–3.4.

A.2. **Lévy-driven models.** In this appendix, by a Lévy-driven model we will mean an infinite time horizon, continuous time model with a constant killing rate $q > 0$ and a stock whose price follows a stochastic process in continuous time that has the form $S_t = S_0 e^{X_t}$, where $S_0 > 0$ and $X = \{X_t\}_{t\geq 0}$ is a 1-dimensional Lévy process.

Let us recall [45, Def. 1.6] that a 1-dimensional Lévy process on a probability space[16] $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection $X = \{X_t\}_{t\geq 0}$ of $\mathbb{R}$-valued random variables on Ω satisfying the following properties:

(1) Given an integer $n \geq 1$ and a collection of times $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.
(2) $X_0 = 0$ almost surely.
(3) For any $t \geq 0$, the distribution of $X_{s+t} - X_s$ is independent of $s \geq 0$.
(4) Stochastic continuity: given $t \geq 0$ and $\epsilon > 0$, we have $\lim_{s\to t} \mathbb{P}\big[|X_s - X_t| > \epsilon\big] = 0$.
(5) There exists a subset $\Omega_0 \subset \Omega$ such that $\mathbb{P}[\Omega_0] = 1$ and for every $\omega \in \Omega_0$, the trajectory $t \mapsto X_t(\omega)$ is right continuous in $t \geq 0$, and has left limits for all $t > 0$.

Remark A.1. It is known, and easy to show, that property (4) in this definition follows from (3) and (5). Indeed, if (3) holds, then (4) is equivalent to the condition that $\lim_{s\to 0^+} \mathbb{P}\big[|X_s - X_0| > \epsilon\big] = 0$ for all $\epsilon > 0$. Assume that it fails. Then there exist $\epsilon, \delta > 0$ and a sequence $s_n > 0$, $s_n \to 0$, such that $\mathbb{P}\big[|X_{s_n} - X_0| > \epsilon\big] \geq \delta$ for all n. It follows that there exist an event $\Omega' \in \mathcal{F}$ such that $\mathbb{P}[\Omega'] > 0$ and a subsequence $\{s_{n_j}\}_{j=1}^{\infty}$ of $\{s_n\}$ such that $|X_{s_{n_j}}(\omega) - X_0(\omega)| > \epsilon$ for all $\omega \in \Omega'$ and all j. Since $\Omega' \cap \Omega_0 \neq \varnothing$, this contradicts the right continuity condition in (5).

Some examples of Lévy processes used in financial modeling were given in §1.6.

Remark A.2. We do *not* assume that the discounted price process $\{e^{-qt}S_t\}_{t\geq 0}$ is a martingale. The reason for this, as well as for using the letter "q" in place of the more standard "r" to denote the killing rate, is clear from the discussion in §2. There, we saw that a pricing problem for a finite-lived double barrier option in an arbitrage-free Lévy-driven model with riskless rate r can be approximated, by means of Carr's randomization, with a sequence of perpetual option pricing problems of the type considered in this appendix, in which the price dynamics of the underlying is the same as in the original model, but the killing rate, q, is different from r.

A.3. **Wiener-Hopf factorization.** The WHF formula for Lévy processes has many different forms. The goal of this subsection is to recall two of the forms that are the most relevant to pricing barrier options. Another form will be recalled in §A.5.

A.3.1. *Probability form of WHF.* We remain in the setup of §A.2. Thus $X = \{X_t\}_{t\geq 0}$ is a 1-dimensional Lévy process and $q > 0$ is fixed. The *supremum process* $\overline{X}$ and the *infimum process* $\underline{X}$ of X are the 1-dimensional stochastic processes defined by

$$\overline{X}_t = \sup_{0\leq s\leq t} X_s, \qquad \underline{X}_t = \inf_{0\leq s\leq t} X_s. \tag{A.1}$$

Let $T_q \sim \operatorname{Exp} q$ denote an exponentially distributed random variable with mean q^{-1} that is independent of the process X. The form of the WHF formula that is commonly used in probability theory reads as follows:

$$\mathbb{E}\big[e^{i\xi X_{T_q}}\big] = \mathbb{E}\big[e^{i\xi \overline{X}_{T_q}}\big] \cdot \mathbb{E}\big[e^{i\xi \underline{X}_{T_q}}\big] \qquad \forall\, \xi \in \mathbb{R}. \tag{A.2}$$

[16] According to standard conventions, Ω is a set equipped with a σ-algebra $\mathcal{F}$ and a probability measure $\mathbb{P}$ on $\mathcal{F}$. Given an event $A \in \mathcal{F}$, its probability will be denoted by $\mathbb{P}[A]$. The expectation operator with respect to the probability measure $\mathbb{P}$ will be denoted by $\mathbb{E}[\cdot]$.

In fact, (A.2) follows easily from Lemma A.5 below and the fact that the characteristic function of the sum of two independent random variables is equal to the product of their characteristic functions.

A.3.2. *Operator form of the WHF formula.* For our purposes, another form of the WHF formula is more fundamental. To state it, let us introduce three *normalized expected present value* (EPV) operators associated to the process X, defined by

$$(\mathcal{E}_q f)(x) = \mathbb{E}\big[f(x + X_{T_q})\big], \tag{A.3}$$

$$(\mathcal{E}_q^+ f)(x) = \mathbb{E}\big[f(x + \overline{X}_{T_q})\big], \qquad (\mathcal{E}_q^- f)(x) = \mathbb{E}\big[f(x + \underline{X}_{T_q})\big]. \tag{A.4}$$

The "operator form" of the WHF formula is as follows:

$$\mathcal{E}_q = \mathcal{E}_q^+ \mathcal{E}_q^- = \mathcal{E}_q^- \mathcal{E}_q^+. \tag{A.5}$$

It can be viewed as special case of the combination of Propositions A.6 and A.7 (cf. Remark A.9). The question of which classes of functions $f(x)$ the action of the operators $\mathcal{E}_q$ and $\mathcal{E}_q^\pm$ can be defined on is briefly addressed in §A.3.3 below.

Remark A.3. The reader may be used to a slightly different system of notational conventions, where, for instance, one would write the expression on the right hand side of (A.3) as $\mathbb{E}^x\big[f(X_{T_q})\big]$, rather than $\mathbb{E}\big[f(x+X_{T_q})\big]$. We prefer the latter notation because it is always unambiguous, whereas we will encounter certain situations later on where notation of the former sort could lead to a serious misinterpretation. Rather than carefully explaining the meaning of the symbol $\mathbb{E}^x$ every time when its use could cause confusion, we decided to avoid it altogether.

The terminology "normalized EPV operators" can be explained as follows. Since the probability distribution of T_q is $qe^{-qt}\mathbb{1}_{[0,+\infty)}(t)\,dt$, and since T_q is independent of the Lévy process X, we obtain

$$(\mathcal{E}_q f)(x) = \mathbb{E}\left[\int_0^\infty qe^{-qt} f(x + X_t)\,dt\right], \tag{A.6}$$

$$(\mathcal{E}_q^+ f)(x) = \mathbb{E}\left[\int_0^\infty qe^{-qt} f(x + \overline{X}_t)\,dt\right] \tag{A.7}$$

and

$$(\mathcal{E}_q^- f)(x) = \mathbb{E}\left[\int_0^\infty qe^{-qt} f(x + \underline{X}_t)\,dt\right]. \tag{A.8}$$

In the framework of §A.2, suppose that the initial spot price of the stock equals $S_0 = e^x$, and consider a perpetual stream of payoffs whose instantaneous payoff at time t is equal to $f(\ln S_t)$. Then the right hand side of (A.6) calculates the expected present value of this stream, multiplied by q. This extra factor of q is responsible for the property that if $f(x) = 1$ for all $x \in \mathbb{R}$, then $(\mathcal{E}_q f)(x) = 1$ for all $x \in \mathbb{R}$ as well, which explains the use of the adjective "normalized." In a similar vein, the operators $\mathcal{E}_q^+$ and $\mathcal{E}_q^-$ can be interpreted as the normalized expected present value operators of the supremum process and the infimum process of X, respectively.

A.3.3. *Domains of the normalized EPV operators.* It is useful to keep in mind the following three situations where the operators $\mathcal{E}_q$ and $\mathcal{E}_q^{\pm}$ introduced in §A.3.2 are well defined. To save space, we will only consider $\mathcal{E}_q$, but everything we will say applies verbatim to the operators $\mathcal{E}_q^+$ and $\mathcal{E}_q^-$ as well.

(1) Suppose that f is a measurable and nonnegative function on $\mathbb{R}$. Then, for each $x \in \mathbb{R}$, the expectation $(\mathcal{E}_q f)(x) = \mathbb{E}\big[f(x + X_{T_q})\big]$ is defined, and is either a nonnegative real number or $+\infty$. Moreover, it is easy to check (using Fubini's theorem) that $(\mathcal{E}_q f)(x)$ is measurable as a function of x. Thus $\mathcal{E}_q$ can be viewed as acting on the collection of all nonnegative measurable functions on the real line.

(2) Suppose now that f is an arbitrary (real or complex-valued) measurable function on $\mathbb{R}$, and that f is bounded: $|f(x)| \le C$ for all $x \in \mathbb{R}$, where $C > 0$ is fixed. Then, for each $x \in \mathbb{R}$, the function $\omega \mapsto f(x + X_{T_q}(\omega))$ is a measurable function on Ω, and it is bounded in absolute value by C, whence it is integrable. Moreover, $\Big|\mathbb{E}\big[f(x + X_{T_q})\big]\Big| \le C$, so $\mathcal{E}_q$ can be viewed as an operator on the space of all bounded measurable functions on the real line. In addition, if we write $||f||_\infty = \sup\big\{|f(x)| \,\big|\, x \in \mathbb{R}\big\}$ for a bounded measurable function f on $\mathbb{R}$, we see that $||\mathcal{E}_q f||_\infty \le ||f||_\infty$, so $\mathcal{E}_q$ is a bounded operator with respect to $||\cdot||_\infty$.

(3) Finally, suppose that $f \in L_1(\mathbb{R})$, i.e., f is a (real or complex-valued) measurable function on $\mathbb{R}$ and
$$||f||_1 \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} |f(x)|\, dx$$
is finite. Let $g(x,\omega)$ be the function on the product space $\mathbb{R} \times \Omega$ defined by $g(x,\omega) = f(x + X_{T_q}(\omega))$. It is clear that g is measurable. Since the function $|g|$ is nonnegative, we can apply Fubini's theorem:
$$\int_{-\infty}^{\infty} \mathbb{E}\big[|g(x,\omega)|\big]\, dx = \mathbb{E}\left[\int_{-\infty}^{\infty} |g(x,\omega)|\, dx\right] = \mathbb{E}\big[||f||_1\big] = ||f||_1 < \infty.$$
In particular, the function g on $\mathbb{R} \times \Omega$ is integrable. This allows us to apply Fubini's theorem to g itself, and we see that $(\mathcal{E}_q f)(x) = \mathbb{E}\big[g(x,\omega)\big]$ is well defined and finite for almost every $x \in \mathbb{R}$, and, moreover, the function $\mathcal{E}_q f$ lies in $L_1(\mathbb{R})$, with $||\mathcal{E}_q f||_1 \le ||f||_1$. We conclude that $\mathcal{E}_q$ can also be viewed as a bounded operator on $L_1(\mathbb{R})$.

Remark A.4. In the context of pricing double barrier options, it is enough to consider the action of the operators $\mathcal{E}_q$ and $\mathcal{E}_q^{\pm}$ on bounded functions. Under suitable additional assumptions on the process, the action of these operators can also be defined in certain spaces of functions of exponential growth. For the details, see, e.g., [10, 13].

A.4. **Value of a perpetual payoff stream with a single barrier.** We will now derive formulas for the expected present value of a perpetual stream of payoffs that is abandoned when the price of the underlying exceeds a certain barrier (the "up-and-out" case), or falls below a certain barrier (the "down-and-out" case). The formulas involve the normalized EPV operators of a Lévy process defined in §A.3, and their derivation is based on the following key lemma, the first part of which is a rather deep result in the fluctuation theory for Lévy processes.

Lemma A.5 (See [23], Lemma 2.1(i) and [44], p. 81)**.** *Let $X = \{X_t\}_{t\ge 0}$ be a 1-dimensional Lévy process, let $q > 0$, and let T_q be an exponentially distributed random variable with mean q that is independent of the process X. Then*

(a) the random variables $\overline{X}_{T_q}$ and $X_{T_q} - \overline{X}_{T_q}$ are independent; and

(b) the random variables $\underline{X}_{T_q}$ and $X_{T_q} - \overline{X}_{T_q}$ are identical in law.

Let the assumptions of the lemma hold. Let us fix a real number h. We will denote by τ_h^+ the *first entrance time of X into* $[h, +\infty)$, which is defined as follows:
$$\tau_h^+(\omega) = \inf\big\{t \ge 0 \,\big|\, X_t(\omega) \in [h, +\infty)\big\} \qquad \forall\, \omega \in \Omega.$$
We recall that Ω is the underlying probability space on which the random variables X_t and T_q are defined. By convention, $\inf \varnothing = +\infty$. Let $\Omega_0 \subset \Omega$ be a subset of full

measure such that the trajectories $t \mapsto X_t(\omega)$ are right continuous with left limits for all $\omega \in \Omega_0$. If $\omega \in \Omega_0$ and $\tau_h^+(\omega) < \infty$, then, by right continuity, we see that $X_{\tau_h^+(\omega)}(\omega) \in [h, +\infty)$, and $X_t(\omega) \notin [h, +\infty)$ for all $t < \tau_h^+(\omega)$ by definition, which justifies the terminology "first entrance time."

Let $f(x)$ be a measurable function defined on $\mathbb{R}$ that is either nonnegative or bounded. In the setup of the Lévy-driven model discussed in §A.2, let us consider a perpetual stream of payoffs whose instantaneous payoff at time t equals $f(\ln S_t)$, and which is abandoned as soon as the price, S_t, of the underlying, reaches or exceeds the barrier $H = e^h$. Let the initial spot price of the underlying be $S_0 = e^x$. By definition, this means that the expected present value of this stream is equal to[17]

$$v_{p.u.o.}(x; f) = \mathbb{E}\left[\int_0^{\tau_{h-x}^+} e^{-qt} f(x + X_t)\, dt\right]. \tag{A.9}$$

Proposition A.6 (See §B.1 for a proof)**.** *In this situation, we have*

$$v_{p.u.o.}(x; f) = q^{-1} \cdot \mathcal{E}_q^+\big(\mathbb{1}_{(-\infty,h)}(x) \cdot (\mathcal{E}_q^- f)(x)\big). \tag{A.10}$$

Down-and-out perpetual streams can be treated similarly. Namely, in the situation above, the first entrance time of the process X into $(-\infty, h]$ is defined as follows:

$$\tau_h^-(\omega) = \inf\big\{t \geq 0 \,\big|\, X_t(\omega) \in (-\infty, h]\big\} \qquad \forall\, \omega \in \Omega.$$

We consider a perpetual stream of payoffs whose instantaneous payoff at time t equals $f(\ln S_t)$, and which is abandoned as soon as the price, S_t, of the underlying, reaches or falls below the barrier $H = e^h$. If the initial spot price of the underlying is $S_0 = e^x$, then, by definition, the expected present value of this stream is given by[18]

$$v_{p.d.o.}(x; f) = \mathbb{E}\left[\int_0^{\tau_{h-x}^-} e^{-qt} f(x + X_t)\, dt\right]. \tag{A.11}$$

Proposition A.7. *In this situation, we have*

$$v_{p.d.o.}(x; f) = q^{-1} \cdot \mathcal{E}_q^-\big(\mathbb{1}_{(h,+\infty)}(x) \cdot (\mathcal{E}_q^+ f)(x)\big). \tag{A.12}$$

Remark A.8. It is clear that Proposition A.7 follows formally from Proposition A.6, by applying the latter to the Lévy process $\{-X_t\}_{t\geq 0}$ in place of X.

Remark A.9. As we already noted, (A.2) can be easily deduced from Lemma A.5. Furthermore, (A.5) can then be deduced from (A.2) and the easily verifiable identities

$$\mathcal{E}_q\big(e^{i\xi x}\big) = \mathbb{E}\big[e^{i\xi X_{T_q}}\big] \cdot e^{i\xi x} \qquad \forall\, \xi \in \mathbb{R} \tag{A.13}$$

and

$$\mathcal{E}_q^+\big(e^{i\xi x}\big) = \mathbb{E}\big[e^{i\xi \overline{X}_{T_q}}\big] \cdot e^{i\xi x}, \qquad \mathcal{E}_q^-\big(e^{i\xi x}\big) = \mathbb{E}\big[e^{i\xi \underline{X}_{T_q}}\big] \cdot e^{i\xi x} \qquad \forall\, \xi \in \mathbb{R}. \tag{A.14}$$

[17] To justify this formula, note that $\ln S_t = x + X_t$, and that S_t reaches $[H, +\infty)$ if and only if X_t reaches $[h - x, +\infty)$. The letters "p.u.o." stand for "perpetual up-and-out."

[18] The letters "p.d.o." stand for "perpetual down-and-out."

The method by which we prove Proposition A.6 is based on the same idea as the derivation of (A.2) and (A.5) from Lemma A.5. Conversely, (A.5) can be viewed as the limiting case of Propositions A.6 and A.7 obtained by taking $h = +\infty$ and $h = -\infty$, respectively.

A.5. **The third form of the Wiener-Hopf factorization and proof of** (1.15)**.** Both are intimately related to the question of efficient numerical realization of the operators $\mathcal{E}_q^{\pm}$ (see §§3.3–3.4 above). Recall that the Wiener-Hopf factors $\phi_q^{\pm}(\xi)$ are defined by (1.14). It follows immediately from (A.4) that

$$\mathcal{E}_q^{\pm}\left(e^{i\xi x}\right) = \phi_q^{\pm}(\xi) \cdot e^{i\xi x}. \tag{A.15}$$

Hence, one can use the Fourier transform and inverse Fourier transform to realize $\mathcal{E}_q^{\pm}$ as pseudo-differential operators with the symbols $\phi_q^{\pm}(\xi)$, and FFT-technique makes the calculations efficient. However, (1.14) is not convenient for the numerical calculations at all, whereas (1.15) is. Unfortunately, we do not know in advance that the functions $\phi_q^{\pm}(\xi)$ in (1.14) coincide with the functions in the RHS of (1.15). For the time being, we denote the latter by $\phi_{q,1}^{\pm}(\xi)$. In fact, the equality

$$\phi_q^{\pm}(\xi) = \phi_{q,1}^{\pm}(\xi) \tag{A.16}$$

may not hold for an arbitrary Lévy process simply because the RHS in (1.15) is not defined unless the characteristic exponent is analytic in a strip around the real axis. In [10, Thm. 3.2], (A.16) was derived from the Spitzer identities by direct calculations, for wide classes of processes which contain almost all model classes of Lévy processes used in Financial Engineering. The only exception was the important class of VG processes. In this subsection, we fill in this gap.

The general tool, which allows one to prove (A.16), is the analytical form of the Wiener-Hopf factorization, which can be defined for wide classes of functions (not necessarily related to any process), and which is, in fact, the initial form of the Wiener-Hopf factorization. Using (1.15) and the residue theorem, it is straightforward to prove that the integrals under the exponential sign (for the signs "+" and "-") sum up to $\ln(q/(q + \psi(\xi))$. Hence,

$$\frac{q}{q + \psi(\xi)} = \phi_{q,1}^{+}(\xi)\phi_{q,1}^{-}(\xi) \tag{A.17}$$

for ξ in the strip $\operatorname{Im}\xi \in [\omega_-, \omega_+]$. Note that $\phi_q^{+}(\xi)$ (resp., $\phi_q^{-}(\xi)$) admits the analytic continuation into the upper (resp., lower) half-plane and does not vanish there, and $\phi_q^{+}(0) = 1$. Thus, (A.17) is a special case of the Wiener-Hopf factorization in analysis.

The same formula holds with $\phi_q^{\pm}(\xi)$ instead of $\phi_{q,1}^{\pm}(\xi)$:

$$\frac{q}{q + \psi(\xi)} = \phi_q^{+}(\xi)\phi_q^{-}(\xi), \tag{A.18}$$

which is well-known in probability. Indeed, formula (A.6) and the identity $\mathbb{E}\left[e^{i\xi(x+X_t)}\right] = e^{ix\xi}e^{-t\psi(\xi)}$, which follows from the definition of the characteristic exponent, $\psi(\xi)$, imply that

$$\mathcal{E}_q\left(e^{ix\xi}\right) = q\cdot(q+\psi(\xi))^{-1}\cdot e^{ix\xi}. \tag{A.19}$$

Taking into account (A.5), we obtain (A.18). Since the trajectories of the supremum process $\overline{X}$ are nondecreasing, $\phi^+_q(\xi)$ admits analytic continuation into the upper half-plane. Similarly, $\phi^-_q(\xi)$ admits analytic continuation into the lower half-plane. It follows from the Spitzer identities that both do not vanish in the corresponding half-planes.

Now, if we know that the Wiener-Hopf factors and their reciprocals are polynomially bounded in the corresponding half-planes, then the well-known theorem about the uniqueness of the Wiener-Hopf factorization gives (A.16) (the normalization $\phi^+_q(0) = 1 = \phi^+_{q,1}(0)$ is important; in general, the equality of the factors in two Wiener-Hopf factorization holds up to scalar multiples). Unfortunately, we do not know how to prove the polynomial boundedness of the reciprocals $\phi^\pm_q(\xi)^{-1}$.

Luckily, it is not difficult to study the behavior of the factors $\phi^\pm_{q,1}(\xi)$ for a VG process, and this is the first step. Although formally excluded from the consideration in [10, Thm. 3.3], the formulas for the Wiener-Hopf factors derived in this theorem are valid for VG processes with non-zero drift μ, and it is an easy exercise to show that these formulas coincide with (A.16). Moreover, the formulas in [10, Thm. 3.3] imply that

if $\mu > 0$, then, as $\xi \to \infty$ in the corresponding half-plane,

$$\phi^+_{q,1}(\xi)^{\pm 1} \sim (-i\mu\xi)^{\pm 1} \tag{A.20}$$

$$\phi^-_{q,1}(\xi)^{\pm 1} = O(1); \tag{A.21}$$

if $\mu < 0$, then, as $\xi \to \infty$ in the corresponding half-plane,

$$\phi^+_{q,1}(\xi)^{\pm 1} = O(1) \tag{A.22}$$

$$\phi^-_{q,1}(\xi)^{\pm 1} \sim (-i\mu\xi)^{\pm 1}. \tag{A.23}$$

If $\mu = 0$, then it is fairly straightforward to show that

$$\phi^\pm_{q,1}(\xi) + 1/\phi^\pm_{q,1}(\xi) = O((\ln|\xi|)^C). \tag{A.24}$$

Now, consider the function given by (A.11), where $f(x) = e^{\beta x}$, $\beta < -\omega_-$, and $h = 0$. Applying (A.12), we obtain

$$v_{p.d.o.}(x; f) = q^{-1}\phi^+_q(-i\beta)\mathcal{E}^-_q\mathbb{1}_{[0,+\infty)}(x)e^{\beta x}. \tag{A.25}$$

A similar formula in terms of $\phi^{\pm}_{q,1}$ can be derived using the analytical tools. First, by [10, Thm. 2.12], function $v_{p.d.o.}(x; f)$ is a solution of the boundary problem

$$(q - L)v(x) = f(x), \quad x > 0, \tag{A.26}$$

$$v(x) = 0, \quad x \le 0. \tag{A.27}$$

(Note that (A.26) is understood in the sense of the generalized functions.) It is proven in [10, Thm. 5.2] that, under conditions (A.22)–(A.23), the solution of the problem (A.26)–(A.27) is unique in the class of continuous functions which grow not faster than $e^{\beta x}$ at infinity, and it is given by

$$v_{p.d.o.}(x; f) = q^{-1}\phi^+_{q,1}(-i\beta)\phi^-_{q,1}(D)\mathbb{1}_{[0,+\infty)}(x)e^{\beta x}. \tag{A.28}$$

Under conditions (A.20)–(A.21), the solution of the problem (A.26)–(A.27) is unique in the class of piece-wise continuous functions which grow not faster than $e^{\beta x}$ at infinity, and it is given by (A.28). The case $\mu = 0$, when (A.24) holds, is not covered by [10, Thm. 5.2] but its proof can be repeated essentially word by word to obtain the same conclusion as in the case (A.22)–(A.23).

Thus, we derived two formulas (A.25) and (A.28) for the same function. It follows that the symbols of the PDO $\mathcal{E}^-_q$ and $\phi^-_{q,1}(D)$, which are ϕ^-_q and $\phi^-_{q,1}$, must equal up to a scalar multiple. Since both functions equals 1 at 0, they are equal. Hence, ϕ^+_q and $\phi^+_{q,1}$ are equal as well.

A.6. **Value of a perpetual first-touch claim.** We remain in the setup of §§A.2–A.4. Let us fix a real number h, and let $G(x)$ be a measurable function on $[h, +\infty)$ that is either nonnegative or bounded. We consider a first-touch up-and-in contingent claim on the underlying $\{S_t\}_{t\ge 0}$ that pays its owner $G(\ln S_t)$ as soon as S_t reaches or exceeds the barrier $H = e^h$. The expected present value of this claim equals[19]

$$v_{p.f.t.u.}(x; G) = \mathbb{E}\left[e^{-q\tau^+_{h-x}}G(x + X_{\tau^+_{h-x}})\right], \tag{A.29}$$

assuming, as usual, that the initial spot price of the underlying equals $S_0 = e^x$. We would like to give a formula for the function $v_{p.f.t.u.}(x; G)$ in terms of the normalized EPV operators of the process X (see §A.3). First we will write down the formula itself in a concise form that can be easily remembered; then we will explain how the formula should be interpreted, and state the conditions under which it is valid (cf. Proposition A.10, Remark A.11 and Example A.12(2)).

Our formula for the function $v_{p.f.t.u.}(x; G)$ defined by (A.29) is as follows:

$$v_{p.f.t.u.}(x; G) = \mathcal{E}^+_q\Big(\mathbb{1}_{[h,+\infty)}(x) \cdot \big((\mathcal{E}^+_q)^{-1}G\big)(x)\Big). \tag{A.30}$$

Its meaning is explained by the next result, proved in §B.3.

[19]The notation is as in §A.4. The letters "p.f.t.u." stand for "perpetual first-touch up-and-in."

Proposition A.10. *Suppose that there exists a bounded measurable function $f(x)$ on $\mathbb{R}$ that satisfies $G(x) = (\mathcal{E}_q f)(x)$ for all $x \geq h$. Then*

$$v_{p.f.t.u.}(x; G) = \mathcal{E}_q^+ \big(\mathbb{1}_{[h,+\infty)}(x) \cdot (\mathcal{E}_q^- f)(x)\big).$$

Remark A.11. By way of explanation, let the assumption of Proposition A.10 be satisfied. Formula (A.5) implies that the function $g = \mathcal{E}_q^- f$ satisfies $(\mathcal{E}_q^+ g)(x) = G(x)$ for all $x \geq h$. In other words, the restriction of $\mathcal{E}_q^- f$ to the interval $[h, +\infty)$ deserves the notation $(\mathcal{E}_q^+)^{-1} G$. However, we stress that, for our purposes, the expression $(\mathcal{E}_q^+)^{-1} G$ only has meaning when the hypothesis of Proposition A.10 holds.

Examples A.12. (1) Suppose that $G(x) = 1$ for all $x \geq h$, so that the claim under consideration is a perpetual up-and-in first-touch digital option. In this case, the assumption of Proposition A.10 holds with $f(x) = 1$ for all $x \in \mathbb{R}$, and the right hand side of (A.30) reduces to the formula $\mathcal{E}_q^+ \big(\mathbb{1}_{[h,+\infty)}(x)\big)$ for the value function of such a claim. This formula was known before (see, e.g., [10]).

(2) Suppose that $G \in C_0^2([h, +\infty))$, which means that the function $G(x)$ is twice continuously differentiable on $[h, +\infty)$, and we have

$$\lim_{x\to+\infty} G(x) = \lim_{x\to+\infty} G'(x) = \lim_{x\to+\infty} G''(x) = 0.$$

As we explain in §B.4, the hypothesis of Proposition A.10 is satisfied under this assumption as well. This fact will become important for us in §A.7 below.

The treatment of perpetual first-touch down-and-in contingent claims is entirely analogous. In the same setup as before, suppose that G is a measurable function on $(-\infty, h]$ that is either nonnegative or bounded, and consider the contingent claim on the underlying $\{S_t\}_{t\geq 0}$ that pays its owner $G(\ln S_t)$ as soon as S_t reaches or falls below the barrier $H = e^h$. The expected present value of this claim equals[20]

$$v_{p.f.t.d.}(x; G) = \mathbb{E}\big[e^{-q\tau^-_{h-x}} G(x + X_{\tau^-_{h-x}})\big], \tag{A.31}$$

assuming, as before, that the initial spot price of the underlying equals $S_0 = e^x$. It can be calculated by means of the formula

$$v_{p.f.t.d.}(x; G) = \mathcal{E}_q^- \Big(\mathbb{1}_{(-\infty,h]}(x) \cdot \big((\mathcal{E}_q^-)^{-1} G\big)(x)\Big), \tag{A.32}$$

whose interpretation is provided by

Proposition A.13. *Suppose that there exists a bounded measurable function $f(x)$ on $\mathbb{R}$ that satisfies $G(x) = (\mathcal{E}_q f)(x)$ for all $x \leq h$. Then*

$$v_{p.f.t.d.}(x; G) = \mathcal{E}_q^- \big(\mathbb{1}_{(-\infty,h]}(x) \cdot (\mathcal{E}_q^+ f)(x)\big).$$

[20] The letters "p.f.t.d." stand for "perpetual first-touch down-and-in."

As before, Proposition A.13 follows formally from Proposition A.10, by applying the latter to the Lévy process $\{-X_t\}_{t\geq 0}$ in place of X. The obvious analogues of Remark A.11 and Examples A.12 are valid for perpetual first-touch down-and-in contingent claims as well. The computational aspects of option pricing formulas that involve expressions such as $(\mathcal{E}_q^+)^{-1}G$ and $(\mathcal{E}_q^-)^{-1}G$ were treated in §3.5 above.

A.7. **Value of a perpetual knock-out payoff stream with two barriers.** In this subsection we derive formulas for the expected present values of a perpetual knock-out double barrier payoff stream and some related securities in the setting of an arbitrary tame Lévy-driven model. The financial motivation for the constructions that we employ here is provided in §1.2.

A.7.1. We remain in the setup of §A.2. In particular, a 1-dimensional Lévy process $X = \{X_t\}_{t\geq 0}$ and a real number $q > 0$ are given, and we consider a risky asset whose price process has the form $\{S_t = S_0 e^{X_t}\}_{t\geq 0}$, where $S_0 > 0$. Let us fix $0 < H_- < H_+$, and write $h_\pm = \ln H_\pm$. Let $g(x)$ be a bounded, nonnegative, measurable function defined on (h_-, h_+), and consider a perpetual stream of payoffs characterized by the following two properties.

- The stream is abandoned as soon as S_t leaves the open interval (H_-, H_+).
- Prior to the time when this happens, the instantaneous payoff of the stream at time $t \geq 0$ is equal to $g(\ln S_t)$.

By definition, the expected present value of this stream of payoffs is given by[21]

$$v_{p.k.o.d.b.}(x; g) = \mathbb{E}\left[\int_0^{\tau_{h_- - x, h_+ - x}} e^{-qt} g(x + X_t)\, dt\right], \tag{A.33}$$

where $\tau_{h_-,h_+} = \min\{\tau^-_{h_-}, \tau^+_{h_+}\}$ is the first entrance time of the process X into the set $(-\infty, h_-] \cup [h_+, +\infty)$ (the random times $\tau^\pm_{h_\pm}$ were defined in §A.4). In other words,

$$\tau_{h_-,h_+}(\omega) = \inf\{t \geq 0 \,|\, X_t(\omega) \geq h_+ \text{ or } X_t(\omega) \leq h_-\} \qquad \forall\, \omega \in \Omega.$$

Note that in (A.33), as elsewhere, the initial spot price of the underlying is $S_0 = e^x$.

A.7.2. Let us also introduce the first-touch analogue of this claim. Let $G^0_+(x)$ (respectively, $G^0_-(x)$) be a bounded measurable function on $[h_+, +\infty)$ (respectively, $(-\infty, h_-]$). We consider a first-touch contingent claim that pays its owner $G^0_\pm(\ln S_t)$ at the first moment that S_t leaves the open interval (H_-, H_+), where the subscript "+" or "−" is chosen according to whether $S_t \geq H_+$ or $S_t \leq H_-$. By definition, the expected present value of this contingent claim is given by [22]

$$v_{p.f.t.d.b.}(x; G^0_\pm) = \mathbb{E}\left[e^{-q\tau_{h_- - x, h_+ - x}} G^0_\pm(x + X_{\tau_{h_- - x, h_+ - x}})\right], \tag{A.34}$$

where $x = \ln S_0$.

[21] The letters "p.k.o.d.b." stand for "perpetual knock-out double barrier."

[22] The letters "p.f.t.d.b." stand for "perpetual first-touch double barrier."

A.7.3. Our main result on perpetual first-touch double barrier contingent claims is

Theorem A.14. *Let X, q, $H_\pm$, $h_\pm$ and $G^0_\pm$ be as above.*

Let $G^1_+(x)$ (respectively, $G^1_-(x)$) denote the value function of the perpetual first-touch down-and-in (respectively, up-and-in) contingent claim that pays its owner $G^0_-(\ln S_t)$ (respectively, $G^0_+(\ln S_t)$) at the first moment that S_t reaches H_- or falls below H_- (respectively, reaches or exceeds H_+).

Let $G^2_+(x)$ (respectively, $G^2_-(x)$) denote the value function of the perpetual first-touch down-and-in (respectively, up-and-in) contingent claim that pays its owner $G^1_-(\ln S_t)$ (respectively, $G^1_+(\ln S_t)$) at the first moment that S_t reaches H_- or falls below H_- (respectively, reaches or exceeds H_+).

(*a*) *We have*

$$v_{p.f.t.d.b.}(x; G^0_\pm) = G^1_+(x) + G^1_-(x) - G^2_+(x) - G^2_-(x) + v_{p.f.t.d.b.}(x; G^2_\pm). \tag{A.35}$$

(*b*) *There exists a number $0 < \epsilon < 1$, depending only on X, q and $H_\pm$, with the following property: if $C > 0$ is a constant such that $|G^0_+(x)| \le C$ for all $x \ge h_+$ and $|G^0_-(x)| \le C$ for all $x \le h_-$, then $|G^1_+(x)| \le \epsilon C$ for all $x \ge h_+$ and $|G^1_-(x)| \le \epsilon C$ for all $x \le h_-$.*

(*c*) *Suppose that the Lévy process X is tame, and assume that the function $G^0_+(x)$ on $[h_+, +\infty)$ satisfies the hypothesis of Proposition A.10, and the function $G^0_-(x)$ on $(-\infty, h_-]$ satisfies the hypothesis of Proposition A.13. Then the same is true for the function $G^1_+(x)$ on $[h_+, +\infty)$ and the function $G^1_-(x)$ on $(-\infty, h_-]$.*

The theorem is proved in §B.6.

Remark A.15. Let us suppose that the Lévy process X is tame, and that the second hypothesis of part (c) of the theorem is satisfied (as we remarked in Example A.12(2), this is the case, for instance, when G^0_+ is either constant or belongs to $C^2_0([h_+, +\infty))$, and when G^0_- is either constant or belongs to $C^2_0((-\infty, h_-]))$.

Then formula (A.35) and part (b) of Theorem A.14 yield a convergent iterative procedure for the computation of the value function $v_{p.f.t.d.b.}(x; G^0_\pm)$. One viewpoint on this iterative procedure is that it represents $v_{p.f.t.d.b.}(x; G^0_\pm)$ as the sum of the following series of functions that converges absolutely and uniformly on the real line:

$$\begin{aligned} v_{p.f.t.d.b.}(x; G^0_\pm) &= G^1_+(x) + G^1_-(x) - G^2_+(x) - G^2_-(x) \\ &\quad + G^3_+(x) + G^3_-(x) - G^4_+(x) - G^4_-(x) \\ &\quad + \cdots, \end{aligned} \tag{A.36}$$

where, using the notation of §A.6, we can write

$$G^n_+(x) = \mathcal{E}^-_q\Big(\mathbb{1}_{(-\infty,h_-]}(x) \cdot \big((\mathcal{E}^-_q)^{-1} G^{n-1}_-\big)(x)\Big) \qquad \forall\, n \ge 1 \tag{A.37}$$

and

$$G^n_-(x) = \mathcal{E}^+_q\Big(\mathbb{1}_{[h_+,+\infty)}(x)\cdot\big((\mathcal{E}^+_q)^{-1}G^{n-1}_+\big)(x)\Big) \qquad \forall\, n\geq 1. \tag{A.38}$$

A.7.4. Our main result on perpetual knock-out payoff streams with two barriers is

Theorem A.16. *Let X, q, $H_\pm$, $h_\pm$ and g be as in §A.7.1, and let $v_{p.k.o.d.b.}$ be defined by* (A.33). *Let $G^0_+(x)$ and $G^0_-(x)$ be the restrictions of the function $(\mathcal{E}_q g)(x)$ to the intervals $[h_+,+\infty)$ and $(-\infty,h_-]$, respectively. Then*

$$q\cdot v_{p.k.o.d.b.}(x;g) = (\mathcal{E}_q g)(x) - v_{p.f.t.d.b.}(x;G^0_\pm). \tag{A.39}$$

If the Lévy process X is tame, the assumption of Theorem A.14(c) is satisfied.

In fact, (A.39) follows easily from Dynkin's formula (B.1), while second statement of the theorem is obvious from the definition of $G^0_\pm$. The theorem leads to the following formula for $v_{p.k.o.d.b.}(x;g)$ as a sum of absolutely and uniformly convergent series (this formula is easily seen to be equivalent to (1.5)):

$$\begin{aligned} q\cdot v_{p.k.o.d.b.}(x;g) =&(\mathcal{E}_q g)(x) - G^1_+(x) - G^1_-(x) + G^2_+(x) + G^2_-(x) \\ &- G^3_+(x) - G^3_-(x) + G^4_+(x) + G^4_-(x) - \cdots, \end{aligned} \tag{A.40}$$

whose individual terms can be calculated using formulas (A.37)–(A.38) above.

Appendix B. Proofs of auxiliary results

B.1. **Proof of Proposition A.6.** It is clear that both sides of (A.10) are linear with respect to f, so it is enough to treat the case where f is measurable and nonnegative. We rewrite (A.9) as follows:

$$\begin{aligned} v_{p.u.o.}(x;f) &= \mathbb{E}\left[\int_0^\infty e^{-qt}\cdot\mathbb{1}_{\{t<\tau^+_{h-x}\}}\cdot f(x+X_t)\,dt\right] \\ &= \mathbb{E}\left[\int_0^\infty e^{-qt}\cdot\mathbb{1}_{\{x+\overline{X}_t<h\}}\cdot f(x+X_t)\,dt\right], \end{aligned}$$

where the second identity is immediate from the definition of τ^+_{h-x}. Using the fact that the probability distribution of T_q equals $qe^{-qt}\mathbb{1}_{[0,+\infty)}(t)\,dt$ and T_q is independent of the process X, we obtain

$$v_{p.u.o.}(x;f) = q^{-1}\cdot\mathbb{E}\big[\mathbb{1}_{(-\infty,h)}(x+\overline{X}_{T_q})\cdot f(x+X_{T_q})\big].$$

To make our notation more concise, let us write $Y = x+\overline{X}_{T_q}$ and $Z = X_{T_q}-\overline{X}_{T_q}$. It follows from Lemma A.5(a) that the random variables Y and Z are independent. By definition, this amounts to the statement that the probability distribution of the $\mathbb{R}^2$-valued random variable (Y,Z) is equal to the product (in the sense of "product measure") of the distribution of Y and the distribution of Z. By Fubini's theorem,

$$\begin{aligned} v_{p.u.o.}(x;f) &= q^{-1}\cdot\mathbb{E}_\omega\big[\mathbb{1}_{(-\infty,h)}(Y(\omega))\cdot f(Y(\omega)+Z(\omega))\big] \\ &= q^{-1}\cdot\mathbb{E}_{\omega_1}\Big[\mathbb{E}_{\omega_2}\big[\mathbb{1}_{(-\infty,h)}(Y(\omega_1))\cdot f(Y(\omega_1)+Z(\omega_2))\big]\Big], \end{aligned}$$

where $\mathbb{E}_\omega$ (respectively, $\mathbb{E}_{\omega_1}$ and $\mathbb{E}_{\omega_2}$) denote the expectation operators defined by integrating with respect to ω (respectively, ω_1 and ω_2). By Lemma A.5(b), the random variables Z and $\underline{X}_{T_q}$ have the same distribution, which implies that

$$v_{p.u.o.}(x;f) = q^{-1}\cdot\mathbb{E}_\omega\big[\mathbb{1}_{(-\infty,h)}(Y(\omega))\cdot(\mathcal{E}_q^- f)(Y(\omega))\big].$$

Finally, by the definition of $\mathcal{E}_q^+$, the last expression coincides with the right hand side of (A.10), which completes the proof of Proposition A.6.

B.2. **Dynkin's formula.** In what follows, we will need (a special case of) Dynkin's formula, which we state as (B.1) below. Let us first introduce the following notation. Let $X = \{X_t\}_{t\geq 0}$ be a 1-dimensional Lévy process, let B be a closed subset of the real line, and let τ_B be the first entrance time of the process X into B:

$$\tau_B(\omega) = \inf\big\{t\geq 0\,\big|\, X_t(\omega)\in B\big\} \qquad \forall\,\omega\in\Omega,$$

with the usual convention that $\inf\varnothing = +\infty$. Further, for any $x\in\mathbb{R}$, we let $B-x$ denote the translate of B consisting of all points of the form $y-x$, with $y\in B$. We refer to the following identity as Dynkin's formula: for every $q>0$,

$$(\mathcal{E}_q f)(x) = \mathbb{E}\Big[e^{-q\tau_{B-x}}(\mathcal{E}_q f)(x+X_{\tau_{B-x}})\Big] + \mathbb{E}\left[\int_0^{\tau_{B-x}} qe^{-qt}f(x+X_t)\,dt\right]. \tag{B.1}$$

It is a special case of formula (41.3) on page 283 of [45].

Remark B.1. It is shown in *loc. cit.* that (B.1) is valid for any nonnegative *universally measurable* function $f(x)$. The definition of universally measurable functions can be found on p. 274 of *op. cit.* It is clear from this definition that Borel measurable functions are universally measurable, so there is no reason to repeat the general definition here. Furthermore, the case where f is Borel measurable and bounded reduces to the case where f is nonnegative by linearity.

B.3. **Proof of Proposition A.10.** As a special case of (B.1), we obtain

$$(\mathcal{E}_q f)(x) = \mathbb{E}\Big[e^{-q\tau^+_{h-x}}(\mathcal{E}_q f)(x+X_{\tau^+_{h-x}})\Big] + \mathbb{E}\left[\int_0^{\tau^+_{h-x}} qe^{-qt}f(x+X_t)\,dt\right]. \tag{B.2}$$

In view of the assumption made in Proposition A.10, the first summand on the right hand side of (B.2) equals $v_{p.f.t.u.}(x;G)$. On the other hand, by Proposition A.6, the second term on the right hand side of (B.2) equals $\mathcal{E}_q^+\big(\mathbb{1}_{(-\infty,h)}(x)\cdot(\mathcal{E}_q^- f)(x)\big)$. Thus we can rewrite (B.2) as follows:

$$v_{p.f.t.u.}(x;G) = (\mathcal{E}_q f)(x) - \mathcal{E}_q^+\big(\mathbb{1}_{(-\infty,h)}(x)\cdot(\mathcal{E}_q^- f)(x)\big).$$

Since $\mathcal{E}_q f = \mathcal{E}_q^+\mathcal{E}_q^- f$ by (A.5), the last identity implies Proposition A.10.

B.4. **Infinitesimal generator of a Lévy process.** In this subsection we recall some results concerning the infinitesimal generator of a Lévy process and its domain that can be used to find convenient sufficient conditions for the hypotheses of Propositions A.10 and A.13 to be satisfied. As before, we let $X = \{X_t\}_{t\geq 0}$ be a 1-dimensional Lévy process, and we fix $q > 0$. Let $C_0(\mathbb{R})$ denote the space of all continuous functions $f : \mathbb{R} \longrightarrow \mathbb{C}$ that vanish at infinity (i.e., $\lim\limits_{x\to\pm\infty} f(x) = 0$). Every such function is, *a fortiori*, measurable and bounded, so, as we remarked in §A.3.3, the function $\mathcal{E}_q f$ is well defined. It is easy to show, using the dominated convergence theorem, that $\mathcal{E}_q f$ is continuous and vanishes at infinity as well. Thus $\mathcal{E}_q$ can be defined as a linear operator on the space $C_0(\mathbb{R})$. Further, as we also saw in §A.3.3, this operator is bounded with respect to the norm $||f||_\infty = \sup\big\{|f(x)| \,\big|\, x \in \mathbb{R}\big\}$.

The following results are contained in [45, §31].

Proposition B.2. (*a*) *For each $q > 0$, the operator $\mathcal{E}_q : C_0(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$ is one-to-one, i.e., if $f \in C_0(\mathbb{R})$ and $(\mathcal{E}_q f)(x) = 0$ for all x, then $f = 0$.*

(*b*) *The range of $\mathcal{E}_q$, i.e., the subspace $\mathcal{D} \subset C_0(\mathbb{R})$ consisting of functions of the form $\mathcal{E}_q f$ for $f \in C_0(\mathbb{R})$, does not depend on $q > 0$. It is dense in $C_0(\mathbb{R})$.*

(*c*) *There exists a unique linear map $L : \mathcal{D} \longrightarrow C_0(\mathbb{R})$ with $q^{-1}(q - L)\mathcal{E}_q f = f$ for all $f \in C_0(\mathbb{R})$ and all $q > 0$.*

(*d*) *We have $C_0^2(\mathbb{R}) \subset \mathcal{D}$, where $C_0^2(\mathbb{R}) \subset C_0(\mathbb{R})$ is defined as the subspace of functions $f(x)$ such that $f(x)$ is twice continuously differentiable and $f(x)$, $f'(x)$ and $f''(x)$ vanish at infinity.*

The linear map L is called the *infinitesimal generator* of the Lévy process X, while $\mathcal{D}$ is referred to as its *domain*. One usually views L as an unbounded linear operator on the Banach space $C^0(\mathbb{R})$.

Remark B.3. Let $h \in \mathbb{R}$, let $q > 0$, and let $G \in C_0^2([h, +\infty))$, i.e., $G(x)$ is a function that is twice continuously differentiable on $[h, +\infty)$, and

$$\lim_{x\to+\infty} G(x) = \lim_{x\to+\infty} G'(x) = \lim_{x\to+\infty} G''(x) = 0.$$

By a standard argument, there exists a (non-unique) function $\widetilde{G} \in C_0^2(\mathbb{R})$ such that $\widetilde{G}(x) = G(x)$ for all $x \geq h$. By Proposition B.2(d), the function $f = q^{-1}(q - L)\widetilde{G}$ is well defined and lies in $C_0(\mathbb{R})$. By parts (a) and (c) of Proposition B.2, we have $\mathcal{E}_q f = \widetilde{G}$. *A fortiori*, this implies that $(\mathcal{E}_q f)(x) = G(x)$ for all $x \geq h$. This justifies the claim made in Example A.12(2).

B.5. **Kernels of the normalized EPV operators.** Let $X = \{X_t\}_{t\geq 0}$ be a 1-dimensional Lévy process, and let $q > 0$ be fixed. The supremum and infimum processes, $\overline{X}$ and $\underline{X}$, were defined in §A.3.1, while the normalized EPV operators $\mathcal{E}_q^\pm$ were defined in §A.3.2. Let $T_q \sim \operatorname{Exp} q$ denote an exponentially distributed random variable with mean q^{-1} that is independent of the process X, and let $p_q^{\pm}(dx)$ denote the probability distributions of the random variables $\overline{X}_{T_q}$ and $\underline{X}_{T_q}$, respectively.

It is clear from the definitions of $\overline{X}$ and $\underline{X}$ that $p_q^+(dx)$ is supported on $[0,+\infty)$, and $p_q^-(dx)$ is supported on $(-\infty, 0]$. Moreover, by definition, $\mathcal{E}_q^{\pm}$ can be viewed as the operator of convolution with the probability measure $p_q^{\pm}(dx)$ – see (1.13).

Proposition B.4. *Assume that the Lévy process X is tame. Then the restriction of $p_q^+(dx)$ to $(0,+\infty)$ (respectively, the restriction of $p_q^-(dx)$ to $(-\infty,0)$) has the form $g_q^{\pm}(x)\,dx$, where $g_q^{\pm}(x)$ is an infinitely differentiable function that decays exponentially as $x \to \pm\infty$ along with all of its derivatives.*

Note that we do not make any statement about $p_q^{\pm}(\{0\})$. In fact, it could happen that $p_q^+(\{0\}) \neq 0$ or $p_q^-(\{0\}) \neq 0$. The functions $g_q^{\pm}(x)$ will be referred to, somewhat imprecisely, as the *kernels of the normalized EPV operators $\mathcal{E}_q^{\pm}$*.

Proof of Proposition B.4. By symmetry, it suffices to consider $p_q^+(dx)$. Let us use the normalization of the Fourier transform commonly used in probability theory. With this convention, the Fourier transform of the probability measure $p_q^+(dx)$ equals

$$\widehat{p_q^+}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} p_q^+(dx) = \phi_q^+(\xi),$$

where we simply used the definition (1.14) of the Wiener-Hopf factors $\phi_q^{\pm}(\xi)$.

Let us define, for all $x > 0$,

$$g_q^+(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\phi_q^+(\xi)\,d\xi \tag{B.3}$$

as a generalized function. The RHS is an oscillating integral, which may not converge absolutely. However, using the third property (iii) of the tame processes and integrating by parts, it is straightforward to show that the function $g_q^+(x)$ is infinitely differentiable, and that it decays exponentially at $+\infty$ together with all of its derivatives. Once this is established, it will follow from (B.3) and the Fourier inversion formula that the restriction of $p_q^+(dx)$ to $(0,+\infty)$ coincides with $g_q^+(x)\,dx$.

□

Corollary B.5. *Let $X = \{X_t\}_{t\geq 0}$ be a tame Lévy process, let $q > 0$, and let $\mathcal{E}_q^{\pm}$ be the normalized EPV operators, defined by (A.4). Further, let $f(x)$ be a bounded measurable function on $\mathbb{R}$ that is supported on $[0,+\infty)$ (respectively, on $(-\infty,0]$). Then the function $\mathcal{E}_q^+ f$ (respectively, $\mathcal{E}_q^- f$) is infinitely differentiable on $(-\infty,0)$ (respectively, on $(0,+\infty)$), and this function, together with all of its derivatives, decays exponentially as $x \to -\infty$ (respectively, $x \to +\infty$).*

Proof. By symmetry, it suffices to consider the case where f is supported on $[0,+\infty)$. Fix any $h>0$. By Proposition B.4, for all $x\le -h$, we have

$$(\mathcal{E}_q^+ f)(x)=\int_h^{+\infty} f(x+y)g_q^+(y)\,dy=\int_{h+x}^{+\infty} f(y)g_q^+(y-x)\,dy=\int_0^{+\infty} f(y)g_q^+(y-x)\,dy,$$

where we used the assumption that $f(y)=0$ for all $y<0$. By Proposition B.4, the function $g_q^+(-x)$ is infinitely differentiable and decays exponentially, together with all of its derivatives, as $x\to-\infty$. Hence, in the formula for $(\mathcal{E}_q^+ f)(x)$ given above, differentiation under the integral sign is permissible (because f is bounded), and the corollary follows easily. □

B.6. **Proof of Theorem A.14.** The assertion of part (a) of the theorem follows from the argument given in §1.2. There is no need to reproduce it here.

Next, we prove part (b) of the theorem. By symmetry, it suffices to prove that there exists $0<\epsilon<1$, depending only on X, q and $H_\pm$, such that if $C>0$ and $|G_+^0(x)|\le C$ for all $x\ge h_+$, then $|G_-^1(x)|\le \epsilon C$ for all $x\le h_-$. Inspecting the definition of G_-^1, we see that it suffices to consider the case where $C=1$ and $G_+^0(x)=1$ for all $x\ge h_+$. In this case, it follows from Example A.12(1) that $G_-^1(x)=\mathcal{E}_q^+\mathbb{1}_{[h_+,+\infty)}(x)$ for all x.

Thus the function $G_-^1(x)$ is nonnegative and nondecreasing, so for all $x\le h_-$, we have $0\le G_-^1(x)\le G_-^1(h_-)$. Letting $\epsilon=\mathcal{E}_q^+\mathbb{1}_{[h_+,+\infty)}(h_-)$, it suffices to show that $\epsilon<1$, since ϵ visibly depends only on X, q and $h_\pm$. By definition, it is clear that

$$\epsilon=\mathbb{P}[\overline{X}_{T_q}\ge h_+-h_-]=\int_0^\infty qe^{-qt}\mathbb{P}[\overline{X}_t\ge h_+-h_-]\,dt.$$

Hence $\epsilon\le 1$, and if $\epsilon=1$, then $\mathbb{P}[\overline{X}_t\ge h_+-h_-]=1$ for almost all $t>0$. On the other hand, using the definition of the process $\overline{X}$, we see that almost every trajectory of $\overline{X}$ is right continuous. The argument given in Remark A.1 implies that the process $\overline{X}$ is stochastically continuous at $t=0$, which is a contradiction, due to the fact that $h_+-h_->0$. This proves Theorem A.14(b).

Finally, we prove part (c) of the theorem. By symmetry, it suffices to show that if there exists a bounded measurable function $f(x)$ on $\mathbb{R}$ such that $G_+^0(x)=(\mathcal{E}_q f)(x)$ for all $x\ge h_+$, then there exists a bounded measurable function $f_1(x)$ on $\mathbb{R}$ such that $G_-^1(x)=(\mathcal{E}_q f_1)(x)$ for all $x\le h_-$. By Proposition A.10, we have

$$G_-^1(x)=\mathcal{E}_q^+\big(\mathbb{1}_{[h_+,+\infty)}(x)\cdot(\mathcal{E}_q^- f)(x)\big).$$

By Corollary B.5, the restriction of the function $G_-^1(x)$ to $(-\infty,h_+)$ is infinitely differentiable, and, along with all of its derivatives, decays exponentially as $x\to-\infty$. *A fortiori*, we have $G_-^1\in C_0^2((-\infty,h_-])$. Using an obvious analogue of Example A.12(2), the proof is complete.

B.7. **Proof of** (1.9)**.** Put $a_\pm = \left(\mu \pm \sqrt{\mu^2 + 2q\sigma^2}\right)/\sigma^2$. By definition, the functions $G^n_\pm(x)$ on the right hand side of (1.4) are given by $G^n_+(x) = V^{BS}_{down-and-in}(x; G^{n-1}_-)$ and $G^n_-(x) = V^{BS}_{up-and-in}(x; G^{n-1}_+)$ for all $n \geq 1$. Using formulas (1.7) and (1.8), we obtain

$$G^n_+(x) = G^{n-1}_-(h_-) \cdot \exp(a_+(h_- - x)) \qquad \forall\, n \geq 1,\ x > h_- \tag{B.4}$$

and

$$G^n_-(x) = G^{n-1}_+(h_+) \cdot \exp(a_-(h_+ - x)) \qquad \forall\, n \geq 1,\ x < h_+. \tag{B.5}$$

In particular, we see that:

$$G^n_+(h_+) = G^{n-1}_-(h_-) \cdot \exp(-a_+(h_+ - h_-)) \qquad \forall\, n \geq 1$$

and

$$G^n_-(h_-) = G^{n-1}_+(h_+) \cdot \exp(a_-(h_+ - h_-)) \qquad \forall\, n \geq 1.$$

Let us define

$$\delta = \exp(-a_+(h_+ - h_-)) \cdot \exp(a_-(h_+ - h_-)) = \exp((a_- - a_+)(h_+ - h_-)).$$

It follows that

$$G^1_+(h_+) = G^0_-(h_-) \cdot \exp(-a_+(h_+ - h_-)), \qquad G^n_+(h_+) = \delta \cdot G^{n-2}_+(h_+) \quad \forall\, n \geq 2 \tag{B.6}$$

and

$$G^1_-(h_-) = G^0_+(h_+) \cdot \exp(a_-(h_+ - h_-)), \qquad G^n_-(h_-) = \delta \cdot G^{n-2}_-(h_-) \quad \forall\, n \geq 2. \tag{B.7}$$

Now let us substitute (B.4)–(B.5) into (1.4). We obtain, for all $h_- < x < h_+$,

$$\begin{aligned} V^{BS}_{f.t.}(x) = {} & \left(G^0_-(h_-) - G^1_-(h_-) + G^2_-(h_-) - G^3_-(h_-) + \cdots\right) \cdot \exp(-a_+(x - h_-)) \\ & + \left(G^0_+(h_+) - G^1_+(h_+) + G^2_+(h_+) - G^3_+(h_+) + \cdots\right) \cdot \exp(a_-(h_+ - x)) \end{aligned} \tag{B.8}$$

Since $0 < \delta < 1$, the two infinite series appearing in (B.8) converge (absolutely) and can be summed explicitly. Using identities (B.6)–(B.7), we obtain

$$G^0_\pm(h_\pm) + G^2_\pm(h_\pm) + \cdots = \frac{G^0_\pm(h_\pm)}{1-\delta}, \qquad G^1_\pm(h_\pm) + G^3_\pm(h_\pm) + \cdots = \frac{G^0_\mp(h_\mp) \cdot \exp(\mp a_\pm(h_+ - h_-))}{1-\delta}.$$

Substituting the latter identities into (B.8) yields (1.9).

Appendix C. Computational techniques

In this appendix we recall the standard definitions of fast Fourier transforms, and present several elementary FFT-based algorithms that were used as the building blocks of the option pricing method developed in §3. Throughout the appendix, we use the normalization of the ordinary Fourier transform of a function $f(x) \in L_1(\mathbb{R})$ that is common in analysis (for the sake of notational compatibility with [5, 6]):

$$(\mathcal{F}_{x\to\xi} f)(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\, dx. \tag{C.1}$$

With this convention, the inverse Fourier transform, $\mathcal{F}^{-1}_{\xi\to x}$, is given by

$$(\mathcal{F}^{-1}_{\xi\to x} g)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} g(\xi)\, d\xi. \tag{C.2}$$

C.1. **Fast Fourier transforms.** We begin by recalling the most common approach to the numerical realization of Fourier and inverse Fourier transforms. Let us consider two uniformly spaced grids, $\vec{x} = (x_j)_{j=1}^M$ and $\vec{\xi} = (\xi_k)_{k=1}^M$, so that

$$x_j = x_1 + (j-1)\Delta, \quad 1 \le j \le M \qquad \text{and} \qquad \xi_k = \xi_1 + (k-1)\zeta, \quad 1 \le k \le M,$$

where $\Delta, \zeta > 0$ are fixed. Using a simplified version of the trapezoid rule, we replace $(\mathcal{F}_{x\to\xi} f)(\xi)$ and $(\mathcal{F}^{-1}_{\xi\to x} g)(x)$ (see (C.1), (C.2)) with the following two functions:

$$(\mathcal{F}_{fast} f)(\xi) = \Delta \cdot \sum_{j=1}^M f(x_j) e^{-i\xi x_j} \tag{C.3}$$

and

$$(\mathcal{F}^{-1}_{fast} g)(x) = \frac{\zeta}{2\pi} \cdot \sum_{k=1}^M g(\xi_k) e^{i\xi_k x}. \tag{C.4}$$

We refer to $\mathcal{F}_{fast}$ and $\mathcal{F}^{-1}_{fast}$ as the fast Fourier transform (FFT) and the inverse fast Fourier transform (iFFT), respectively. In §C.3 we recall how the values of $\mathcal{F}_{fast} f$ (respectively, $\mathcal{F}^{-1}_{fast} g$) on the grid $\vec{\xi}$ (respectively, $\vec{x}$) can be calculated efficiently under the assumption that $M\Delta\zeta = 2\pi$.

C.2. **Discrete Fourier transforms.** If $\vec{f} = (f_j)_{j=1}^M$ is an array of complex numbers, the *discrete* Fourier transform of $\vec{f}$ is another array of complex numbers, $\texttt{fft}(\vec{f})$, of the same length, whose entries are given by the formula

$$\texttt{fft}(\vec{f})_k = \sum_{j=1}^M f_j \cdot e^{-2\pi i (j-1)(k-1)/M}, \qquad 1 \le k \le M. \tag{C.5}$$

Similarly, the *inverse discrete* Fourier transform of an array $\vec{g} = (g_k)_{k=1}^M$ is given by

$$\texttt{ifft}(\vec{g})_j = \frac{1}{M} \sum_{k=1}^M g_k \cdot e^{2\pi i (j-1)(k-1)/M}, \qquad 1 \le j \le M. \tag{C.6}$$

The reason we use the notation `fft` and `ifft` for the discrete Fourier transforms is that most of the standard numerical algorithms for computing fast Fourier and inverse Fourier transforms are designed for the calculation of (C.5) and (C.6).

C.3. **Numerical realization of FFT and iFFT.** We assume that two uniformly spaced grids, $\vec{x} = (x_j)_{j=1}^M$ and $\vec{\xi} = (\xi_k)_{k=1}^M$, are given, so that

$$x_j = x_1 + (j-1)\Delta, \quad 1 \le j \le M \qquad \text{and} \qquad \xi_k = \xi_1 + (k-1)\zeta, \quad 1 \le k \le M,$$

where $\Delta, \zeta > 0$ are fixed. We also assume that $M\Delta\zeta = 2\pi$.

If $f(x)$ is a function whose domain contains the points x_j, and $(\mathcal{F}_{fast}f)(\xi)$ is defined by (C.3), the array of values $\big((\mathcal{F}_{fast}f)(\xi_k)\big)_{k=1}^M$ can be computed as follows.

- Calculate the array $\vec{f} = (f_j)_{j=1}^M$ whose entries are given by

$$f_j = f(x_j) \cdot \exp(-i \cdot \Delta \cdot \xi_1 \cdot (j-1)), \qquad 1 \le j \le M.$$

- The desired values of $\mathcal{F}_{fast}f$ are given by

$$(\mathcal{F}_{fast}f)(\xi_k) = \Delta \cdot \exp(-i \cdot x_1 \cdot \xi_k) \cdot \texttt{fft}(\vec{f})_k, \qquad 1 \le k \le M.$$

If $g(\xi)$ is a function whose domain contains the points ξ_k, and $(\mathcal{F}^{-1}_{fast}g)(x)$ is defined by (C.4), the array of values $\big((\mathcal{F}^{-1}_{fast}g)(x_j)\big)_{j=1}^M$ can be computed as follows.

- Calculate the array $\vec{g} = (g_k)_{k=1}^M$ whose entries are given by

$$g_k = g(\xi_k) \cdot \exp(i \cdot \zeta \cdot x_1 \cdot (k-1)), \qquad 1 \le k \le M.$$

- The desired values of $\mathcal{F}^{-1}_{fast}g$ are given by

$$(\mathcal{F}^{-1}_{fast}g)(x_j) = (1/\Delta) \cdot \exp(i \cdot \xi_1 \cdot x_j) \cdot \texttt{ifft}(\vec{g})_j, \qquad 1 \le j \le M.$$

C.4. **Numerical realization of refined iFFT.** As in §C.3, we consider a uniformly spaced grid $\vec{x} = (x_j)_{j=1}^M$ of points in $\mathbb{R}$, where $x_j = x_1 + (j-1)\Delta$, and both M and $\Delta > 0$ are fixed. We set $\zeta = 2\pi/(M\Delta)$. Our goal is to recall, following [5, 6], the numerical realization of inverse FFT in the case where the dual grid $\vec{\xi}$ is obtained by refining and stretching the dual grid that was used in §C.3 above.

First, let us briefly review the construction of the refined dual grid. One should choose two positive integers, M_2 and M_3, that are responsible, respectively, for refining and stretching the dual grid. One should also choose $\xi_1 \in \mathbb{C}$, the desired initial point of the ξ-grid. The parameters M_2, M_3 and ξ_1 can be varied independently of M and Δ. The total number of points in the ξ-grid equals $M_1 = MM_2M_3$, and the mesh of the ξ-grid equals $\zeta_1 = \zeta/M_2$. Explicitly, the ξ-grid is given by

$$\vec{\xi} = (\xi_k)_{k=1}^{M_1}, \qquad \xi_k = \xi_1 + (k-1)\zeta_1 = \xi_1 + (k-1) \cdot \frac{\zeta}{M_2}.$$

Let $g(\xi)$ be a function whose domain contains the grid $\vec{\xi}$. We would like to calculate the values of the function $\mathcal{F}^{-1}_{fast}g$ (defined by (C.4), with M_1 in place of M) on the grid $\vec{x}$. To this end, we let $g_{j,\ell}$ be the restriction of g to the sub-grid

$$\vec{\xi}(j,\ell) = \big(\xi_{M_2\cdot(k-1+(j-1)M)+\ell}\big)_{k=1}^M,$$

for each $1 \le j \le M_3$ and each $1 \le \ell \le M_2$. Since $\vec{\xi}(j,\ell)$ has mesh ζ, the values of $\mathcal{F}^{-1}_{fast}g_{j,\ell}$ on the grid $\vec{x}$ can be calculated using the algorithm of §C.3, and it follows immediately from the definitions that

$$\mathcal{F}^{-1}_{fast} g = \frac{1}{M_2} \sum_{j=1}^{M_3} \sum_{\ell=1}^{M_2} \mathcal{F}^{-1}_{fast}(g_{j,\ell}),$$

which allows us to calculate the values of $\mathcal{F}^{-1}_{fast} g$ on the grid $\vec{x}$.

C.5. **Discrete convolution.** The enhanced realization of EPV operators (and their inverses) described in §§3.4–3.5, and the algorithm for the approximate calculation of the Wiener-Hopf factors presented in §3.3, involve calculations of sums of the form

$$h_k = \sum_{j=1}^{M} f_j g_{k-j}, \tag{C.7}$$

where $\vec{f} = (f_j)_{j=1}^M$ and $\vec{g} = (g_\ell)_{\ell=1-M}^{M-1}$ are arrays of complex numbers of lengths M and $2M-1$, respectively. In this subsection we explain how the calculation of the sums (C.7) can be reduced to three applications of FFT to arrays of length $2M$.

- Let $\widetilde{f}$ be the array of length $2M$ with entries
$$\widetilde{f}_j = \begin{cases} f_j, & 1 \le j \le M; \\ 0, & M+1 \le j \le 2M. \end{cases}$$
- Let $\widetilde{g}$ be the array of length $2M$ with entries
$$g_0, g_1, \ldots, g_{M-1}, 0, g_{1-M}, g_{2-M}, \ldots, g_{-1}$$
- Calculate the array $\widetilde{h} = (\widetilde{h}_\ell)_{\ell=1}^{2M}$ with entries
$$\widetilde{h}_\ell = \mathtt{fft}(\widetilde{f})_\ell \cdot \mathtt{fft}(\widetilde{g})_\ell.$$
- The sums (C.7) are given by
$$h_k = \mathtt{ifft}(\widetilde{h})_k, \qquad 1 \le k \le M.$$

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TABLE 1. Prices of a knock-out double barrier put option and of a double-no-touch option in the KoBoL model

Spot	Knock-out double barrier put			Double-no-touch option		
price	Our price	MC price	MC error	Our price	MC price	MC error
81%	222.5256	222.5448	0.0001	0.4194	0.4192	-0.0006
82%	302.2846	301.8782	-0.0013	0.5778	0.5768	-0.0017
83%	344.5279	344.6685	0.0004	0.6799	0.6799	0.0000
84%	364.5539	364.1193	-0.0012	0.7504	0.7500	-0.0004
85%	370.3752	370.9815	0.0016	0.8009	0.8004	-0.0007
86%	366.6665	367.2755	0.0017	0.8383	0.8394	0.0013
87%	356.3501	356.6483	0.0008	0.8665	0.8670	0.0006
88%	341.3540	341.7309	0.0011	0.8881	0.8887	0.0006
89%	323.0086	323.8835	0.0027	0.9048	0.9046	-0.0003
90%	302.2708	303.3371	0.0035	0.9179	0.9179	0.0001
91%	279.8571	280.7147	0.0031	0.9280	0.9288	0.0008
92%	256.3280	257.1714	0.0033	0.9358	0.9355	-0.0004
93%	232.1444	232.8017	0.0028	0.9419	0.9415	-0.0004
94%	207.7075	208.4755	0.0037	0.9464	0.9463	0.0000
95%	183.3905	183.8869	0.0027	0.9496	0.9501	0.0006
96%	159.5662	160.1593	0.0037	0.9517	0.9522	0.0006
97%	136.6340	137.2298	0.0044	0.9527	0.9531	0.0003
98%	115.0501	115.5238	0.0041	0.9529	0.9534	0.0005
99%	95.3578	95.7482	0.0041	0.9522	0.9524	0.0003
100%	78.1900	78.6724	0.0062	0.9506	0.9508	0.0002
101%	64.0702	64.5565	0.0076	0.9481	0.9487	0.0006
102%	52.8802	53.2834	0.0076	0.9448	0.9451	0.0003
103%	44.0286	44.2211	0.0044	0.9404	0.9413	0.0009
104%	36.9574	37.1324	0.0047	0.9351	0.9357	0.0007
105%	31.2445	31.3603	0.0037	0.9285	0.9294	0.0010
106%	26.5798	26.9603	0.0143	0.9206	0.9207	0.0001
107%	22.7348	22.7653	0.0013	0.9112	0.9115	0.0003
108%	19.5381	19.5361	-0.0001	0.8999	0.9005	0.0006
109%	16.8595	17.0190	0.0095	0.8866	0.8865	-0.0001
110%	14.5985	14.5777	-0.0014	0.8708	0.8703	-0.0005
111%	12.6764	12.6848	0.0007	0.8519	0.8515	-0.0004
112%	11.0305	11.1604	0.0118	0.8293	0.8295	0.0003
113%	9.6097	9.5769	-0.0034	0.8020	0.8026	0.0008
114%	8.3718	8.2587	-0.0135	0.7688	0.7696	0.0010
115%	7.2798	7.1954	-0.0116	0.7280	0.7285	0.0006
116%	6.2990	6.2712	-0.0044	0.6771	0.6773	0.0002
117%	5.3919	5.2579	-0.0249	0.6121	0.6134	0.0021
118%	4.5056	4.5382	0.0072	0.5264	0.5270	0.0013
119%	3.5222	3.5406	0.0052	0.4049	0.4063	0.0036

The first column contains the spot price as a percentage of 3500. The errors reported in columns 4 and 7 are the relative errors. If V_{MC} denotes the Monte-Carlo price of an option, and V denotes the price obtained using our algorithm, the relative error is defined as $(V_{MC} - V)/V$.

KoBoL parameters: $\nu = 0.5$, $c = 1$, $\lambda_+ = 9$, $\lambda_- = -8$, $\mu \approx -0.0423$.
Option parameters: $K = 3500$ (for the double barrier put), $H_- = 2800$, $H_+ = 4200$, $r = 0.03$, $T = 0.1$.
Algorithm parameters: $n = 812$ (number of points on the "main" x-grid), $\Delta = \frac{\ln H_+ - \ln H_-}{n-1} \approx 0.005$, $M = 4096$, $M_2 = 4$, $M_3 = 16$, $\zeta_1 \approx 0.767$, $m = 8$ (for the calculation of the Wiener-Hopf factors), $N = 80$ (number of time steps), $\epsilon = 10^{-7}$ (error tolerance for the iterative procedure).