Fast and accurate pricing of barrier options under Lévy processes

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Historical background

The problem of pricing barrier options under Lévy processes has attracted much attention in the recent years.

Popular R.Cont and E.Volchkova (2005) method looks attractive due to its simplicity.

However, the method uses models with a tangible diffusion component to approximate models with zero diffusion component.

The main goals of the paper

Fast Wiener-Hopf factorization method (FWH-method)

Our goals:

- to suggest universal, simple, fast and efficient method for pricing barrier options under a wide class of Lévy processes
- to demonstrate the accuracy and fast convergence of the method comparing with the results obtained by other methods
- to show that R.Cont and E.Volchkova (2005) method leads to sizable errors, especially near the barrier

Publications

The main results can be found in
KUDRYAVTSEV, O. AND LEVENDORSKIĬ, S.,(November 30, 2007) "Fast and Accurate Pricing of Barrier Options Under Levy Processes" submitted to Finance and Stochastic. Preprint: available at SSRN: http://ssrn.com/abstract=1040061

FWH-method: theoretical background

Carr's randomization reduces the pricing problem to a sequence of stationary problems.

Each problem in the sequence can be solved explicitly using Wiener-Hopf factorization method.

Explicit formulas in terms of EPV-operators are available.

However, no efficient numerical realization of the action of EPV–operators was suggested.

FWH-method: main ideas

FWH-method

Fast Wiener-Hopf factorization method (FWH-method) is based on an efficient approximation of the Wiener-Hopf factors in the exact formula for the solution and real FFT and iFFT.

Key ideas

- symbol of EPV-operator in the exact formula for the solution – characteristic function of i.d.d.
- factorization of the characteristic function into the product of the leading function and the subordinate function
- the leading function is the characteristic function of VG distribution without drift
- the subordinate function can be factorized by using Poisson type approximation
- real FFT and iFFT can be applied

The closely related method

In Boyarchenko, M. and Levendorskiĭ, S. Z. (2008a,b)

the EPV operators are realized as convolution operators, instead of using the realization based on Fourier transforms employed in our paper.

the enhanced FFT technique is applied, instead of using the ordinary FFT technique.

approximate formulas of the factors are obtained by using the trapezoidal rule to discretize the integral formulas, instead of our approach.

Publications

Boyarchenko, M. and Levendorskiǐ, S. Z., "Refined and Enhanced Fast Fourier Transform Techniques, with an Application to the Pricing of Barrier Options" (2008a). Available at SSRN

Boyarchenko, M. and Levendorskiǐ, S. Z., "Prices and sensitivities of barrier and first-touch digital options in Lévy driven models" (2008b). Available at SSRN

The generalized Black-Scholes equation for the down-and-out put option

S.I.Boyarchenko and S.Z.Levendorskiĭ (2000, 2002): the option price is the solution of the boundary problem

$$
\begin{aligned}
(\partial_t + L - r)V(t,x) &= 0, \quad t < T, x > h, && (1)\\
V(T,x) &= (K - e^x)_+, && (2)\\
V(t,x) &= 0, \quad t < T,\ x \leqslant h. && (3)
\end{aligned}
$$

$X_t = \log S_t$ follows a Lévy process,
L – the infinitesimal generator,
K – strike,
T – maturity,
$H < K$ – barrier,
$V(t, X_t)$ – option value at time $t < T$ and spot price e^{X_t}.

Analytic method of lines or Carr's randomization

Carr's randomization

Carr's randomization reduces the pricing problem to a sequence of stationary problems.

$$\begin{aligned} q^{-1}(q-L)V^s(x) &= \frac{1}{q\Delta t}V^{s+1}(x), \quad x > h, \\ V^s(x) &= 0, \quad x \leqslant h. \end{aligned}$$

$[0, T]$ – time interval,
N - number of time steps,
$\Delta t = T/N$ – time step,
$q = r + 1/\Delta t$,
$V^N(x) = (K - e^x)_+$,
V^s – approximation to the price of the barrier option at time $t_s = s\Delta t$, $s = N-1, N-2, \ldots$.

The Wiener-Hopf factorization formula and EPV-operators

X_t: a Lévy process with characteristic exponent $\psi(\xi)$
$\bar{X}_t = \sup_{0\leqslant s\leqslant t} X_s$, $\underline{X}_t = \inf_{0\leqslant s\leqslant t} X_s$: supremum and infimum processes

$$\begin{aligned} \phi_q^+(\xi) &= qE\left[\int_0^\infty e^{-qt} e^{i\xi \bar{X}_t} dt\right], \\ \phi_q^-(\xi) &= qE\left[\int_0^\infty e^{-qt} e^{i\xi \underline{X}_t} dt\right]. \end{aligned}$$

$\mathcal{E} := q/(q-L) = q(q+\psi(D))^{-1}$ and $\mathcal{E}^\pm = \phi_q^\pm(D)$ - PDO with the symbols $q(q+\psi(\xi))^{-1}$ and $\phi^\pm(\xi)$.
The operator form of WHF: $\mathcal{E} = \mathcal{E}^+\mathcal{E}^- = \mathcal{E}^-\mathcal{E}^+$.

Wiener-Hopf factorization method

S.I.Boyarchenko and S.Z.Levendorskiĭ (2002):

$$V^s = \frac{1}{q\Delta t}\mathcal{E}^- \mathbf{1}_{[h,+\infty)}\mathcal{E}^+ V^{s+1}, \quad (4)$$

$\mathbf{1}_{[h,+\infty)}$ is the indicator function of $[h,+\infty)$,
$\mathcal{E}^+$ and $\mathcal{E}^-$ are EPV-operators under supremum and infimum processes.

However, no efficient numerical realization of the action of EPV–operators was suggested.

Finite difference schemes

Construction of any finite difference scheme involves:

- discretization in space and time
- truncation of large jumps
- approximation of small jumps

The result is a linear system that needs to be solved at each time step.

Implementation of finite difference schemes to option pricing

2005 CONT, R., AND E.VOLTCHKOVA:

- small jumps are approximated by additional diffusion component;
- the integral term is treated explicitly;
- the differential term is treated implicitly;
- the result is the explicit-implicit scheme, with tridiagonal system.

2005 LEVENDORSKIĬ, S., KUDRYAVTSEV, O., AND V.ZHERDER:

- small jumps are approximated by additional drift;
- the integral and the differential terms are treated implicitly;
- the result is the implicit scheme, with dense matrix;
- the iteration method is used.

Iterative Wiener-Hopf factorization method (IWH-method)

IWH-method use a decomposition of the operator $q^{-1}(q - L)$ into the leading part and the subordinate part.

The leading part is a simple operator that admits an analytical factorization.

An appropriate iterative scheme is designed.

Note the method does not entail the errors typical for CV-method.

Fast Wiener-Hopf factorization method (FWH–method)

The main contribution of our method is an efficient numerical realization of EPV-operators.

For $s = N-1, N-2, \dots, 0$, define

$$W^s = \mathbf{1}_{[h;+\infty)}\mathcal{E}^+ V^{s+1}.$$

Then

$$V^s = (q\Delta t)^{-1}\mathcal{E}^- W^s(x).$$

$$W^s = (q\Delta t)^{-1}\mathbf{1}_{[h;+\infty)}\mathcal{E} W^{s+1}.$$

Construction of approximation of factors in the Wiener-Hopf factorization formula

Setup

ν – order of Lévy process
$\psi(\xi)$ – characteristic exponent

$$\Lambda_-(\xi) = \lambda_+^{\nu/2}(\lambda_+ + i\xi)^{-\nu/2}; \quad (5)$$

$$\Lambda_+(\xi) = (-\lambda_-)^{\nu/2}(-\lambda_- - i\xi)^{-\nu/2}; \quad (6)$$

$$\Phi(\xi) = q\Big((q+\phi(\xi))\Lambda_+(\xi)\Lambda_-(\xi)\Big)^{-1}. \quad (7)$$

$$\Phi(\xi) = \Phi^+(\xi)\Phi^-(\xi). \quad (8)$$

Explicit formulas for approximations of $\phi^{\pm}$

For small positive d and large $M(= 2^n)$, set

$$
\begin{aligned}
b_k^d &= \frac{d}{2\pi}\int_{-\pi/d}^{\pi/d} \ln\Phi(\xi)e^{-i\xi kd}d\xi, \quad k \neq 0, && (9)\\
b_{d,M}^{+}(\xi) &= \sum_{k=1}^{M/2} b_k^d(\exp(i\xi kd) - 1), && (10)\\
b_{d,M}^{-}(\xi) &= \sum_{k=-M/2+1}^{-1} b_k^d(\exp(i\xi kd) - 1); && (11)\\
\Phi^{\pm}(\xi) &\approx \exp(b_{d,M}^{\pm}(\xi)), && (12)\\
\phi_q^{\pm}(\xi) &= \Lambda_{\pm}(\xi)\Phi^{\pm}(\xi). && (13)
\end{aligned}
$$

Summary: FWH–method

FWH-method is faster than IWH–method and FDS-method because no iteration is involved

FWH-method is more accurate than CV–method because the approximation does not imply that the option value is of class C^2

approximate formulas in FWH-method for $\phi_q^{\pm}$ are needed at the first and last steps

At all intermediate steps, the exact analytic expression $q/(q+\psi(\xi))$ is used

The results obtained by FWH, IWH and FDS methods and FWH-method in Boyarchenko, M. and Levendorskiĭ, S. Z. (2008), are in extremely good agreement.

Numerical examples

We use FDS, FWH and IWH-methods to demonstrate that R.Cont and E.Volchkova (2005) method leads to sizable relative errors, especially near the barrier and strike.

We consider the down-and-out put option with strike K, barrier H and time to expiry T.

PC characteristics:
Intel Core(TM)2 Due CPU, 1.8GHz, RAM 1024Mb, under Windows Vista.

KoBoL as example

"KoBoL processes" were defined in S.I.Boyarchenko and S.Z.Levendorskiĭ (2000).

Later the same family of Lévy processes was used in P. Carr, H. Geman, D.B. Madan and M. Yor, (2002), under the name "CGMY-model".

KoBoL

The characteristic exponent of a pure jump KoBoL process of order $\nu \in (0,2), \nu \neq 1$ is given by

$$\psi(\xi) = -i\mu\xi + c\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu + (-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu],$$

where $c > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

KoBoL as example

Down-and-out put prices in KoBoL model, $\nu = 0.5$

	FDS	FWH		CV	
Spot price		$d = 0.00025$	$d = 0.0001$	$d = 5 \cdot 10^{-6}$	$d = 2 \cdot 10^{-6}$
$S = 91$	0.235866	0.233993	0.235040	0.218617	0.223599
$S = 101$	0.566907	0.565697	0.566630	0.552174	0.556747
$S = 111$	0.384982	0.384665	0.385188	0.377460	0.379963
$S = 121$	0.208093	0.208003	0.208265	0.204459	0.205700
$S = 131$	0.107307	0.107262	0.107398	0.105499	0.106115
CPU-time(sec)	97,300	33	77	26,000	116,000

KoBoL parameters: $\nu = 0.5$, $\lambda_+ = 9$, $\lambda_- = -8$, $c = 1$.
Option parameters: $K = 100$, $H = 90$, $r = 0.072310$, $T = 0.5$.
Algorithm parameters: d – space step, $N = 1600$ – number of time steps, S – spot price.

KoBoL as example

Relative errors

	FDS	FWH		CV	
Spot price		$d = 0.00025$	$d = 0.0001$	$d = 5 \cdot 10^{-6}$	$d = 2 \cdot 10^{-6}$
$S = 91$	0.235866	-0.79%	-0.35%	-7.31%	-5.20%
$S = 101$	0.566907	-0.21%	-0.05%	-2.60%	-1.79%
$S = 111$	0.384982	-0.08%	0.05%	-1.95%	-1.30%
$S = 121$	0.208093	-0.04%	0.08%	-1.75%	-1.15%
$S = 131$	0.107307	-0.04%	0.08%	-1.68%	-1.11%
CPU-time(sec)	97,300	33	77	26,000	116,000

KoBoL parameters: $\nu = 0.5$, $\lambda_+ = 9$, $\lambda_- = -8$, $c = 1$.
Option parameters: $K = 100$, $H = 90$, $r = 0.072310$, $T = 0.5$.
Algorithm parameters: d – space step, $N = 1600$ – number of time steps, S – spot price.

KoBoL as example

Down-and-out put prices in KoBoL model, FWH-method

	Option price, $V_{d,N}(S)$			Relative errors w.r.t. FDS		
Spot price	$N = 50$	$N = 100$	$N = 200$	$N = 50$	$N = 100$	$N = 200$
$S = 91$	0.2424	0.2371	0.2345	2.78%	0.53%	-0.57%
$S = 101$	0.5723	0.5681	0.5660	0.95%	0.22%	-0.15%
$S = 111$	0.3806	0.3823	0.3831	-1.14%	-0.71%	-0.49%
$S = 121$	0.2048	0.2062	0.2069	-1.57%	-0.90%	-0.55%
$S = 131$	0.1059	0.1065	0.1068	-1.33%	-0.77%	-0.49%
CPU-time(sec)	0.5	1	2			

KoBoL parameters: $\nu = 0.5$, $\lambda_+ = 9$, $\lambda_- = -8$, $c = 1$.
Option parameters: $K = 100$, $H = 90$, $r = 0.072310$, $T = 0.5$.
Algorithm parameters: $d = 5 \cdot 10^{-4}$ – space step, N – number of time steps, S – spot price.

KoBoL as example

Relative difference between down-and-out put prices in KoBoL model computed with FWH-method and IWH-method

	$N = 50$			$N = 1600$		
Spot price	FWH	IWH	Relative difference	FWH	IWH	Relative difference
$S = 91$	0.2383	0.2386	0.15%	0.2283	0.2300	0.74%
$S = 101$	0.5692	0.5695	0.05%	0.5611	0.5623	0.21%
$S = 111$	0.3789	0.3791	0.04%	0.3821	0.3827	0.16%
$S = 121$	0.2040	0.2041	0.04%	0.2067	0.2070	0.14%
$S = 131$	0.1055	0.1055	0.04%	0.1066	0.1068	0.14%
CPU-time, sec	0.2	7		7	49	

KoBoL parameters: $\nu = 0.5$, $\lambda_+ = 9$, $\lambda_- = -8$, $c = 1$.
Option parameters: $K = 100$, $H = 90$, $r = 0.072310$, $T = 0.5$.
Algorithm parameters: $d = 0.001$ – space step, N – number of time steps, S – spot price.

KoBoL as example

Relative difference between down-and-out put prices in KoBoL model computed with FWH-method and IWH-method

Spot price	$N = 50$ FWH	$N = 50$ IWH	$N = 50$ Relative difference	$N = 1600$ FWH	$N = 1600$ IWH	$N = 1600$ Relative difference
$S = 91$	0.2424	0.2426	0.06%	0.2323	0.2329	0.25%
$S = 101$	0.5723	0.5724	0.02%	0.5642	0.5648	0.11%
$S = 111$	0.3806	0.3807	0.02%	0.3838	0.3841	0.08%
$S = 121$	0.2048	0.2048	0.02%	0.2076	0.2077	0.07%
$S = 131$	0.1059	0.1059	0.02%	0.1070	0.1071	0.07%
CPU-time, sec	0.5	14		15	105	

KoBoL parameters: $\nu = 0.5$, $\lambda_+ = 9$, $\lambda_- = -8$, $c = 1$.
Option parameters: $K = 100$, $H = 90$, $r = 0.072310$, $T = 0.5$.
Algorithm parameters: $d = 0.0005$ – space step, N – number of time steps, S – spot price.

KoBoL as example

Relative difference between down-and-out put prices in KoBoL model computed by FWH-method and CV-method

Spot price	CV/d $2.5 \cdot 10^{-5}$	FWH/d $5 \cdot 10^{-3}$	Relative difference	CV/d 10^{-5}	FWH/d $3.25 \cdot 10^{-3}$	Relative difference
$S = 91$	0.1997	0.2036	2.16%	0.2129	0.2122	-0.33%
$S = 101$	0.5389	0.5374	-0.27%	0.5474	0.5479	0.08%
$S = 111$	0.3703	0.3690	-0.37%	0.3749	0.3747	-0.04%
$S = 121$	0.2009	0.2000	-0.45%	0.2032	0.2030	-0.09%
$S = 131$	0.1037	0.1033	-0.47%	0.1049	0.1048	-0.05%
CPU-time, sec	1,950	1.6		11,000	1.8	

KoBoL parameters: $\nu = 0.5$, $\lambda_+ = 9$, $\lambda_- = -8$, $c = 1$.
Option parameters: $K = 100$, $H = 90$, $r = 0.072310$, $T = 0.5$.
Algorithm parameters: d – space step, S – spot price.

KoBoL as example

Relative difference between down-and-out put prices in KoBoL model computed by FWH-method and CV-method

Spot price	CV/d $5 \cdot 10^{-6}$	FWH/d $2.5 \cdot 10^{-3}$	Relative difference	CV/d $2 \cdot 10^{-6}$	FWH/d $1.75 \cdot 10^{-3}$	Relative difference
$S = 91$	0.2186	0.2184	-0.09%	0.2236	0.2237	0.03%
$S = 101$	0.5522	0.5523	0.02%	0.5567	0.5567	-0.01%
$S = 111$	0.3775	0.3772	-0.08%	0.3800	0.3796	-0.09%
$S = 121$	0.2045	0.2042	-0.11%	0.2057	0.2055	-0.12%
$S = 131$	0.1055	0.1054	-0.09%	0.1061	0.1060	-0.10%
CPU-time, sec	26,000	3.2		116,000	3.5	

KoBoL parameters: $\nu = 0.5$, $\lambda_+ = 9$, $\lambda_- = -8$, $c = 1$.

Option parameters: $K = 100$, $H = 90$, $r = 0.072310$, $T = 0.5$.

Algorithm parameters: d – space step, S – spot price.