

# Wiener-Hopf factorization as a general method for valuation of barrier options, American options and real options

Svetlana Boyarchenko and Sergei Levendorskiĭ

University of Texas at Austin, University of Leicester

November 20, 2008

# Aim of the talk: to explain

**A general approach to pricing** barrier options, real options, American options and other optimal stopping problems in Lévy models, including regime-switching models

**Regime-switching models can be used to approximate models with**

- stochastic interest rates
- stochastic volatility
- mean reversion

# Outline

- 1 Lévy processes: general definitions and examples
- 2 No-arbitrage pricing
- 3 European options
- 4 Barrier options and generalized Black-Scholes equation
- 5 American options and free boundary problems
- 6 Carr's randomization
- 7 Perpetual American options and real options
- 8 EPV-operators and Wiener-Hopf factorization
- 9 Model situations
- 10 Regime-switching models
- 11 Finite time horizon
- 12 3-factor model

# Lévy processes: general definitions

$L$ , infinitesimal generator, and  $\psi$ , characteristic exponent of  $X = (X_t)$ :

$$E \left[ e^{i\xi X_t} \right] = e^{-t\psi(\xi)}, \quad Le^{i\xi x} = -\psi(\xi)e^{i\xi x}$$

Hence,  $L = -\psi(D)$  is the PDO with symbol  $-\psi(\xi)$ . In 1D,

$$Lu(x) = (2\pi)^{-1} \int_{\mathbb{R}} e^{ix\xi} (-\psi(\xi)) \hat{u}(\xi) d\xi.$$

Explicit formulas (Lévy-Khintchine) are (also in 1D)

$$\psi(\xi) = \frac{\sigma^2}{2} \xi^2 - i\mu\xi + \int_{\mathbb{R} \setminus 0} (1 - e^{iy\xi} + iy\xi \mathbb{1}_{|y| < 1}(y)) F(dy),$$

$$Lu(x) = \frac{\sigma^2}{2} u''(x) + \mu u'(x) + \int_{\mathbb{R} \setminus 0} (u(x+y) - \mathbb{1}_{|y| < 1}(y) y u'(x) - u(x)) F(dy),$$

where  $F(dy)$ , the Lévy density, satisfies

$$\int_{\mathbb{R} \setminus 0} \min\{|y|^2, 1\} F(dy) < \infty.$$

# Some examples

1. **Brownian motion** (no jumps)
2. **Double-exponential model** (Kou (2002)): sizes of jumps are exponentially distributed on each half-axis

$$\psi(\xi) = \frac{\sigma^2}{2} \xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi},$$

where  $\sigma > 0$ ,  $\mu \in \mathbb{R}$ ,  $c_{\pm} > 0$ , and  $\lambda_- < 0 < \lambda_+$ .

3. **Extended Koponen's model** Boyarchenko and Levendorskii (1999); a.k.a. CGMY-model (Carr, Madan, Geman, Yor (2002)) and KoBoL model. The Lévy density behaves as  $c|y|^{-\nu-1}$  near 0 and exponentially decays at infinity. For  $\nu \in (0, 2)$ ,  $\nu \neq 1$ ,

$$\psi(\xi) = -i\mu\xi + c\Gamma(-\nu)[\lambda_+^{\nu} - (\lambda_+ + i\xi)^{\nu} + (-\lambda_-)^{\nu} - (-\lambda_- - i\xi)^{\nu}],$$

where  $c > 0$ ,  $\mu \in \mathbb{R}$ , and  $\lambda_- < 0 < \lambda_+$ .

**Markov-modulated Lévy model: e.g.,  
different stock price dynamics in booms and recessions**

$L_S = [\lambda_{jk}]_{j,k=1}^m$  : the generator of a Markov chain

For each  $j$ ,  $L_j$  : the generator of a Lévy process

Markov-modulated Lévy process: the generator  $L_S \otimes I + \text{diag}(L_j)$

**Models with stochastic volatility and/or stochastic interest rate**

Additional stochastic factors with mean-reversion. The infinitesimal generator is of the form  $L_Y + c(y)L_X + \mu(y)\partial_x - r(y)$ , where  $L_Y$  acts w.r.t.  $y$  and  $L_X$  w.r.t.  $x$ .

State space: the cartesian product of  $\mathbb{R}_x$  and a subset of  $\mathbb{R}_y^n$

**Appropriate discretization** of the state space for  $Y$  and discretization of  $L_Y$  leads to a **regime-switching model**.

# No-arbitrage pricing of options on the underlying asset (e.g., stock, with the price process $S_t$ )

Assuming that the riskless rate  $r > 0$  is fixed, the discounted price process is a martingale under  $\mathbb{Q}$ , an equivalent martingale measure (EMM) (chosen by the market):

$$e^{-rt} S_t = \mathbb{E}_t^{\mathbb{Q}}[e^{-rT} S_T],$$

for  $t < T$ , where  $\mathbb{E}_t^{\mathbb{Q}}$  is the expectation operator under  $\mathbb{Q}$  conditioned on the information available at time  $t$ , or

$$S_t = \mathbb{E}_t^{\mathbb{Q}}[e^{-r(T-t)} S_T]$$

Any **derivative security**, e.g., an option on the stock  $S_t$ , with the price process  $V_t$ , must be also a (local) martingale under  $\mathbb{Q}$ : for any stopping time  $T \geq t$ , which does not exceed the (random) life span of the security,

$$V_t = \mathbb{E}_t^{\mathbb{Q}}[e^{-r(T-t)} V_T]$$

# European option with expiry date $T$ and payoff $G(S_T)$

Life-time span is deterministic.

**Call:**  $G(S_T) = (S_T - K)_+$ . **Put:**  $G(S_T) = (K - S_T)_+$

**No-arbitrage price:**

$$V_t = \mathbb{E}_t^{\mathbb{Q}}[e^{-r(T-t)}G(S_T)]$$

**Numerical calculation:**

- i if pdf of  $S_T|S_t$  is available: numerical integration (easy)
- ii in the general case, Monte-Carlo methods are fairly straightforward
- iii if characteristic function of  $S_T|S_t$  is available: inverse Fast Fourier Transform (iFFT). The best choice if calculations need to be done fast and at many points.

**There are important computational issues for iFFT**

Levendorskii and Zherder (2001), Lord (2004);

especially in higher dimensions N.Boyarchenko and Levendorskii (2007)

## Barrier options: life-time is random

Numerous versions, e.g., **down-and-out call option** with strike  $K$ , expiry date  $T$ , barrier  $H$  and rebate  $G_b$ . Typically,  $H < K$ .

Let  $\tau$  be the random time when the barrier is reached or crossed from above. If  $\tau \leq T$ , the option expires and option owner receives the rebate  $G_b$ . Otherwise, at time  $T$ , the option owner receives the payoff  $G(S_T) = (S_T - K)_+$ .

**No-arbitrage price at time 0:**

$$V_0 = \mathbb{E}^{\mathbb{Q}}[e^{-r\tau} \mathbb{1}_{\tau \leq T} G_b] + \mathbb{E}^{\mathbb{Q}}[e^{-rT} \mathbb{1}_{\tau > T} (S_T - K)_+]$$

## Analytical expressions for the price of a barrier option

- i in diffusion models: in many cases, in particular, in the Brownian Motion (BM) model and exponential BM model, explicit analytical formulas are available
- ii for certain classes of diffusions with embedded compound Poisson jumps of a relatively simple structure, explicit formulas are derived by several authors starting with Lipton (2001) and Kou (2002), different probabilistic methods being used
- iii for wide classes of exponential Lévy models, general analytic formulas were derived by Boyarchenko and Levendorskiĭ (2000) using the Wiener-Hopf technique. An important ingredient was the derivation of the generalization of the Black-Scholes equation

# The generalized Black-Scholes equation for barrier options in Lévy models (Boyarchenko and Levendorskiĭ (2002))

The stock price  $S_t = \exp X_t$ , where  $X_t$  is a Lévy process under  $\mathbb{Q}$ .  
The boundary problem for the down-and-out call option:

$$\begin{aligned}(r - \partial_t - L)V(t, x) &= 0, \quad x > h := \log H, \quad t < T; \\ V(T, x) &= (e^x - K)_+, \quad x \geq h := \log H; \\ V(t, x) &= G_b, \quad x \leq h, \quad t \leq T,\end{aligned}$$

plus the natural condition on the rate of growth as  $x \rightarrow +\infty$

**GBS equation:** in the sense of the theory of **generalized functions**

$X$  satisfies **(ACT)-property:** transition densities exist

$\psi$  admits the analytic continuation into the strip  $\text{Im } \xi \in [-1, 0]$

Generalizations for Lévy processes in  $\mathbf{R}^n$  are also proved

## Numerical calculations in the non-gaussian case:

- a. General formulas (Boyarchenko and Levendorskiĭ (2000)): not computationally efficient;
- b. but help to derive the asymptotics of the price near the barrier: needed to understand why naive approximations cannot work
- c. Explicit formulas obtained by probabilistic methods are applicable to jump-diffusion processes which are not observed in the real markets
- d. If these processes are used as approximations to more realistic ones (e.g., Cont-Volchkova method; Pistorius with different co-authors), sizable errors result near the barrier
- e. Monte-Carlo methods also give sizable errors near the barrier
- f. Efficient realizations using time-discretization, Wiener-Hopf factorization and a refined FFT technique are obtained only recently: Kudryavtsev and Levendorskiĭ (2007), M.Boyarchenko and Levendorskiĭ (2008)

# American options and free boundary problems

The payoff function is the same as in the European case but the exercise time  $\tau \leq T$  can be chosen by the option owner.

**Optimal stopping problem:** find an optimal stopping time  $\tau \leq T$  for

$$V(0, x) = \max_{\tau} \mathbb{E}^{\mathbb{Q}}[e^{-r\tau} G(S_{\tau})].$$

The corresponding **free boundary problem**, for the put option, in the exponential Lévy model, is: find the early exercise boundary  $x = h(t)$  which maximizes the bounded solution of the problem

$$\begin{aligned}(r - \partial_t - L)V(t, x) &= 0, \quad t < T; \\ V(T, x) &= (K - e^x)_+; \\ V(t, x) &= K - e^x, \quad x \leq h(t), t < T.\end{aligned}$$

**No analytical solution even in the Gaussian case**

# Failure of smooth pasting and other irregularities

**Smooth pasting** condition is not mentioned: does not hold in some cases (Boyarchenko and Levendorskiĭ (2002))

For a *typical pure jump process* used in finance,

**The price of a barrier option is not smooth** up to the boundary; in the diffusion case with exponentially distributed compound Poisson jumps, the price is of class  $C^2$  up to the boundary

**For American options**, there may be a **gap** between the early exercise boundary and strike, at expiry, in the classical situations when for diffusion models, there is no gap Boyarchenko and Levendorskiĭ (2002); Levendorskiĭ (2004)

**General formula for the gap, for wide classes of Markov processes with jumps** Levendorskiĭ (2008)

# Carr's randomization or analytical method of lines

Most efficient method for calculation of prices of barrier options and American options with finite time horizon  $T$ : discretize time  $(0 =) t_0 < t_1 < \dots < t_N (= T)$  but not the space variable.

The equivalent probabilistic version: **Carr's randomization**

For **American options**: put, BM - Carr (1998); Lévy processes and general payoff functions - Boyarchenko and Levendorskiĭ (2002); proof of convergence for American options - Bouchard, El Karoui, Touzi (2006)

For **barrier options**: efficient procedure - Kudryavtsev and Levendorskiĭ (2007), M.Boyarchenko and Levendorskiĭ (2008); proof of convergence - M.Boyarchenko (2008)

For the barrier option with payoff  $G(X_T)_+$ , barrier  $h$  and no rebate, the result is a sequence of stationary boundary problems, equivalently, a sequence of perpetual barrier options with the same barrier.

# Algorithm

Calculate the sequence of approximations to the option value:

$v_s(x) = V(t_s, x)$ , and to the early exercise boundary:  $h_s = h(t_s)$ ,  
 $s = 0, 1, \dots, N - 1$ .

1. Set  $v_N(x) = G(x)_+$
2. For  $s = N - 1, N - 2, \dots, 0$ , set  $\Delta_s = t_{s+1} - t_s$ ,  $q^s = r + \Delta_s^{-1}$ , and find an optimal stopping time  $\tau_s$  which maximizes

$$v_s(x) = \mathbb{E}^x \left[ \int_0^{\tau_s} e^{-q^s t} \Delta_s^{-1} v_{s+1}(X_t) dt \right] + \mathbb{E}^x \left[ e^{-q^s \tau_s} G(X_{\tau_s}) \right]$$

**How to solve this sequence**, especially for  $G$  of a general form, and a general Lévy process?

# Main idea (Boyarchenko and Levendorskiĭ (2002, 2005))

- i Assume that  $G(x)$  is the  $q^s$ -resolvent of a non-increasing function  $g_s$ , which changes sign:

$$G(x) = \mathbb{E}^x \left[ \int_0^\infty e^{-q^s t} g_s(X_t) dt \right]$$

- ii Introduce  $\tilde{v}_s = v_s - G$ ; maximization of  $v_s \Leftrightarrow$  maximization of  $\tilde{v}_s$   
iii Using Dynkin's formula,

$$\tilde{v}_s(x) = \mathbb{E}^x \left[ \int_0^{\tau_s} e^{-q^s t} (\Delta_s^{-1} \tilde{v}_{s+1}(X_t) - g_s(X_t)) dt \right]$$

- iv Note that for  $s = N - 1$ ,  $\tilde{v}_{s+1} = (-G)_+$  is non-decreasing, hence,  $g^s := \Delta_s^{-1} \tilde{v}_{s+1} - g_s$  is a non-decreasing function which changes sign  
v Hence, for  $s = N - 1$ , the standard **optimal exit problem**; solution is a non-negative non-decreasing function  
vi By induction, the same argument is valid at each time step

**Basic problem I (perpetual barrier option)** Calculate

$$V(x; \tau) = \mathbb{E}^x \left[ \int_0^\tau e^{-qt} g(X_t) dt \right],$$

where  $\tau$  is a stopping time. Typically,

- $\tau = \tau_h^-$ : the hitting time of  $(-\infty, h]$ , or
- $\tau = \tau_h^+$ : the hitting time of  $[h, +\infty)$

**Basic problem II (optimal exit problem)**

Find an optimal stopping time  $\tau$  which maximizes  $V(x; \tau)$ .

**Main technical tools**

- expected present value operators (EPV-operators) under a Lévy process and its supremum and infimum processes
- operator form of the Wiener-Hopf factorization

# EPV-operators and Wiener-Hopf factorization

## Supremum and infimum processes

- $\overline{X}_t = \sup_{0 \leq s \leq t} X_s$  - the supremum process
- $\underline{X}_t = \inf_{0 \leq s \leq t} X_s$  - the infimum process

## Normalized EPV operators under $X$ , $\overline{X}$ , and $\underline{X}$ :

- $$(\mathcal{E}g)(x) := q\mathbb{E}^x \left[ \int_0^{+\infty} e^{-qt} g(X_t) dt \right]$$
- $$(\mathcal{E}^+g)(x) := q\mathbb{E}^x \left[ \int_0^{+\infty} e^{-qt} g(\overline{X}_t) dt \right],$$
- $$(\mathcal{E}^-g)(x) := q\mathbb{E}^x \left[ \int_0^{+\infty} e^{-qt} g(\underline{X}_t) dt \right].$$

**Example 1: Brownian Motion.** EPV-operators are of the form

$$\mathcal{E}^+ u(x) = \beta^+ \int_0^{+\infty} e^{-\beta^+ y} u(x+y) dy,$$

$$\mathcal{E}^- u(x) = (-\beta^-) \int_{-\infty}^0 e^{-\beta^- y} u(x+y) dy,$$

where  $\beta^- < 0 < \beta^+$  are the roots of  $q + \psi(-i\beta) = 0$ .

**Example 2: Kou's model.** EPV-operators are of the form

$$\mathcal{E}^+ u(x) = \sum_{j=1,2} a_j^+ \beta_j^+ \int_0^{+\infty} e^{-\beta_j^+ y} u(x+y) dy,$$

$$\mathcal{E}^- u(x) = \sum_{j=1,2} a_j^- (-\beta_j^-) \int_{-\infty}^0 e^{-\beta_j^- y} u(x+y) dy,$$

where  $\beta_2^- < \lambda_- < \beta_1^- < 0 < \beta_1^+ < \lambda_+ < \beta_2^+$  are the roots of the "characteristic equation"  $q + \psi(-i\beta) = 0$ , and  $a_j^\pm > 0$  are constants.

# Wiener-Hopf factorization formula

Three versions:

1. Let  $T \sim \text{Exp}(q)$  be the exponential random variable of mean  $q^{-1}$ , independent of process  $X$ . For  $\xi \in \mathbb{R}$ ,

$$\mathbb{E}[e^{i\xi X_T}] = \mathbb{E}[e^{i\xi \bar{X}_T}] \mathbb{E}[e^{i\xi \underline{X}_T}];$$

2. For  $\xi \in \mathbb{R}$ ,

$$\frac{q}{q + \psi(\xi)} = \phi_q^+(\xi) \phi_q^-(\xi),$$

where  $\phi_q^\pm(\xi)$  admits the analytic continuation into the corresponding half-plane and does not vanish there

3.  $\mathcal{E}_q = \mathcal{E}_q^- \mathcal{E}_q^+ = \mathcal{E}_q^+ \mathcal{E}_q^-$ .

3 is valid in appropriate function spaces, and can be either proved as 1 or deduced from 2 because  $\mathcal{E}_q = q(q + \psi(D))^{-1}$ ,  $\mathcal{E}_q^\pm = \phi_q^\pm(D)$ .

## Fixed barrier (perpetual barrier option)

### Theorem 1

Let  $g$  be a measurable locally bounded function satisfying certain conditions on growth at  $\infty$ . Then for any  $h$ ,

$$V(x; h) = q^{-1} \mathcal{E}_q^- \mathbb{1}_{(h, +\infty)} \mathcal{E}_q^+ g(x). \quad (1)$$

The proof is based on the following result

### Lemma 2

Let  $X$  and  $T$  be as above. Then

- (a) the random variables  $\bar{X}_T$  and  $X_T - \bar{X}_T$  are independent (deep!); and
- (b) the random variables  $\underline{X}_T$  and  $X_T - \bar{X}_T$  are identical in law.

# Proof of Theorem 1

Let  $T \sim \text{Exp } q$  be independent of  $X$ . Then  $\mathcal{E}g(x) = \mathbb{E}[g(x + X_T)]$ ,

$$\mathcal{E}^+g(x) = \mathbb{E}[g(x + \bar{X}_T)], \quad \mathcal{E}^-g(x) = \mathbb{E}[g(x + \underline{X}_T)],$$

and, by definition,

$$\begin{aligned} V(x; h) &:= \mathbb{E} \left[ \int_0^{\tau_h^-} e^{-qt} g(x + X_t) dt \right] \\ &= \mathbb{E} \left[ \int_0^\infty \mathbb{1}_{x + \underline{X}_t > h} e^{-qt} g(x + X_t) dt \right] \\ &= q^{-1} \mathbb{E}[g(x + X_T) \mathbb{1}_{x + \underline{X}_T > h}]. \end{aligned}$$

Applying Lemma 2, we continue

$$\begin{aligned} V(x; h) &= q^{-1} \mathbb{E}[g(x + \underline{X}_T + X_T - \underline{X}_T) \mathbb{1}_{x + \underline{X}_T > h}] \\ &= q^{-1} \mathbb{E}[\mathbb{1}_{x + \underline{X}_T > h} \mathcal{E}^+g(x + \underline{X}_T)] \\ &= q^{-1} \mathcal{E}^- \mathbb{1}_{(h, +\infty)} \mathcal{E}^+g(x). \end{aligned}$$

# Option to abandon a non-decreasing stream

**Find an optimal stopping time  $\tau$ , which maximizes**

$$V(x) = \sup_{\tau} E^x \left[ \int_0^{\tau} e^{-qt} g(X_t) dt \right]$$

In many cases, for non-decreasing  $g$ , the optimal stopping time is  $\tau_h^-$ , the hitting time of a semi-infinite interval  $(-\infty, h]$ .

From

$$V(x; h) = q^{-1} \mathcal{E}^- \mathbb{1}_{(h, +\infty)} \mathcal{E}^+ g(x),$$

a natural candidate for the optimal exercise boundary is  $h_*$  s.t.

$$\mathcal{E}^+ g(h_*) = 0. \tag{2}$$

### Theorem 3

Assume that

- $X$  satisfies (ACP)-condition,
- trajectories of the supremum (resp., infimum) process reach any subinterval of  $(0, +\infty)$  (resp.,  $(-\infty, 0)$ ) with non-zero probability, and
- $g$  is a continuous non-decreasing function that changes sign.

Then

- equation  $\mathcal{E}^+ g(h_*) = 0$  has a unique solution;
- $\tau_{h_*}^-$  is an optimal stopping time, and  $h_*$  is the unique optimal threshold;
- the rational value of the stream (with the option to abandon it) is

$$V_*(x) = q^{-1} \mathcal{E}^- \mathbb{1}_{(h_*, +\infty)} \mathcal{E}^+ g(x); \quad (3)$$

- there exists a non-increasing function  $W_*$  which is non-negative below  $h_*$  and 0 above  $h_*$  such that  $V_* = q^{-1} \mathcal{E}(W_* + g)$ .

# Bad News Principle for irreversible investment

Assume  $g$  is non-decreasing and changes sign. Find optimal  $\tau$

$$V(x) = \sup_{\tau} \mathbb{E}^x \left[ \int_{\tau}^{\infty} e^{-qt} g(X_t) dt \right]$$

Rewrite as

$$\mathbb{E}^x \left[ \int_0^{\infty} e^{-qt} g(X_t) dt \right] + \sup_{\tau} \mathbb{E}^x \left[ \int_0^{\tau} e^{-qt} (-g(X_t)) dt \right]$$

Second term: mirror reflection of the exit problem with  $-g$  in place of  $g$ .  
The final answer:

$$\begin{aligned} V^*(x) &= q^{-1} \mathcal{E} g(x) - q^{-1} \mathcal{E}^+ \mathbb{1}_{(-\infty, h^*)} \mathcal{E}^- g(x) \\ &= q^{-1} \mathcal{E}^+ \mathbb{1}_{[h^*, +\infty)} \mathcal{E}^- g(x), \end{aligned}$$

where  $h^*$  is a unique solution to

$$\mathcal{E}^- g(h) = \mathbb{E} \left[ \int_0^{\infty} g(h + \underline{X}_t) dt \right] = 0.$$

# Regime-switching models

## General set-up and notation

- finite state Markov chain
- $\lambda_{jk}$  – transition rates;  $\Lambda_j = \sum_{k \neq j} \lambda_{jk}$
- $X_t^{(j)}$  – Lévy process in state  $j$
- $\psi_j$  and  $L_j$ : characteristic exponent and infinitesimal generator of  $X_t^{(j)}$
- jumps and switches do not happen simultaneously, a.s.
- $q_j$  – state- $j$  riskless rate
- $g_j$  – payoff stream in state  $j$
- $G_j$  – instantaneous payoff in state  $j$

## Regularity conditions: for each $j$

- (i)  $X_t^{(j)}$  satisfies (ACP)-property, and trajectories of the supremum (resp., infimum) process reach any subinterval of  $(0, +\infty)$  (resp.,  $(-\infty, 0)$ ) with non-zero probability
- (ii)  $g_j$  is continuous, monotone, and does not grow too fast at infinity
- (iii)  $(q_j + \Lambda_j - L_j)G_j$  is continuous, monotone, and does not grow too fast at infinity

# General scheme in the regime-switching case

- I. reduce an optimal stopping problem to option to abandon a stream
- II. assuming that the value functions in each state but state  $j$  are known and monotone, apply Theorem 3 and find the state- $j$  optimal exercise boundary and option value
- III. using this conditional result as a guide, construct an iteration scheme, for all states, and prove that the boundaries and value functions converge to some limits
- IV. using general sufficient conditions of optimality, prove that the limits give a solution to the optimal stopping problem
- V. using Theorem 3 in each state once again, conclude that this solution is unique in the class of stopping times of threshold type.

## Step II

Assuming that the reduction to the exit problem has been made, consider the option to abandon a stream

$$g(X_t) = \left( g(j_t, X_t^{(j)}) \right) = \left( g_j(X_t^{(j)}) \right)_{j=1}^m,$$

with a slight abuse of notation.

Fix  $j$  and assume that value functions  $V_k, k \neq j$ , are sufficiently regular. Then

$$V_j(x) = E^x \left[ \int_0^{\tau_j^-} e^{-(q_j + \Lambda_j)t} \left( g_j(X_t^{(j)}) + \sum_{k \neq j} \lambda_{jk} V_k(X_t^{(j)}) \right) dt \right], \quad (4)$$

where  $\tau_j^-$  is an optimal time to exit in state  $j$ .

Using  $q_j + \Lambda_j$  and  $X^{(j)}$  in place of  $q$  and a Lévy process  $X$ , we construct the normalized EPV-operators  $\mathcal{E}_j, \mathcal{E}_j^-, \mathcal{E}_j^+$ .

#### Theorem 4

- Function  $w_{j,*} = \mathcal{E}_j^+(g_j + \sum_{k \neq j} \lambda_{jk} V_{k,*})$  is continuous; it increases and changes sign;
- equation  $w_{j,*}(h) = 0$  has a unique solution, denote it  $h_{j,*}$ ;
- the hitting time of  $(-\infty, h_{j,*}]$  is an optimal stopping time, and  $h_{j,*}$  is the optimal threshold;
- state- $j$  value of the stream with the option to abandon the stream is given by

$$V_{j,*} = (q_j + \Lambda_j)^{-1} \mathcal{E}_j^- \mathbb{1}_{(h_{j,*}, +\infty)} w_{j,*}; \quad (5)$$

- function  $V_{j,*}$  vanishes below  $h_{j,*}$  and increases on  $[h_{j,*}, +\infty)$ .

The resulting system can be solved using the iteration procedure with 0 as the initial approximations for the value functions

# Finite time horizon: iteration procedure for put-like options

Assumptions: for  $j = 1, \dots, m$ ,

- (i)  $G_j$  decrease and change sign;
- (ii)  $L_j G_j$  is well-defined;
- (iii)  $\tilde{g}_j = \sum_{k \neq j} \lambda_{jk} G_k - (q_j + \Lambda_j - L_j) G_j$  does not decrease, and  $\tilde{g}_j(-\infty) < 0$

Example: for  $G_j(x) = K_j - B_j e^x$

- (i)  $B_j, K_j > 0$ ;
- (ii)  $\psi_j$  admits the analytic continuation into the strip  $\text{Im } \xi \in [-1, 0]$ ;
- (iii)  $q_j + \sum_{k \neq j} \lambda_{jk} (1 - K_k / K_j) > 0 \geq -\psi_j(-i) - q_j + \sum_{k \neq j} \lambda_{jk} (B_k / B_j - 1)$

RHS: minus dividend rate. LHS: effective discount rate for the strike price.

# Algorithm

1. Choose  $(0 =) t_0 < t_1 < \dots < t_N (= T)$ .
2. For  $s = N - 1, N - 2, \dots$ ,
  - ▶ set  $\Delta_s = t_{s+1} - t_s$ ,  $q_j^s = q_j + \Lambda_j + \Delta_s^{-1}$
  - ▶ construct the EPV-operators  $\mathcal{E}_j^{s,-}, \mathcal{E}_j^{s,+}$  using  $X^{(j)}$  and  $q = q_j^s$
3. Set  $v_{j,*}^N = (G_j)_+$ ,  $j = 1, 2, \dots, m$ .
4. For  $s = N - 1, N - 2, \dots$ , denote by  $v_{j,*}^s$  and  $h_{j,*}^s$  the Carr's randomization approximations to state- $j$  option value and exercise boundary for interval  $[t_s, t_{s+1})$ :
  - ▶ assume that  $v_{j,*}^{s+1}, j = 1, 2, \dots, m$ , are known
  - ▶ calculate the (approximations to the) exercise boundaries  $h_{j,*}^s$  and option values  $v_{j,*}^s, j = 1, 2, \dots, m$

# Backward induction:

For each  $s = N - 1, N - 2, \dots$  and  $j = 1, 2, \dots, m$ , we construct sequences  $\{h_j^{sn}\}_{n=0}^{\infty}$  and  $\{v_j^{sn}\}_{n=0}^{\infty}$ , s.t.

$$h_{j,*}^s = \lim_{n \rightarrow +\infty} h_j^{sn}, \quad v_{j,*}^s = \lim_{n \rightarrow +\infty} v_j^{sn} \quad (6)$$

Thus, for each  $s$ , we need to introduce an additional cycle in  $n$ ; and inside the cycle in  $n$ , we use additional cycles in  $j = 1, 2, \dots, m$ .

At step  $s = N - 1, N - 2, \dots$

- for  $j = 1, \dots, m$ , set  $v_j^{s0} = 0, h_j^{s0} = +\infty$ ,
- for  $j = 1, \dots, m$ , calculate

$$w_{0j}^s = -\mathcal{E}_j^{s,+} (q_j^s - L_j) G_j = q_j^s (\mathcal{E}_j^{s,-})^{-1} G_j,$$

- for  $n = 1, 2, \dots$ , for each  $j$ , calculate
  - (i)  $w_j^{sn} = \mathcal{E}_j^{s,+} \left( \sum_{k \neq j} \lambda_{jk} v_k^{s,n-1} + \Delta_s^{-1} v_{j,*}^{s+1} \right)$
  - (ii)  $\tilde{w}_j^{sn} = w_j^{sn} + w_{0j}^s$ ;
  - (iii)  $h_j^{sn}$ , the solution of the equation  $\tilde{w}_j^{sn}(h) = 0$ ;
  - (iv)  $v_j^{sn} = (q_j^s)^{-1} \mathcal{E}_j^{s,-} \left( \mathbb{1}_{(-\infty, h_j^{sn}]} (-w_{j0}^s) + \mathbb{1}_{(h_j^{sn}, +\infty)} w_j^{sn} \right)$
- stop cycle in  $n$  when  $\max_j \sup_x |v_j^{s,n+1}(x) - v_j^{sn}(x)| < \epsilon$

# American put in Heston model with CIR interest rates

## Non-dividend paying stock

The stock dynamics,  $S_t$ , stock volatility,  $\hat{v}_t$ , and the riskless interest rate,  $r_t$ , follow the system of SDE

$$\begin{aligned}\frac{dS_t}{S_t} &= r_t dt + \sqrt{\hat{v}_t} d\hat{W}_{1,t}, \\ d\hat{v}_t &= \hat{\kappa}_v(\hat{\theta}_v - \hat{v}_t)dt + \hat{\sigma}_v \sqrt{\hat{v}_t} dW_{2,t}, \\ dr_t &= \kappa_r(\theta_r - r_t)dt + \sigma_r \sqrt{r_t} dW_{3,t},\end{aligned}$$

where  $\hat{W}_{1,t}$ ,  $W_{2,t}$ ,  $W_{3,t}$  are components of the Brownian motion in 3D, with unit variances. For simplicity, only  $\hat{W}_{1,t}$  and  $W_{2,t}$  are correlated, the correlation coefficient being  $\rho$ . The coefficients  $\hat{\kappa}_v$ ,  $\kappa_r$ ,  $\hat{\theta}_v$ ,  $\theta_r$ ,  $\hat{\sigma}_v$  and  $\sigma_r$  are positive.

A. Medvedev and O. Scaillet (M&S) “Pricing American options under stochastic volatility and stochastic interest rates”, Finance Swiss Institute, Res. Paper (2007)

- M&S: “Up to now, there exist no feasible methodology for pricing American options when volatility and interest rates are stochastic.”
- M&S develop an asymptotic method based on a certain suboptimal exercise rule, which they regard as intuitively natural.
- The method leads to an asymptotic expansion with terms given by analytic expressions, therefore, *formally*, Greeks can be calculated quite easily.
- M&S produce a numerical example of Heston model with CIR interest rates, and compare the results with the results obtained by Longstaff-Schwarz algorithm with 200,000 sample paths, 500 time steps and 50 exercise dates.

# American put in Heston model with CIR interest rates

We consider the same example. The parameters are  $\hat{\kappa}_v = 1.58, \hat{\theta}_v = 0.03, \hat{\sigma}_v = 0.2, \kappa_r = 0.26, \theta_r = 0.04, \sigma_r = 0.08, \rho = -0.26$ , the spot value of the riskless rate is 0.04, the spot stock price is  $S = 100$ , the spot volatility  $v$  and strike  $K$  vary:  $v = 0.04, 0.09, 0.16, K = 90, 100, 110$ , and time to expiry is  $T = 1/12, 0.25, 0.5$ .

## In the tables below

- (AP): prices of the American put calculated using discretization of  $(t, v, r)$ -space and the reduction to a regime-switching model;
- (MC): prices of the American put calculated in Medvedev and Scaillet (2007) using the Longstaff-Schwarz algorithm with 200, 000 sample paths, 500 time steps and 50 exercise dates;
- (MS): prices of the American put, which Medvedev and Scaillet (2007) calculated using their asymptotic method with 5 terms of the asymptotic expansion.

# American put in Heston model with CIR interest rates

$T = 1/12$									
price	$v = 0.04$			$v = 0.09$			$v = 0.16$		
	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$
(AP)	0.0754	2.1378	10.002	0.4037	3.2308	10.356	0.9716	4.3325	11.059
(MS)	0.076	2.135	10	0.403	3.228	10.353	0.97	4.329	11.056
(MC)	0.075	2.15	10.018	0.404	3.238	10.352	0.969	4.341	11.065
relative difference									
difference	$v = 0.04$			$v = 0.09$			$v = 0.16$		
	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$
$\epsilon(AP, MS)$	-0.0076	0.0013	0.0002	0.0018	0.0009	0.0003	0.0016	0.0008	0.0003
$\epsilon(AP, MC)$	0.0056	-0.0057	-0.0016	-0.0007	-0.0022	0.0004	0.0027	-0.0020	-0.0006
$\epsilon(MS, MC)$	0.0133	-0.0070	-0.0018	-0.0025	-0.0031	0.0001	0.0010	-0.0028	-0.0008
$T = 0.25$									
price	$v = 0.04$			$v = 0.09$			$v = 0.16$		
	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$
(AP)	0.6131	3.4650	10.321	1.6216	5.1758	11.548	2.9077	6.9254	13.096
(MS)	0.612	3.459	10.312	1.619	5.169	11.537	2.904	6.919	13.085
(MC)	0.621	3.462	10.308	1.627	5.171	11.573	2.903	6.936	13.063
relative difference									
difference	$v = 0.04$			$v = 0.09$			$v = 0.16$		
	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$
$\epsilon(AP, MS)$	0.0019	0.0017	0.0009	0.0016	0.0013	0.0010	0.0013	0.0009	0.0008
$\epsilon(AP, MC)$	-0.0127	0.0009	0.0013	-0.0033	0.0009	-0.0022	0.0016	-0.0015	0.0025
$\epsilon(MS, MC)$	-0.0145	-0.0009	0.0004	-0.0049	-0.0004	-0.0031	0.0003	-0.0025	0.0017

# American put in Heston model with CIR interest rates

(cont-d)

$T = 0.5$	$v = 0.04$			$v = 0.09$			$v = 0.16$		
price	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$
(AP)	1.3615	4.5737	10.896	2.8290	6.6689	12.708	4.5809	8.8552	14.807
(MS)	1.357	4.559	10.879	2.822	6.654	12.68	4.575	8.84	14.781
(MC)	1.351	4.578	10.912	2.819	6.678	12.678	4.573	8.843	14.785
relative difference	$v = 0.04$			$v = 0.09$			$v = 0.16$		
	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$
$\epsilon(AP, MS)$	0.0033	0.0032	0.0015	0.0025	0.0022	0.0022	0.0013	0.0017	0.0017
$\epsilon(AP, MC)$	0.0078	-0.0009	-0.0015	0.0035	-0.0014	0.0023	0.0017	0.0014	0.0015
$\epsilon(MS, MC)$	0.0044	-0.0042	-0.0030	0.0011	-0.0036	0.0002	0.0004	-0.0003	-0.0003

# Early exercise threshold

