American options in the Heston model with stochastic interest rate

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Outline

- Heston model with CIR stochastic interest rate
- Numerical example: comparison with Medevedev-Scaillet method (2007) and Longstaff-Schwartz method
- Outline of our approach to jump-diffusion models with regime-switching, stochastic volatility and/or stochastic interest rates

American put in Heston model with CIR interest rates

Non-dividend paying stock

The stock dynamics, S_t, stock volatility, $\hat{v}_t$, and the riskless interest rate, r_t, follow the system of SDE

$$\begin{aligned} \frac{dS_t}{S_t} &= r_t dt + \sqrt{\hat{v}_t}\, d\hat{W}_{1,t}, \\ d\hat{v}_t &= \hat{\kappa}_v(\hat{\theta}_v - \hat{v}_t)dt + \hat{\sigma}_v\sqrt{\hat{v}_t}\, dW_{2,t}, \\ dr_t &= \kappa_r(\theta_r - r_t)dt + \sigma_r\sqrt{r_t}\, dW_{3,t}, \end{aligned}$$

where $\hat{W}_{1,t}, W_{2,t}, W_{3,t}$ are components of the Brownian motion in 3D, with unit variances. For simplicity, only $\hat{W}_{1,t}$ and $W_{2,t}$ are correlated, the correlation coefficient being ρ. The coefficients $\hat{\kappa}_v, \kappa_r$, $\hat{\theta}_v$, θ_r, $\hat{\sigma}_v$ and σ_r are positive.

A. Medvedev and O. Scaillet (M&S) "Pricing American options under stochastic volatility and stochastic interest rates", Finance Swiss Institute, Res. Paper (2007)

- M&S: "Up to now, there exist no feasible methodology for pricing American options when volatility and interest rates are stochastic."
- M&S develop an asymptotic method based on a certain suboptimal exercise rule, which can be regarded as intuitively natural.
- M&S produce a numerical example of Heston model with CIR interest rates, and compare the results with the results obtained by Longstaff-Schwarz algorithm with 200,000 sample paths, 500 time steps and 50 exercise dates.

American put in Heston model with CIR interest rates

We consider the same example.

The parameters are

$\hat{\kappa}_v = 1.58, \hat{\theta}_v = 0.03, \hat{\sigma}_v = 0.2, \kappa_r = 0.26, \theta_r = 0.04, \sigma_r = 0.08, \rho = -0.26$, the spot value of the riskless rate is 0.04, the spot stock price is $S = 100$, the spot volatility v and strike K vary: $v = 0.04, 0.09, 0.16$, $K = 90, 100, 110$, and time to expiry is $T = 1/12, 0.25, 0.5$.

American put in Heston model with CIR interest rates

- (AP): prices of the American put calculated using discretization of (t, v, r)–space and the reduction to a regime-switching model;
- (MC): prices of the American put calculated in Medvedev and Scaillet (2007) using the Longstaff-Schwarz algorithm with 200, 000 sample paths, 500 time steps and 50 exercise dates;
- (MS): prices of the American put, which Medvedev and Scaillet (2007) calculated using their asymptotic method with 5 terms of the asymptotic expansion.

American put in Heston model with CIR interest rates

$T = 1/12$	$v = 0.04$			$v = 0.09$			$v = 0.16$		
price	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$
(AP)	0.0754	2.1378	10.002	0.4037	3.2308	10.356	0.9716	4.3325	11.059
(MS)	0.076	2.135	10	0.403	3.228	10.353	0.97	4.329	11.056
(MC)	0.075	2.15	10.018	0.404	3.238	10.352	0.969	4.341	11.065

relative	$v = 0.04$			$v = 0.09$			$v = 0.16$		
difference	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$
$\epsilon(AP, MS)$	-0.0076	0.0013	0.0002	0.0018	0.0009	0.0003	0.0016	0.0008	0.0003
$\epsilon(AP, MC)$	0.0056	-0.0057	-0.0016	-0.0007	-0.0022	0.0004	0.0027	-0.0020	-0.0006
$\epsilon(MS, MC)$	0.0133	-0.0070	-0.0018	-0.0025	-0.0031	0.0001	0.0010	-0.0028	-0.0008

$T = 0.25$	$v = 0.04$			$v = 0.09$			$v = 0.16$		
price	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$
(AP)	0.6131	3.4650	10.321	1.6216	5.1758	11.548	2.9077	6.9254	13.096
(MS)	0.612	3.459	10.312	1.619	5.169	11.537	2.904	6.919	13.085
(MC)	0.621	3.462	10.308	1.627	5.171	11.573	2.903	6.936	13.063

relative	$v = 0.04$			$v = 0.09$			$v = 0.16$		
difference	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$
$\epsilon(AP, MS)$	0.0019	0.0017	0.0009	0.0016	0.0013	0.0010	0.0013	0.0009	0.0008
$\epsilon(AP, MC)$	-0.0127	0.0009	0.0013	-0.0033	0.0009	-0.0022	0.0016	-0.0015	0.0025
$\epsilon(MS, MC)$	-0.0145	-0.0009	0.0004	-0.0049	-0.0004	-0.0031	0.0003	-0.0025	0.0017

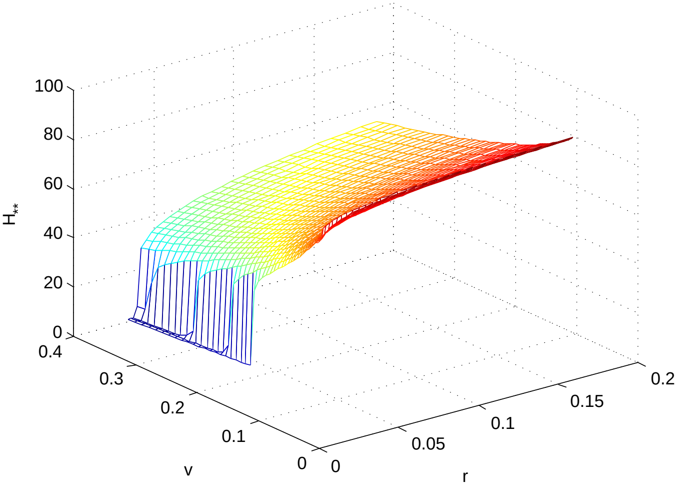
American put in Heston model with CIR interest rates

(cont-d)

$T = 0.5$	$v = 0.04$			$v = 0.09$			$v = 0.16$		
price	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$
(AP)	1.3615	4.5737	10.896	2.8290	6.6689	12.708	4.5809	8.8552	14.807
(MS)	1.357	4.559	10.879	2.822	6.654	12.68	4.575	8.84	14.781
(MC)	1.351	4.578	10.912	2.819	6.678	12.678	4.573	8.843	14.785

relative	$v = 0.04$			$v = 0.09$			$v = 0.16$		
difference	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$
$\epsilon(AP, MS)$	0.0033	0.0032	0.0015	0.0025	0.0022	0.0022	0.0013	0.0017	0.0017
$\epsilon(AP, MC)$	0.0078	-0.0009	-0.0015	0.0035	-0.0014	0.0023	0.0017	0.0014	0.0015
$\epsilon(MS, MC)$	0.0044	-0.0042	-0.0030	0.0011	-0.0036	0.0002	0.0004	-0.0003	-0.0003

Early exercise threshold



Scheme of the method

General set-up:

- regime-switching model, finite number of states
- in each state j, a 2-3 factor process $(X_t^{(j)}, Y_t^{(j)})$
- $Y_t^{(j)}$ is a mean-reverting process, which defines the dynamics of the interest rate and/or volatility
- jump-diffusion innovations in $X_t^{(j)}$ are independent of innovations in $Y_t^{(j)}$, and both are independent of the modulating Markov chain
- drift of $X_t^{(j)}$ depends on $Y_t^{(j)}$
- log-stock price $\log S_t^{(j)} = X_t^{(j)} +$ affine or quadratic function of $Y_t^{(j)}$

Basic examples:

- $Y_t^{(j)}$ (or its components) are of the OU-type or CIR-type, with embedded jumps
- interest rate and/or volatility are affine or quadratic function of $Y_t^{(j)}$
- SDE for $X_t^{(j)}$ is of the form

$$dX_t^{(j)} = \mu^{(j)}(Y_t^{(j)})dt + \sigma(Y_t^{(j)})dZ_t^{(j)}$$

where $Z_t^{(j)}$ is a Lévy process

General scheme of solution. Part I.

In each state of the modulating Markov chain

- discretize the state space for and the infinitesimal generator of $Y_t^{(j)}$
- freeze the riskless rate and coefficients in SDE for $X_t^{(j)}$ at points of the discretized state space
- truncate the discretized state space for $Y_t^{(j)}$
- impose appropriate boundary conditions so that the discretized part of the infinitesimal generator can be interpreted as the infinitesimal generator of a Markov chain

General scheme of solution. Part II.

- Apply to the resulting regime-switching Lévy model (with a larger modulating Markov chain) a natural generalization of the method of lines (equivalently, Carr's randomization method)
- Carr's randomization reduces the pricing of an American option with finite time horizon to a sequence of perpetual options in regime-switching models
- Apply to each perpetual option in the sequence an iteration procedure
- The main building block of the iteration procedure is a general optimal stopping theorem for American and real options in Lévy models based on the Wiener-Hopf method

Non-switching Lévy models

Notation

- $q > 0$ – the riskless rate
- $X = \{X_t\}_{t \geq 0}$ – a Lévy process
- Ψ – Lévy exponent of X
- L – infinitesimal generator of X
- $g(X_t)$ – payoff stream
- $G(X_t)$ – instantaneous payoff
- x – the current realization of X
- $E^x[f(X_t)] := E[f(X_t) | X_0 = x]$

Non-switching Lévy models

Moment generating function of a Lévy process

$$E\left[e^{zX_t}\right] = e^{t\Psi(z)}$$

$$Le^{zx} = \Psi(z)e^{zx}$$

Lévy process with jump component of finite variation: infinitesimal generator and Lévy Khintchine formula. b – drift, σ^2 – variance, $F(dy)$ – density of jumps

$$Lu(x) = \frac{\sigma^2}{2}u''(x) + bu'(x) + \int_{-\infty}^{+\infty} \left(u(x+y) - u(x)\right) F(dy)$$

$$E\left[e^{zX_t}\right] = exp\left[t\left(\frac{\sigma^2}{2}z^2 + bz + \int_{-\infty}^{+\infty} \left(e^y - 1\right) F(dy)\right)\right]$$

Perpetual American options

Find the optimal stopping time τ which maximizes

$$V(x;\tau) = E^x\left[\int_0^\tau e^{-qt} g(X_t)dt\right] + E^x\left[e^{-r\tau} G(X_\tau)\right].$$

- Can be reduced to the case $G = 0$.
- In many cases, τ is the hitting time of a semi-infinite interval.
- Explicit formula for $V(x;\tau)$ and equation for the optimal exercise threshold are available provided g is monotone and changes sign (after reduction to the case $G = 0$).
- **At each step of the backward induction, each step of the iteration procedure, and for each state, we will solve an optimal stopping problem of this form**

Regime-switching models

General set-up and notation

- finite state Markov chain
- λ_{jk} - transition rates; $\Lambda_j = \sum_{k \neq j} \lambda_{jk}$
- $X_t^{(j)}$ - Lévy process in state j
- Ψ_j and L_j: Lévy exponent and infinitesimal generator of $X_t^{(j)}$
- q_j - state-j riskless rate
- g_j - payoff stream in state j
- G_j - instantaneous payoff in state j

Regime-switching models. Finite time horizon

Assumptions: for $j = 1, \dots, m$,

(i) the payoff functions G_j decrease and change sign;

(ii) $L_j G_j$ is well-defined;

(iii) $\tilde{g}_j = \sum_{k \neq j} \lambda_{jk} G_k - (q_j + \Lambda_j - L_j) G_j$ does not decrease, and $\tilde{g}_j(-\infty) < 0$

Example: for $G_j(x) = K_j - B_j e^x$

(i) $B_j, K_j > 0$;

(ii) $E\left[\exp(X_1^{(j)})\right] < \infty$;

(iii) $\sum_{k \neq j} \lambda_{jk}(K_k/K_j - 1) - q_j < 0 \leq \sum_{k \neq j} \lambda_{jk}(1 - B_k/B_j) + (q_j - \Psi_j(1))$

The RHS: minus the rate of instantaneous discounted gains

First step of Algorithm

1. Choose $(0 =)t_0 < t_1 < \cdots < t_N(= T)$.
2. For $s = N-1, N-2, \ldots$, set $\Delta_s = t_{s+1} - t_s$, $q_j^s = q_j + \Lambda_j + \Delta_s^{-1}$
3. Set $v_{j,*}^N(x) = G_j(x)_+$, $j = 1, 2, \ldots, m$.
4. For $s = N-1, N-2, \ldots$, denote by $v_{j,*}^s$ and $h_{j,*}^s$ the Carr's randomization approximations to state-j option value and exercise boundary for interval $[t_s, t_{s+1})$.

Backward induction and iteration procedure

- for $s = l < N$, calculate the (approximations to the) exercise boundaries $h^s_{j,*}$ and option values $v^s_{j,*}$, $j = 1, 2, \dots, m$, assuming that for $l + 1 \le s \le N$ and $j = 1, 2, \dots, m$, $h^s_{j,*}$ and $v^s_{j,*}$ are known
- for each $s = N - 1, N - 2, \dots$ and $j = 1, 2, \dots, m$, construct sequences $\{h^{sn}_j\}^{\infty}_{n=0}$ and $\{v^{sn}_j\}^{\infty}_{n=0}$, s.t.

$$h^s_{j,*} = \lim_{n\to+\infty} h^{sn}_j, \quad v^s_{j,*} = \lim_{n\to+\infty} v^{sn}_j \tag{1}$$

Thus, for each s, we need to introduce an additional cycle in n; and inside the cycle in n, we use additional cycles in $j = 1, 2, \dots, m$.

At step $s = N-1, N-2, \ldots$

- set $v_j^{s0} = 0, h_j^{s0} = +\infty$, $j = 1, \ldots, m$,
- for $n = 1, 2, \ldots$, and $j = 1, 2, \ldots, m$, find an optimal $\tau_j^{sn,-}$, which maximizes

$$v_j^{sn}(x) = E^{j,x}\left[\int_0^{\tau_j^{sn,-}} e^{-q_j^s t}(\Delta_s^{-1} v_{j,*}^{s+1}(X_t^{(j)}) + \sum_{k \neq j} \lambda_{jk} v_k^{s,n-1}(X_t^{(j)}))dt\right]$$

$$+E^{j,x}\left[e^{-q_j^s \tau_j^{sn,-}} G_j(X_{\tau_j^{sn,-}}^{(j)})\right].$$

We find $\tau_j^{sn,-}$ as the hitting time of the unique interval of the form $(-\infty, h_j^{sn}]$ solving the problem in *non-regime-switching Lévy model*

Main technical tools

- expected present value operators (EPV-operators) under a Lévy process and its supremum and infimum processes
- Wiener-Hopf factorization
- general theorem about optimal timing to abandon a monotone stream that is a monotone function of a Lévy process

Wiener-Hopf factorization

Supremum and infimum processes

- $\bar{X}_t = \sup_{0\le s\le t} X_s$ - the supremum process
- $\underline{X}_t = \inf_{0\le s\le t} X_s$ - the infimum process

Normalized EPV operators under X, $\bar{X}$, and $\underline{X}$:

- $$(\mathcal{E}g)(x) := qE^x\left[\int_0^{+\infty} e^{-qt} g(X_t)dt\right]$$
- $$(\mathcal{E}^+g)(x) := qE^x\left[\int_0^{+\infty} e^{-qt} g(\bar{X}_t)dt\right],$$
- $$(\mathcal{E}^-g)(x) := qE^x\left[\int_0^{+\infty} e^{-qt} g(\underline{X}_t)dt\right].$$

Wiener-Hopf factorization formula

Three versions:

1. $T \sim \mathrm{Exp}(q)$ – the exponential random variable of mean q^{-1}, independent of process X. For $z \in i\mathbf{R}$,

$$E[e^{zX_T}] = E[e^{z\bar{X}_T}]E[e^{z\underline{X}_T}];$$

2. For $z \in i\mathbf{R}$,

$$\frac{q}{q - \Psi(z)} = \kappa_q^+(z)\kappa_q^-(z);$$

3. $\mathcal{E} = \mathcal{E}^-\mathcal{E}^+ = \mathcal{E}^+\mathcal{E}^-$.

1 and 2 are the same, 3 is valid in the space of locally bounded measurable function which do not grow too fast at infinity

Example 1: Brownian motion with variance σ^2 and drift b

The *Lévy exponent is*

$$\Psi(z) = \frac{\sigma^2}{2} z^2 + bz$$

EPV-operators are of the form

$$\mathcal{E}^+ u(x) = \beta^+ \int_0^{+\infty} e^{-\beta^+ y} u(x+y) dy,$$

$$\mathcal{E}^- u(x) = (-\beta^-) \int_{-\infty}^{0} e^{-\beta^- y} u(x+y) dy,$$

where $\beta^- < 0 < \beta^+$ are the roots of the “characteristic equation” $q - \Psi(\beta) = 0$.

Example 2: Kou's model

The *Lévy exponent*

$$\Psi(z) = \frac{\sigma^2}{2}z^2 + bz + \frac{c^+z}{\lambda^+ - z} + \frac{c^-z}{\lambda^- - z}.$$

EPV-operators are of the form

$$\mathcal{E}^+u(x) = \sum_{j=1,2} a_j^+\beta_j^+ \int_0^{+\infty} e^{-\beta_j^+ y}u(x+y)dy,$$

$$\mathcal{E}^-u(x) = \sum_{j=1,2} a_j^-(-\beta_j^-) \int_{-\infty}^{0} e^{-\beta_j^- y}u(x+y)dy,$$

where $\beta_2^- < \lambda^- < \beta_1^- < 0 < \beta_1^+ < \lambda^+ < \beta_2^+$ are the roots of the "characteristic equation" $q - \Psi(\beta) = 0$, and $a_j^\pm > 0$ are constants.

Option to abandon a non-decreasing stream.

Objective:

Find an optimal stopping time τ

$$V(x) = \sup_{\tau} E^x \left[\int_0^{\tau} e^{-qt} g(X_t) dt \right]$$

In many cases, for non-decreasing g, the optimal stopping time is the hitting time of a semi-infinite interval $(-\infty, h]$. Notation: τ_h^-.

Theorem 1

Let X satisfy the (ACP)-property, and let g be a measurable locally bounded function satisfying certain regularity conditions. Then for any h,

$$V(x;h) = q^{-1}\mathcal{E}^{-}\mathbf{1}_{(h,+\infty)}\mathcal{E}^{+}g(x). \tag{2}$$

Corollary.

A natural candidate for the optimal exercise boundary is h_* s.t.

$$\mathcal{E}^{+}g(h_*) = 0. \tag{3}$$

Theorem 2

Assume, in addition, that g is monotone and changes sign. Then

(i) *equation $\mathcal{E}^+g(h_*) = 0$ has a unique solution;*

(ii) *$\tau^-_{h_*}$ is an optimal stopping time, and h_* is the unique optimal threshold;*

(iii) *the rational value of the stream (with the option to abandon it) is*

$$V_*(x) = q^{-1}\mathcal{E}^-\mathbf{1}_{(h_*,+\infty)}\mathcal{E}^+g(x); \tag{4}$$

(iv) *there exists a non-increasing function W_* which is non-negative below h_* and 0 above h_* such that $V_* = \mathcal{E}(W_* + g)$.*