Carr's Randomization and New FFT Techniques for the Fast and Accurate Pricing of Barrier Options

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Forthcoming Attractions

Objects of study

Knock-out options with one or two barriers, which for brevity will simply be called "barrier options." They are among the most popular OTC options currently traded on financial markets.

Our goal

To present fast and accurate algorithms for calculating the prices and sensitivities of these options in a wide class of asset pricing models

Advantages of our approach

Our methods are very efficient on the one hand, and easy to implement in practice on the other hand

Brief Reminder on Barrier Options

Knock-out options with one or two barriers

A knock-out barrier option on an asset $\{S_t\}_{t\geq 0}$ expires if at any time $t \leq T =$ maturity, the price, S_t, of the underlying leaves a pre-specified open interval (L, U), where $0 \leq L < U \leq +\infty$. Otherwise, at $t = T$, the option's owner receives payoff $G(S_T)$, where G is a certain function.

Examples of terminal payoff functions

$G(S) = (S - K)_+ = \max\{S - K, 0\}$, where $K > 0$ is fixed (barrier call);
$G(S) = (K - S)_+$ (barrier put); $G(S) = 1$ (double-no-touch, or DNT).

Continuous vs. discrete monitoring

For discretely monitored options, the expiry condition $S_t \notin (L, U)$ is only checked at a finite collection of times $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$.

A 40,000 ft. View of Our Approach to Barrier Options

- We focus on continuously monitored barrier options
- We work with a wide class of jump-diffusion models, including: Black-Scholes, Kou's model, hyper-exponential jump-diffusions, Variance Gamma, Normal Inverse Gaussian, CGMY/KoBoL
- Carr's randomization approximation allows us to replace the original finite-lived pricing problem with a sequence of perpetual ones
- Perpetual pricing problems for barrier options are solved using the "Wiener-Hopf method"
- To implement the Wiener-Hopf method in practice, we use *enhanced realization of expected present value operators*, as well as the *enhanced and refined Fast Fourier Transform* (*FFT*) *techniques*

Strong Points of Our Approach

- We allow rather general terminal payoff functions.
- Option prices are calculated for a (rather fine) uniformly spaced grid of initial log-spot prices (as opposed to one initial spot price).
- This allows us to calculate the deltas and gammas of the option at the points of the same grid using numerical differentiation.
- The prices and sensitivities corresponding to log-spot prices that do not lie on the grid are found using interpolation (the additional computational cost of interpolation is negligible).
- High accuracy is maintained even in the regions where the initial spot price of the underlying is very close to the barrier(s).

Acknowledgements and (incomplete) Credits

Discretely monitored barrier options in Lévy-driven models

The definitive work in this area is the article by L. Feng and V. Linetsky

However, there are situations where monitoring occurs so frequently that it must be treated as occurring continuously, e.g., foreign exchange

Continuously monitored barrier options in the Black-Scholes model

These were studied by many authors. The single barrier case was pioneered by R.C. Merton (1973), while the double barrier case was pioneered by N. Kunimoto and M. Ikeda (1992).

Continuously monitored barrier options in Kou's model

S.G. Kou and H. Wang (2002, 2003, 2004) and A. Sepp (2004)

Acknowledgements and Credits, continued

Continuously monitored barrier options in HEJD models

S.B. (2006, perpetual case); M. Jeannin and M. Pistorius (2007, barrier and first-touch digitals); P. Carr and J. Crosby (2008, double barrier)

Approximations of other jump-diffusion models by HEJD

M. Jeannin and M. Pistorius (2007); S. Asmussen, D. Madan and M. Pistorius (2007); J. Crosby, N. Le Saux and A. Mijatović (2008)

Acknowledgements

Numerical results reproduced in the works by Jeannin-Pistorius and Carr-Crosby were very useful for benchmarking. We also particularly benefited from email correspondence with Peter Carr and John Crosby.

What is Carr's Randomization?

- Carr's randomization (a.k.a. "Canadization") was originally discovered (P. Carr, 1998) as a probabilistic interpretation of the "analytic method of lines" (used by P. Carr and D. Faguet, 1996).
- Carr proposed to approximate a finite-lived option pricing problem by replacing a deterministic maturity date T with a suitably chosen random maturity date whose mean is equal to T.
- When this random maturity is a sum of independent exponentially distributed maturity dates, the new pricing problem often reduces to a sequence of perpetual pricing problems, which are easier to solve.
- We believe that this idea has a very wide scope of applications. For the time being, the efficiency of Carr's randomization for American

Some Numerical Examples

Table: Prices of DNT options on Cable (STG/USD) under a HEJD process with 7 double exponential summands. The initial spot price is $S = 2.006$ in all cases. J. Crosby reports computational times of 1–1.5 seconds per option price.

Maturity (in years)	Domestic (USD) interest rate	Foreign (GBP) interest rate	Lower barrier	Upper barrier	Our price	Carr-Crosby price	Relative difference	CPU time (seconds)
0.085	0.0537	0.0589	1.95	2.05	0.87546	0.87547	-9.30e-06	0.094
0.170	0.0539	0.0591	1.95	2.05	0.72668	0.72665	3.43e-05	0.094
0.258	0.0539	0.0597	1.95	2.05	0.58146	0.58140	1.08e-04	0.078
0.337	0.0539	0.0601	1.95	2.05	0.46872	0.46870	2.99e-05	0.094
0.419	0.0539	0.0604	1.95	2.05	0.37261	0.37265	-9.64e-05	0.109
0.507	0.0539	0.0607	1.95	2.05	0.29070	0.29076	-2.04e-04	0.125
0.756	0.0538	0.0613	1.95	2.05	0.14239	0.14247	-6.15e-04	0.172
1.005	0.0536	0.0618	1.95	2.05	0.06943	0.06952	-1.30e-03	0.234
0.085	0.0537	0.0589	1.97	2.04	0.78752	0.78755	-4.11e-05	0.047
0.258	0.0539	0.0597	1.97	2.04	0.38150	0.38152	-6.59e-05	0.063
0.019	0.0535	0.0589	1.98	2.03	0.93403	0.93400	2.33e-05	0.046
0.085	0.0537	0.0589	1.98	2.03	0.66446	0.66440	9.14e-05	0.047

Table: Prices and sensitivities of a down-and-out barrier put option in the NIG model: comparison with the results of M. Jeannin and M. Pistorius

Spot price	Option price BL	Option price JP	Delta BL	Delta JP	Gamma BL	Gamma JP	Theta BL	Theta JP
64%	507.0212	486.8291	0.929	0.907	-0.00877	-0.00856	-342.5	-359.8
66%	554.2226	532.6638	0.443	0.437	-0.00546	-0.00532	-359.2	-373.2
68%	573.7149	551.8006	0.123	0.127	-0.00377	-0.00368	-355.0	-368.6
70%	574.2871	552.6467	-0.102	-0.092	-0.00271	-0.00265	-340.4	-354.0
72%	561.3327	540.3645	-0.265	-0.251	-0.00196	-0.00193	-320.0	-333.5
74%	538.5875	518.5689	-0.383	-0.366	-0.00140	-0.00137	-296.0	-308.9
76%	508.8566	490.0120	-0.465	-0.445	-0.00095	-0.00092	-269.1	-281.1
78%	474.3919	456.9306	-0.518	-0.496	-0.00056	-0.00053	-239.6	-250.2
80%	437.1020	421.2329	-0.545	-0.521	-0.00022	-0.00019	-207.6	-216.7

Spot prices are reported as percentages of the strike price $K = 3500$.

Other parameters: $H = 2100$ (barrier), $r = 0.03$ (riskless rate), $T = 1$ (maturity).

Some Comments on the Previous Table

- NIG parameters: $\alpha = 8.858$, $\beta = -5.808$, $\delta = 0.174$.
- Jeannin and Pistorius report computational times of 55 sec. per option price on a 2GHz machine. Our computational time: ≈ 8 sec. for all prices and sensitivities in the last table (also on a 2GHz PC).
- J&P use approximation of NIG by a HEJD process, which changes the order of the leading term of the asymptotics of the value function of the option near the barrier and leads to *underpricing* of the option.
- Carr's randomization approximation *does not* change the order, so it is expected to be more accurate on theoretical grounds.
- All our numerical tests are consistent with this theoretical expectation.

Lévy Processes (a.k.a. Jump-Diffusions)

- A Lévy process is a stochastic process $\{X_t\}_{t\geq 0}$ in continuous time that has time-homogenous (or stationary) independent increments.
- We work with a Lévy process in terms of its *characteristic exponent* $\psi(\xi)$, determined by $\mathbb{E}[e^{i\xi X_t}] = e^{-t\psi(\xi)}$ (for all $t \geq 0$ and $\xi \in \mathbb{R}$).
- A Lévy-driven model is a frictionless market with one riskless asset $\{B_t = e^{rt}\}_{t\geq 0}$ (e.g., a bond), where $r > 0$ is fixed, and one risky asset $\{S_t = S_0 e^{X_t}\}_{t\geq 0}$ (e.g., a stock), where X is a Lévy process.
- Example: Brownian Motion (which gives rise to the Black-Scholes model) has $\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi$, where $\sigma =$ volatility and $\mu =$ drift.
- All other Lévy processes have "jumps" (discontinuous sample paths).

Lévy Processes with Jumps

- Typically, Lévy processes with jumps provide a better fit to real data.
- Work with a chosen *risk-neutral measure*, a.k.a. *equivalent martingale measure* (EMM) on the space of all possible future trajectories of the price process of the underlying (an EMM is not unique).
- Lévy-Khintchine representation (or formula):

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{i\xi y} + i\xi y \mathbb{1}_{(-1,1)}(y)\right) \nu(dy),$$

where ν is the Lévy measure of X, which controls the sizes and the intensity of the jumps of X. It must satisfy the conditions

$$\nu\big(\{0\}\big) = 0 \quad \text{and} \quad \int_{\mathbb{R}\setminus\{0\}} \min\{1, y^2\}\, \nu(dy) < \infty.$$

Kou's Model (a.k.a. Double-Exponential Jump-Diffusion)

- Introduced by S.G. Kou (2002), it fits real data much better than the Black-Scholes model, while remaining rather tractable analytically.
- It is very popular among both practitioners and academicians.
- The underlying Lévy process is a sum $X_t = W_t + Z_t^+ + Z_t^-$, where $W = \{W_t\}$ is a Brownian motion and $Z^\pm = \{Z_t^\pm\}_{t\geq 0}$ are pure jump processes with Lévy measures $\nu^+(dy) = c^+\alpha^+ e^{-\alpha^+ y}\mathbb{1}_{(0,+\infty)}(y)\,dy$ and $\nu^-(dy) = c^-\alpha^- e^{\alpha^- y}\mathbb{1}_{(-\infty,0)}(y)\,dy$, respectively ($c^\pm, \alpha^\pm > 0$).
- The characteristic exponent is given by

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi - \frac{ic^+\xi}{\alpha^+ - i\xi} - \frac{ic^-\xi}{\alpha^- + i\xi}$$

Hyper-Exponential Jump-Diffusions (HEJD processes)

- Parameters: volatility $\sigma \geq 0$, drift $\mu \in \mathbb{R}$, and four collections of positive real numbers $\{\alpha_j^+\}_{j=1}^{n^+}$, $\{c_j^+\}_{j=1}^{n^+}$, $\{\alpha_k^-\}_{k=1}^{n^-}$ and $\{c_k^-\}_{k=1}^{n^-}$.
- Let $W = \{W_t\}_{t\geq 0}$ be a Brownian motion with volatility σ and drift μ.
- For all $1 \leq j \leq n^+$ and all $1 \leq k \leq n^-$, construct pure jump processes $Z_j^+ = \{Z_j^+(t)\}_{t\geq 0}$ and $Z_k^- = \{Z_k^-(t)\}_{t\geq 0}$ with Lévy measures $\nu_j^+(dy) = c_j^+ \alpha_j^+ e^{-\alpha_j^+ y} \mathbb{1}_{(0,+\infty)}(y)\, dy$ and $\nu_k^-(dy) = c_k^- \alpha_k^- e^{\alpha_k^- y} \mathbb{1}_{(-\infty,0)}(y)\, dy$, respectively, so that the processes W, Z_j^+, Z_k^- are all independent.
- Set $X_t = W_t + \sum_{j=1}^{n^+} Z_j^+(t) + \sum_{k=1}^{n^-} Z_k^-(t)$. This is a HEJD.
- The characteristic exponent of X is a *rational* function, given by

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi - \sum_{j=1}^{n^+} \frac{ic_j^+\xi}{\alpha_j^+ - i\xi} - \sum_{k=1}^{n^-} \frac{ic_k^-\xi}{\alpha_k^- + i\xi}$$

- These processes (and their WHF) were considered by S.L. in 2002.

Some Other Examples of Lévy-Driven Models

- Processes of the extended Koponen family (S.B. and S.L., 1999), a.k.a. "CGMY model" or "KoBoL processes": for $\nu \in (0,2)$, $\nu \neq 1$, $\mu \in \mathbb{R}$, $c > 0$, $\lambda_- < 0 < \lambda_+$:

$$\psi(\xi) = -i\mu\xi + c \cdot \Gamma(-\nu) \cdot \left[(-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu + \lambda_+^\nu - (\lambda_+ + i\xi)^\nu\right].$$

- Variance Gamma (used by D.B. Madan with collaborators, 1990):

$$\psi(\xi) = -i\mu\xi + c \cdot \left[\ln(-\lambda_- - i\xi) - \ln(-\lambda_-) + \ln(\lambda_+ + i\xi) - \ln(\lambda_+)\right].$$

- Normal Inverse Gaussian (O.E. Barndorff-Nielsen): for $\mu \in \mathbb{R}$, $\delta > 0$ and $\alpha > |\beta| > 0$,

$$\psi(\xi) = -i\mu\xi + \delta \cdot \left[\left(\alpha^2 - (\beta + i\xi)^2\right)^{1/2} - (\alpha^2 - \beta^2)^{1/2}\right].$$

Value Function of a Barrier Option: Finite Time Horizon

- Under an EMM, which is chosen once and for all, the price of the underlying is $\{S_t = S_0 e^{X_t}\}_{t \geq 0}$, where $X = \{X_t\}_{t \geq 0}$ is a Lévy process.
- EMM condition: $\mathbb{E}\left[e^{-rt_2} S_{t_2} \,\middle|\, S_{t_1} = S\right] = e^{-rt_1} S \;\forall\, t_2 > t_1 \geq 0$, where $r > 0$ is the (constant) riskless rate.
- Specify maturity $T > 0$, barriers $0 \leq L < U \leq +\infty$ and terminal payoff function $g(x) = G(e^x)$. (E.g., $g(x) = (K - e^x)_+$.)
- Write $h_- = \ln L$, $h_+ = \ln U$ and $x = \ln S_0$.
- Introduce $\tau_{h_-,h_+}(\omega) = \inf\left\{t \geq 0 \,\middle|\, X_t(\omega) \geq h_+ \text{ or } X_t(\omega) \leq h_-\right\}$.
- The no-arbitrage price of the option is given by

$$V_{k.o.}(x, T; h_\pm; g) = \mathbb{E}\left[e^{-rT} \mathbb{1}_{\{\tau_{h_- - x, h_+ - x} > T\}} g(x + X_T)\right].$$

Value Function of a Barrier Option: Infinite Time Horizon

- The underlying price process $\{S_t = S_0 e^{X_t}\}_{t\geq 0}$ and the barriers $0 \leq L < U \leq +\infty$ are as before.
- Write $h_- = \ln L$, $h_+ = \ln U$ and $x = \ln S_0$.
- Introduce $\tau_{h_-,h_+}(\omega) = \inf\{t \geq 0 \,|\, X_t(\omega) \geq h_+ \text{ or } X_t(\omega) \leq h_-\}$.
- Instead of r, we have a killing rate $q > 0$.
- Consider a payoff stream $\{g(\ln S_t)\}_{t\geq 0}$ that becomes deactivated at the first moment $t \geq 0$ when S_t leaves (L, U).
- The expected present value of this stream equals

$$v_{k.o.}(x; q; h_\pm; g) = \mathbb{E}\left[\int_0^{\tau_{h_- - x, h_+ - x}} e^{-qt} \cdot g(x + X_t)\, dt\right].$$

Carr's Randomization for Finite-Lived Barrier Options

Step I. Choose a partition, $\mathcal{P}$, of the interval $[0, T]$. Thus $\mathcal{P}$ is a finite collection of points $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$, where N is a positive integer. (Typically, $t_s = sT/N$ for all s.)

Step II. For every $0 \le s \le N-1$, set $\Delta_s = t_{s+1} - t_s$ and $q_s = r + \Delta_s^{-1}$.

Step III. Put $V^N(x) = g(x)$.

Step IV. In a cycle with respect to $s = N-1, N-2, \dots, 1, 0$, calculate

$$V^s(x) = \Delta_s^{-1} \cdot v_{k.o.}(x; q_s; h_\pm; V^{s+1}).$$

Step V. Put $V_{\mathcal{P}}(x, T; h_\pm; g) = V^0(x)$, where $V^0(x)$ is obtained at the end of the cycle in Step IV. Then $V_{\mathcal{P}}$ is *Carr's randomization approximation* to $V_{k.o.}(x, T; h_\pm; g)$, defined by the partition $\mathcal{P}$.

Additional Comments on Carr's Randomization

- It is proved that for all examples of Lévy processes used in financial modeling, $V_{\mathcal{P}}(x, T; h_\pm; g)$ converges to $V_{k.o.}(x, T; h_\pm; g)$ as the mesh, $\max_s \Delta_s$, of the partition $\mathcal{P}$ approaches 0. (M.B., 2008)
- For every positive integer N, choosing the partition $\mathcal{P}_N$ with $t_s = sT/N$ for $0 \leq s \leq N$ is often practically convenient.
- Sometimes, Richardson extrapolation can further increase the speed and accuracy of the calculation. For example, combining Carr's randomization with 3-point Richardson extrapolation yields to the following approximation to $V_{k.o.}(x, T; h_\pm; g)$:

$$0.5 \cdot V_{\mathcal{P}_N}(x, T; h_\pm; g) - 4 \cdot V_{\mathcal{P}_{2N}}(x, T; h_\pm; g) + 4.5 \cdot V_{\mathcal{P}_{3N}}(x, T; h_\pm; g)$$

Normalized EPV operators of a Lévy process

Our next goal is to explain how to price knock-out payoff streams with barriers in a Lévy-driven model. First we recall that the *supremum* process and the *infimum* process of a Lévy process $X = \{X_t\}_{t \geq 0}$ are defined by

$$\overline{X}_t = \sup_{0 \leq s \leq t} X_s \quad \text{and} \quad \underline{X}_t = \inf_{0 \leq s \leq t} X_s.$$

We define the normalized expected present value (EPV) operators by

$$(\mathcal{E}_q f)(x) = \mathbb{E}\left[\int_0^\infty qe^{-qt} f(x + X_t)\, dt\right],$$

$$(\mathcal{E}_q^+ f)(x) = \mathbb{E}\left[\int_0^\infty qe^{-qt} f(x + \overline{X}_t)\, dt\right],$$

$$(\mathcal{E}_q^- f)(x) = \mathbb{E}\left[\int_0^\infty qe^{-qt} f(x + \underline{X}_t)\, dt\right].$$

Wiener-Hopf Factorization (WHF)

- Let $T_q \sim \operatorname{Exp} q$ denote an exponentially distributed random variable with mean q^{-1} that is independent of the process $X = \{X_t\}_{t \geq 0}$.
- Probability form of the WHF formula:

$$\mathbb{E}\left[e^{i\xi X_{T_q}}\right] = \mathbb{E}\left[e^{i\xi \overline{X}_{T_q}}\right] \cdot \mathbb{E}\left[e^{i\xi \underline{X}_{T_q}}\right] \qquad \forall\, \xi \in \mathbb{R}.$$

- The last identity follows from the following facts:
  (1) we have $X_{T_q} = \overline{X}_{T_q} + (X_{T_q} - \overline{X}_{T_q})$;
  (2) the random variables $\overline{X}_{T_q}$ and $X_{T_q} - \overline{X}_{T_q}$ are independent (deep!);
  (3) the random variables $\underline{X}_{T_q}$ and $X_{T_q} - \overline{X}_{T_q}$ are identical in law;
  (4) the characteristic function of the sum of two independent random variables is equal to the product of their characteristic functions.

Two Other WHF Formulas

- Define the *Wiener-Hopf factors* $\phi_q^{\pm}(\xi)$ (for $\xi \in \mathbb{R}$) by the formulas

$$\phi_q^{+}(\xi) = \mathbb{E}\left[e^{i\xi \overline{X}_{T_q}}\right], \qquad \phi_q^{-}(\xi) = \mathbb{E}\left[e^{i\xi \underline{X}_{T_q}}\right].$$

- $\phi_q^{\pm}(\xi)$ admit analytic continuation without zeroes into the upper/lower half plane.
- They are related to the normalized EPV operators $\mathcal{E}_q^{\pm}$ via

$$\mathcal{E}_q^{\pm}\left(e^{i\xi x}\right) = \phi_q^{\pm}(\xi) \cdot e^{i\xi x} \qquad \forall\, \xi \in \mathbb{R}.$$

- One can verify directly using the definitions that

$$\mathcal{E}_q\left(e^{i\xi x}\right) = q \cdot (q + \psi(\xi))^{-1} \cdot e^{i\xi x} \qquad \forall\, \xi \in \mathbb{R}.$$

- Analytic form of the WHF formula: $q \cdot (q + \psi(\xi))^{-1} = \phi_q^{+}(\xi)\phi_q^{-}(\xi)$.
- Operator form of the WHF formula: $\mathcal{E}_q = \mathcal{E}_q^{+}\mathcal{E}_q^{-} = \mathcal{E}_q^{-}\mathcal{E}_q^{+}$.

The Wiener-Hopf Method: Streams with One Barrier

- Now we return to pricing a knock-out payoff stream $\{g(\ln S_t)\}_{t\geq 0}$.
- We assume that there is only one barrier. For concreteness, say $U = +\infty$ and $L > 0$. Thus we have a down-and-out payoff stream.
- Write $x = \ln S_0$ and $h_- = \ln L$. The stream is abandoned as soon as $S_t = S_0 e^{X_t} = e^{x+X_t}$ reaches or falls below $L = e^{h_-}$.
- Let a killing rate $q > 0$ be given.
- The value function of this knock-out stream equals

$$v_{down-and-out}(x; q; h_-; g) = q^{-1} \cdot \mathcal{E}_q^- \left(\mathbb{1}_{(h_-,+\infty)}(x) \cdot (\mathcal{E}_q^+ g)(x) \right).$$

- Similar formula in the up-and-out case ($L = 0$, $U = e^{h_+} < \infty$):

$$v_{up-and-out}(x; q; h_+; g) = q^{-1} \cdot \mathcal{E}_q^+ \left(\mathbb{1}_{(-\infty,h_+)}(x) \cdot (\mathcal{E}_q^- g)(x) \right).$$

The Wiener-Hopf Method: Streams with Two Barriers

- Now assume that $0 < L < U < +\infty$ and write $h_- = \ln L$, $h_+ = \ln U$.
- Value of a knock-out stream $\{g(\ln S_t)\}_{t\geq 0}$ with barriers (L, U):

$$v_{k.o.}(x; q; h_\pm; g) = G^0(x) - G^1_+(x) - G^1_-(x) + G^2_+(x) + G^2_-(x) \\ - G^3_+(x) - G^3_-(x) + G^4_+(x) + G^4_-(x) - \cdots$$

- To find the terms on the RHS, first calculate $G^0(x) = q^{-1} \cdot (\mathcal{E}_q g)(x)$.
- Next, use the formulas

$$G^0_+(x) = G^0(x)\big|_{[h_+,+\infty)}, \qquad G^0_-(x) = G^0(x)\big|_{(-\infty,h_-]},$$

$$G^n_+(x) = \mathcal{E}^-_q\Big(\mathbb{1}_{(-\infty,h_-]}(x) \cdot \big((\mathcal{E}^-_q)^{-1} G^{n-1}_-\big)(x)\Big) \qquad \forall\, n \geq 1,$$

$$G^n_-(x) = \mathcal{E}^+_q\Big(\mathbb{1}_{[h_+,+\infty)}(x) \cdot \big((\mathcal{E}^+_q)^{-1} G^{n-1}_+\big)(x)\Big) \qquad \forall\, n \geq 1.$$

Normalized EPV Operators in the Black-Scholes Model

- Introduce two types of integral operators:

$$(I^+_\beta f)(x) = \int_0^\infty \beta e^{-\beta y} f(x+y)\, dy \qquad (\beta > 0),$$

$$(I^-_\beta f)(x) = \int_{-\infty}^0 (-\beta) e^{-\beta y} f(x+y)\, dy \qquad (\beta < 0).$$

- Assume that $X = \{X_t\}_{t\geq 0}$ is a BM with volatility σ and drift μ.
- Denote by $\beta^- < 0 < \beta^+$ the roots of the *characteristic equation*

$$\frac{\sigma^2}{2}\beta^2 + \mu\beta - q = 0.$$

- Then $\phi^\pm_q(\xi) = \beta^\pm \cdot (\beta^\pm - i\xi)^{-1}$ and $\mathcal{E}^\pm_q = I^\pm_{\beta^\pm}$.
- Also: $\mathcal{E}_q = (\beta^+ - \beta^-)^{-1} \cdot \left(-\beta^- I^+_{\beta^+} + \beta^+ I^-_{\beta^-}\right)$.
- This allows us to calculate the action of $\mathcal{E}^\pm_q$, $\mathcal{E}_q$ very efficiently.

Enhanced Realization of the Operators $I_\beta^\pm$

The idea behind enhancement

Given a function $f(x)$, approximate it with a piecewise linear function. For piecewise linear functions, the action of $I_\beta^\pm$ can be calculated explicitly.

Setup for enhancement (precise formulation)

Choose a grid $\vec{x} = (x_\ell)_{\ell=0}^M$ of points in $\mathbb{R}$, where $x_\ell = x_0 + \ell \cdot \Delta$ for all $0 \le \ell \le M$ and $\Delta > 0$ is fixed. For $0 \le \ell \le M-1$, use the approximation

$$f(x) \approx f_\ell + \Delta^{-1} \cdot (f_{\ell+1} - f_\ell) \cdot (x - x_\ell), \qquad x_\ell \le x \le x_{\ell+1},$$

where $f_\ell = f(x_\ell)$. Further, approximate $f(x)$ by zero outside of $[x_0, x_M]$. The error is controlled by the size of $f''(x)$ for $x_\ell < x < x_{\ell+1}$ (assuming it exists) and by the size of $f(x)$ for $x < x_0$ and $x > x_M$.

Enhanced Realization of $I_\beta^{\pm}$, continued

Fix $\beta > 0$, and let I_ℓ^+ denote the approximation to $(I_\beta^+ f)(x_\ell)$ constructed above. The values I_ℓ^+ can be computed inductively as follows:

- Set $I_M^+ = 0$.
- In a cycle with respect to $\ell = M, M-1, \dots, 2, 1$, calculate

$$I_{\ell-1}^+ = e^{-\beta\Delta} \cdot I_\ell^+ + \frac{e^{-\beta\Delta} - 1 + \beta\Delta}{\beta\Delta} \cdot f_{\ell-1} + e^{-\beta\Delta} \cdot \frac{e^{\beta\Delta} - 1 - \beta\Delta}{\beta\Delta} \cdot f_\ell.$$

Next, let $\beta < 0$, and let I_ℓ^- denote the approximation to $(I_\beta^- f)(x_\ell)$ constructed above. The values I_ℓ^- can be computed inductively as follows:

- Set $I_0^- = 0$.
- In a cycle with respect to $\ell = 1, 2, \dots, M$, calculate

$$I_\ell^- = e^{\beta\Delta} \cdot I_{\ell-1}^- + \frac{1 + \beta\Delta - e^{\beta\Delta}}{\beta\Delta} \cdot f_\ell + e^{\beta\Delta} \cdot \frac{1 - \beta\Delta - e^{-\beta\Delta}}{\beta\Delta} \cdot f_{\ell-1}.$$

Normalized EPV Operators for HEJD Processes

- Let $X = \{X_t\}_{t \geq 0}$ be a HEJD with parameters $\sigma > 0$, μ, $\{\alpha_j^+\}_{j=1}^{n^+}$, $\{c_j^+\}_{j=1}^{n^+}$, $\{\alpha_k^-\}_{k=1}^{n^-}$ and $\{c_k^-\}_{k=1}^{n^-}$. The characteristic exponent of X is

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi - \sum_{j=1}^{n^+} \frac{ic_j^+\xi}{\alpha_j^+ - i\xi} - \sum_{k=1}^{n^-} \frac{ic_k^-\xi}{\alpha_k^- + i\xi}.$$

- Let $\{\beta_j^+\}_{j=1}^{n^++1}$ and $\{\beta_k^-\}_{k=1}^{n^-+1}$ be the positive and negative roots of the *characteristic equation* $q + \psi(-i\beta) = 0$.
- For suitable constants a_j^+, a_k^-, b_j^+, b_k^- (given by explicit formulas),

$$\mathcal{E}_q^+ = \sum_{j=1}^{n^++1} a_j^+ I^+_{\beta_j^+}, \qquad \mathcal{E}_q^- = \sum_{k=1}^{n^-+1} a_k^- I^-_{\beta_k^-},$$

$$\mathcal{E}_q = \sum_{j=1}^{n^++1} b_j^+ I^+_{\beta_j^+} + \sum_{k=1}^{n^-+1} b_k^- I^-_{\beta_k^-}.$$

Barrier Options in HEJD Models

Up-and-out and down-and-out options

The numerical calculation of

$$v_{down-and-out}(x;q;h_-;g) = q^{-1}\cdot\mathcal{E}_q^-\left(\mathbb{1}_{(h_-,+\infty)}(x)\cdot(\mathcal{E}_q^+g)(x)\right)$$

$$\text{and}\quad v_{up-and-out}(x;q;h_+;g) = q^{-1}\cdot\mathcal{E}_q^+\left(\mathbb{1}_{(-\infty,h_+)}(x)\cdot(\mathcal{E}_q^-g)(x)\right)$$

can be realized very efficiently using the formulas on the last two slides

Knock-out options with two barriers

In the formula $v_{k.o.}(x;q;h_\pm;g) = G^0(x) + \sum_{n=1}^\infty (-1)^n\cdot(G_+^n(x)+G_-^n(x))$, the infinite sum can be evaluated *in closed form*.

In both cases, combining Carr's randomization with the Wiener-Hopf method yields extremely fast algorithms (typical CPU time is < 0.1 sec.).

Integral Formulas for the Wiener-Hopf Factors

Under certain regularity conditions on the characteristic exponent $\psi(\xi)$,

$$\phi_q^{\pm}(\xi) = \exp\left[\pm\frac{1}{2\pi i}\int_{\mathrm{Im}\,\eta=\omega_{\mp}} \frac{\xi \cdot \ln(1+q^{-1}\psi(\eta))}{\eta(\xi-\eta)}\,d\eta\right],$$

where $\omega_- < 0 < \omega_+$ are suitably chosen. Main requirements:

- $\psi(\xi)$ admits analytic continuation into an open horizontal strip in $\mathbb{C}$ that contains the closed strip $\left\{\xi \in \mathbb{C} \,\middle|\, \mathrm{Im}\,\xi \in [\omega_-, \omega_+]\right\}$, and
- $\mathrm{Re}(q + \psi(\xi)) > 0$ for all ξ in this closed strip.

Practical applications of the above formula

One calculates the values of $\phi_q^{\pm}(\xi)$ on a suitable grid $\vec{\xi} = (\xi_k)_{k=1}^M$ by applying the trapezoid rule to discretize the integral above and using standard FFT tools (see below) to evaluate the resulting sums.

Fourier Transforms and FFT

Fourier transforms on the real line

$$\widehat{f}(\xi) = (\mathcal{F}f)(\xi) = (\mathcal{F}_{x\to\xi}f)(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\,dx$$

$$(\mathcal{F}^{-1}g)(\xi) = (\mathcal{F}^{-1}_{\xi\to x}g)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x} g(\xi)\,d\xi$$

Fast Fourier transforms

Consider uniformly spaced grids $\vec{x} = (x_j)_{j=1}^{M}$ and $\vec{\xi} = (\xi_k)_{k=1}^{M}$ with mesh Δ and ζ, respectively. Replace $(\mathcal{F}_{x\to\xi}f)(\xi)$ and $(\mathcal{F}^{-1}_{\xi\to x}g)(x)$ with

$$(\mathcal{F}_{fast}f)(\xi) = \Delta \cdot \sum_{j=1}^{M} f(x_j)e^{-i\xi x_j}, \quad (\mathcal{F}^{-1}_{fast}g)(x) = \frac{\zeta}{2\pi} \cdot \sum_{k=1}^{M} g(\xi_k)e^{i\xi_k x}.$$

FFT and Fast Discrete Convolution

- Let $\vec{f} = (f_j)_{j=1}^M$ be an array of complex numbers. Set

$$\mathtt{fft}(\vec{f})_k = \sum_{j=1}^{M} f_j \cdot e^{-2\pi i (j-1)(k-1)/M}, \qquad 1 \le k \le M.$$

  Standard FFT algorithms are designed for fast calculation of the vector $\mathtt{fft}(\vec{f})$ ("fast" means $O(M \cdot \ln M)$ arithmetic operations).

- $\mathcal{F}_{fast}^{\pm 1}$ can be expressed in terms of `fft` provided the identity $M \cdot \Delta \cdot \zeta = 2\pi$ holds ("Nyquist relation" or "uncertainty principle").

- `fft` can also be used for very fast calculation of sums of the form $h_k = \sum_{j=1}^M f_j g_{k-j}$ $(1 \le k \le M)$, where $\vec{f} = (f_j)_{j=1}^M$ and $\vec{g} = (g_\ell)_{\ell=1-M}^{M-1}$ are arrays of complex numbers.

Beyond the Standard FFT Techniques

The main problem one must solve when using standard FFT techniques is how to choose the parameters Δ, ζ and M so that $(\mathcal{F}_{fast} f)(\xi_k)$ is a good approximation to $(\mathcal{F}_{x\to\xi} f)(\xi_k)$ for all k, and $(\mathcal{F}^{-1}_{fast} g)(x_j)$ is a good approximation to $(\mathcal{F}^{-1}_{\xi\to x} g)(x_j)$ for all j. The uncertainty principle often makes it impossible to achieve the two goals simultaneously. For example, Δ is responsible for the "discretization error" that arises when $\mathcal{F}_{fast} f$ is used as an approximation to $\mathcal{F} f$, while ζ is responsible for the discretization error that arises when $\mathcal{F}^{-1}_{fast} g$ is used as an approximation to $\mathcal{F}^{-1} g$. If we wish to decrease both Δ and ζ, then M must be increased. However, this is rather inefficient from the computational viewpoint.

Fractional FFT

There exists a version of the standard FFT technique, called *fractional FFT* (due to D.H. Bailey and P.N. Swarztrauber), that allows one to compute $\mathcal{F}_{fast}f$ and $\mathcal{F}_{fast}^{-1}g$ even when the Nyquist relation is *not* satisfied. However, in this setup, calculating $\mathcal{F}_{fast}f$ using fractional FFT involves three applications of ordinary FFT to arrays with $2M$ elements. By comparison, when the Nyquist relation *is* satisfied, then a single application of FFT to an array with M elements suffices. When M is large, we see that using fractional FFT decreases the computational speed significantly.

Our approach to improving FFT was partially inspired by fractional FFT; however, the underlying ideas are quite different. In particular, our method minimizes the number of arithmetic operations that must be performed.

Explaining Our Idea Using a Simple Example

- Suppose we are given uniformly spaced grids $\vec{x} = (x_j)_{j=1}^{M}$ and $\vec{\xi} = (\xi_k)_{k=1}^{M}$ of mesh Δ and ζ; and the relation $M\Delta\zeta = 2\pi$ holds.
- Given a function $f(x)$, we can (quickly) calculate $(\mathcal{F}_{fast} f)(\xi_k)$ for all k using standard FFT techniques.
- Now suppose we wish to halve the mesh of the ξ-grid and double the number of points in it, while leaving the x-grid intact.
- Call the new grid $\vec{\xi'} = (\xi'_k)_{k=1}^{2M}$. It has mesh equal to $\zeta/2$.
- The points $\{\xi'_1, \xi'_3, \xi'_5, \ldots, \xi'_{2M-1}\}$ and $\{\xi'_2, \xi'_4, \xi'_6, \ldots, \xi'_{2M}\}$ form two uniformly spaced grids with mesh ζ.
- Apply the standard FFT technique twice, and we are in good shape.

An Improved Setup for FFT and Inverse FFT

We assume that a uniformly spaced grid $\vec{x} = (x_j)_{j=1}^{M}$ of points in $\mathbb{R}$ is given, where $x_j = x_1 + (j-1)\Delta$, and both M and $\Delta > 0$ are fixed. One should choose two positive integers, M_2 and M_3, that will be responsible, respectively, for refining and stretching the ξ-grid. One should also choose $\xi_1 \in \mathbb{C}$, the desired initial point of the ξ-grid.

The total number of points in the ξ-grid equals $M_1 = MM_2M_3$. Let us define $\zeta = 2\pi/(M\Delta)$. The mesh of the ξ-grid equals $\zeta_1 = \zeta/M_2$. Hence the length of the ξ-grid equals $M_3 \cdot (2\pi/\Delta)$. Explicitly, the ξ-grid is given by

$$\vec{\xi} = (\xi_k)_{k=1}^{M_1}, \qquad \xi_k = \xi_1 + (k-1)\zeta_1 = \xi_1 + (k-1) \cdot \frac{\zeta}{M_2}.$$

Implementing FFT in the New Setup

We would like to calculate the values of $\mathcal{F}_{fast} f$ at all the points of the grid $\vec{\xi}$. The best one can hope for is to reduce the calculation to $M_2 \cdot M_3$ applications of FFT for arrays of length M (since the input array has length M and the output array has length $M \cdot M_2 \cdot M_3$). To this end, we represent the grid $\vec{\xi} = (\xi_k)_{k=1}^{M_1}$ as a disjoint union of $M_2 \cdot M_3$ grids, each of which has M points and mesh ζ, and apply ordinary FFT to each of them:

$$\left(\xi_{M_2\cdot(k-1)+1}\right)_{k=1}^{M}, \quad \left(\xi_{M_2\cdot(k-1)+2}\right)_{k=1}^{M}, \quad \dots, \quad \left(\xi_{M_2\cdot k}\right)_{k=1}^{M},$$

$$\left(\xi_{M_2\cdot(k-1+M)+1}\right)_{k=1}^{M}, \quad \left(\xi_{M_2\cdot(k-1+M)+2}\right)_{k=1}^{M}, \quad \dots, \quad \left(\xi_{M_2\cdot(k+M)}\right)_{k=1}^{M},$$

$$\dots,$$

$$\left(\xi_{M_2\cdot(k-1+(M_3-1)M)+1}\right)_{k=1}^{M}, \quad \dots, \quad \left(\xi_{M_2\cdot(k+(M_3-1)M)}\right)_{k=1}^{M}.$$

Implementing Inverse FFT in the New Setup

Let $g(\xi)$ be a function whose domain contains the grid $\vec{\xi}$. We would like to calculate the values of the function $\mathcal{F}_{fast}^{-1}g$ on the grid $\vec{x}$. To this end, for each $1 \leq j \leq M_3$ and each $1 \leq \ell \leq M_2$, let $g_{j,\ell}$ be the restriction of g to the sub-grid $\vec{\xi}(j,\ell) = \left(\xi_{M_2\cdot(k-1+(j-1)M)+\ell}\right)_{k=1}^{M}$. The values of $\mathcal{F}_{fast}^{-1}g_{j,\ell}$ on the grid $\vec{x}$ can be calculated using the standard FFT techniques. Note that for each pair (j,ℓ), we only need to calculate a single FFT for a vector of length M. Finally, it follows immediately from the definitions that

$$\mathcal{F}_{fast}^{-1}g = \frac{1}{M_2}\sum_{j=1}^{M_3}\sum_{\ell=1}^{M_2}\mathcal{F}_{fast}^{-1}(g_{j,\ell}).$$

This method of calculating $\mathcal{F}_{fast}^{-1}g$ requires only $O(M_1 \cdot \ln M)$ arithmetic operations, which, again, is the best one can hope for.

Enhancement of FFT

Consider a grid $\vec{x} = (x_j)_{j=1}^{M}$, where $x_j = x_1 + (j-1)\Delta$ for all $1 \leq j \leq M$, and $\Delta > 0$ is fixed. Approximating f with a piecewise linear function yields

$$\begin{aligned}\widehat{f}(\xi) \approx (\mathcal{F}_{enh} f)(\xi) = & \frac{e^{i\xi\Delta} + e^{-i\xi\Delta} - 2}{(i\xi\Delta)^2} \cdot (\mathcal{F}_{fast} f)(\xi) \\ & + \frac{1 + i\xi\Delta - e^{i\xi\Delta}}{(i\xi\Delta)^2} \cdot \Delta \cdot f_1 \cdot e^{-i\xi x_1} \\ & + \frac{1 - i\xi\Delta - e^{-i\xi\Delta}}{(i\xi\Delta)^2} \cdot \Delta \cdot f_M \cdot e^{-i\xi x_M}.\end{aligned}$$

The main advantage of using $(\mathcal{F}_{enh} f)(\xi)$, as opposed to $(\mathcal{F}_{fast} f)(\xi)$, as an approximation to $\widehat{f}(\xi)$, stems from the fact that $\left|\widehat{f}(\xi) - (\mathcal{F}_{enh} f)(\xi)\right|$, the error of the former approximation, can be estimated *independently of the size of* ξ. The analogous statement is *false* for $(\mathcal{F}_{fast} f)(\xi)$.

Normalized EPV Operators and Fourier Transforms

- $X = \{X_t\}_{t \geq 0}$ a Lévy process with characteristic exponent $\psi(\xi)$
- fix $q > 0$ and let $\phi_q^{\pm}(\xi)$ be the Wiener-Hopf factors of $q \cdot (q + \psi(\xi))^{-1}$
- PDO realization of the normalized EPV operators of X:

$$(\mathcal{E}_q^{\pm} f)(x) = \mathcal{F}^{-1}_{\xi \to x}\big(\phi_q^{\pm}(\xi) \cdot \widehat{f}(\xi)\big)$$

- convolution realization of the normalized EPV operators:

$$(\mathcal{E}_q^{+} f)(x) = \int_0^{+\infty} f(x+y)\, p_q^{+}(dy), \quad (\mathcal{E}_q^{-} f)(x) = \int_{-\infty}^{0} f(x+y)\, p_q^{-}(dy),$$

where $p_q^{\pm}(dy)$ are Borel probability measures on $\mathbb{R}$ supported on the positive and the negative half axis, respectively

- the Fourier transforms of $p_q^{\pm}$ are given by $\widehat{p}_q^{\pm}(\xi) = \phi_q^{\pm}(\xi)$

Enhanced Convolution Realization of the EPV Operators

- The idea is the same as the one we used in the HEJD setting.
- Given a function $f(x)$ and a uniformly spaced grid $\vec{x} = (x_j)_{j=1}^M$ of points in $\mathbb{R}$, we approximate $f(x)$ with a linear function on each subinterval $[x_j, x_{j+1}]$, and approximate $f(x)$ with 0 outside of $[x_1, x_M]$.
- Now we must calculate the action of $\mathcal{E}_q^\pm$ on a function of the form $\left(f_j + \Delta^{-1} \cdot (f_{j+1} - f_j) \cdot (x - x_j)\right) \cdot \mathbb{1}_{[x_j, x_{j+1}]}(x)$. This is done using the convolution realization of $\mathcal{E}_q^\pm$, described on the previous slide.
- The answer can be expressed in terms of $\phi_q^\pm(\xi)$ and the inverse Fourier transforms of certain auxiliary functions.
- The resulting explicit formulas (see the next two slides) can be realized efficiently in practice using fast discrete convolution.

Consider a grid $\vec{x} = (x_j)_{j=1}^{M}$, where $x_j = x_1 + (j-1)\Delta$ for all $1 \le j \le M$, and $\Delta > 0$ is fixed. Approximating f with a piecewise linear function yields

$$(\mathcal{E}_q^+ f)(x_k) \approx -d_k^+ \cdot f_M + \sum_{j=k}^{M} c_{k-j}^+ \cdot f_j \qquad (1 \le k \le M),$$

where $f_j = f(x_j)$ for $1 \le j \le M$,

$$d_k^+ = \frac{\Delta}{2\pi} \int_{-\infty}^{\infty} e^{i(k-M)\Delta\xi} \cdot \phi_q^+(\xi) \cdot \frac{e^{-i\xi\Delta} + i\xi\Delta - 1}{(i\xi\Delta)^2} \, d\xi$$

for $1 \le k \le M$,

$$c_\ell^+ = \frac{\Delta}{2\pi} \int_{-\infty}^{\infty} e^{i\ell\Delta\xi} \cdot \phi_q^+(\xi) \cdot \frac{e^{i\xi\Delta} + e^{-i\xi\Delta} - 2}{(i\xi\Delta)^2} \, d\xi$$

for $1 - M \le \ell \le -1$, and

$$c_0^+ = 1 - \sum_{1-M \le \ell \le -1} c_\ell^+.$$

Similar formulas for $\mathcal{E}_q^-$:

$$(\mathcal{E}_q^- f)(x_k) \approx -d_k^- \cdot f_1 + \sum_{j=1}^{k} c_{k-j}^- \cdot f_j \qquad (1 \leq k \leq M),$$

where $f_j = f(x_j)$ for $1 \leq j \leq M$,

$$d_k^- = \frac{\Delta}{2\pi} \int_{-\infty}^{\infty} e^{i(k-1)\Delta\xi} \cdot \phi_q^-(\xi) \cdot \frac{e^{i\xi\Delta} - i\xi\Delta - 1}{(i\xi\Delta)^2}\, d\xi$$

for $1 \leq k \leq M$,

$$c_\ell^- = \frac{\Delta}{2\pi} \int_{-\infty}^{\infty} e^{i\ell\Delta\xi} \cdot \phi_q^-(\xi) \cdot \frac{e^{i\xi\Delta} + e^{-i\xi\Delta} - 2}{(i\xi\Delta)^2}\, d\xi$$

for $1 \leq \ell \leq M-1$, and

$$c_0^- = 1 - \sum_{1 \leq \ell \leq M-1} c_\ell^-.$$

References to Our Works

All the texts listed below can be downloaded from the SSRN webpage:

- “Refined and enhanced fast Fourier transform techniques, with an application to the pricing of barrier options” (M.B. and S.L.)
- “Prices and sensitivities of barrier and first-touch digital options in Lévy-driven models” (M.B. and S.L.)
- “Valuation of continuously monitored double barrier options and related securities” (M.B. and S.L.)
- “User’s guide to double barrier options. Part I: Kou’s model and generalizations” (M.B. and S.B.)
- “Carr’s randomization for finite-lived barrier options: proof of convergence” (M.B.)

Wiener-Hopf Factors for HEJD

Characteristic exponent and the Wiener-Hopf factors of a HEJD process:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi - \sum_{j=1}^{n^+} \frac{ic_j^+\xi}{\alpha_j^+ - i\xi} - \sum_{k=1}^{n^-} \frac{ic_k^-\xi}{\alpha_k^- + i\xi} \qquad (\sigma > 0),$$

$$\phi_q^+(\xi) = \left(\prod_{j=1}^{n^+} \frac{\alpha_j^+ - i\xi}{\alpha_j^+}\right) \cdot \left(\prod_{j=1}^{n^++1} \frac{\beta_j^+}{\beta_j^+ - i\xi}\right) = \sum_{j=1}^{n^++1} \frac{a_j^+\beta_j^+}{\beta_j^+ - i\xi},$$

$$\phi_q^-(\xi) = \left(\prod_{k=1}^{n^-} \frac{\alpha_k^- + i\xi}{\alpha_k^-}\right) \cdot \left(\prod_{k=1}^{n^-+1} \frac{\beta_k^-}{\beta_k^- - i\xi}\right) = \sum_{k=1}^{n^-+1} \frac{a_k^-\beta_k^-}{\beta_k^- - i\xi},$$

where $\{\beta_j^+\}_{j=1}^{n^++1}$ and $\{\beta_k^-\}_{k=1}^{n^-+1}$ are the positive and negative roots of the characteristic equation $q + \psi(-i\beta) = 0$, and the formulas for a_j^+ and a_k^- are given on the next slide.

Normalized EPV Operators for HEJD

With the notation of the previous slide, we have (for any $q > 0$)

$$\mathcal{E}_q^+ = \sum_{j=1}^{n^++1} a_j^+ I_{\beta_j^+}^+, \quad \mathcal{E}_q^- = \sum_{k=1}^{n^-+1} a_k^- I_{\beta_k^-}^-, \quad \mathcal{E}_q = \sum_{j=1}^{n^++1} b_j^+ I_{\beta_j^+}^+ + \sum_{k=1}^{n^-+1} b_k^- I_{\beta_k^-}^-,$$

where the integral operators $I_\beta^\pm$ were defined earlier,

$$a_j^+ = \left(\prod_{\ell=1}^{n^+} \frac{\alpha_\ell^+ - \beta_j^+}{\alpha_\ell^+} \right) \cdot \left(\prod_{\substack{\ell \neq j \\ 1 \le \ell \le n^++1}} \frac{\beta_\ell^+}{\beta_\ell^+ - \beta_j^+} \right),$$

$$a_k^- = \left(\prod_{\ell=1}^{n^-} \frac{\alpha_\ell^- + \beta_k^-}{\alpha_\ell^-} \right) \cdot \left(\prod_{\substack{\ell \neq k \\ 1 \le \ell \le n^-+1}} \frac{\beta_\ell^-}{\beta_\ell^- - \beta_k^-} \right),$$

$$b_j^+ = a_j^+ \cdot \phi_q^-(-i\beta_j^+) \qquad \text{and} \qquad b_k^- = a_k^- \cdot \phi_q^+(-i\beta_k^-).$$

Double Barrier K.O. Payoff Streams under HEJD

- Given: a HEJD $X = \{X_t\}_{t\geq 0}$ so that $S_t = S_0 e^{X_t}$; log-barriers $h_- < h_+$; a bounded measurable function $g(x)$ on (h_-, h_+); and a killing rate $q > 0$.
- Let $\vec{G}^0_+$ and $\vec{G}^0_-$ be column vectors of size $n^- + 1$ and $n^+ + 1$ with entries
$$(\vec{G}^0_+)_k = q^{-1} \cdot b^-_k \cdot (I^-_{\beta^-_k} g)(h_+), \qquad (\vec{G}^0_-)_j = q^{-1} \cdot b^+_j \cdot (I^+_{\beta^+_j} g)(h_-).$$
- Introduce matrices $A^\pm$ of size $(n^\pm + 1) \times (n^\mp + 1)$ with entries
$$A^+_{jk} = \frac{a^+_j \beta^+_j}{\beta^+_j - \beta^-_k} \cdot \frac{e^{-\beta^+_j (h_+ - h_-)}}{\phi^+_q(-i\beta^-_k)} \qquad \text{and} \qquad A^-_{kj} = \frac{a^-_k \beta^-_k}{\beta^-_k - \beta^+_j} \cdot \frac{e^{\beta^-_k (h_+ - h_-)}}{\phi^-_q(-i\beta^+_j)}$$
- Put $B = (I - A^+A^-)^{-1}$ and $C = (I - A^-A^+)^{-1}$, and calculate the vectors
$$\vec{V}^+ = A^- \cdot B \cdot (\vec{G}^0_- - A^+ \vec{G}^0_+), \qquad \vec{V}^- = A^+ \cdot C \cdot (\vec{G}^0_+ - A^- \vec{G}^0_-).$$
- Then for all $h_- < x < h_+$, we have
$$v_{k.o.}(x; q; h_\pm; g) = q^{-1} \cdot (\mathcal{E}_q g)(x) - \sum_{j=1}^{n^+ + 1} \vec{V}^-_j \cdot e^{\beta^+_j (x - h_-)} - \sum_{k=1}^{n^- + 1} \vec{V}^+_k \cdot e^{\beta^-_k (x - h_+)}.$$

Fast Discrete Convolution via FFT

- Goal: compute $h_k = \sum_{j=1}^{M} f_j g_{k-j}$, where $\vec{f} = (f_j)_{j=1}^{M}$ and $\vec{g} = (g_\ell)_{\ell=1-M}^{M-1}$ are complex arrays of lengths M and $2M-1$.
- Let $\widetilde{f}$ be the array of length $2M$ with entries

$$\widetilde{f}_j = \begin{cases} f_j, & 1 \le j \le M; \\ 0, & M+1 \le j \le 2M. \end{cases}$$

- Let $\widetilde{g}$ be the array of length $2M$ with entries

$$g_0, g_1, \ldots, g_{M-1}, 0, g_{1-M}, g_{2-M}, \ldots, g_{-1}$$

- Calculate the array $\widetilde{h} = (\widetilde{h}_\ell)_{\ell=1}^{2M}$ with entries

$$\widetilde{h}_\ell = \mathtt{fft}(\widetilde{f})_\ell \cdot \mathtt{fft}(\widetilde{g})_\ell.$$

- For all $1 \le k \le M$, we have $h_k = \mathtt{ifft}(\widetilde{h})_k$ (where $\mathtt{ifft} = \mathtt{fft}^{-1}$).

How Does the Fractional FFT Work?

- Change the indexing for simplicity: $\vec{x} = (x_j)_{j=0}^{M-1}$ and $\vec{\xi} = (\xi_k)_{k=0}^{M-1}$, where $x_j = x_0 + j\Delta$ and $\xi_k = \xi_0 + k\zeta$ (no relation between M, Δ, ζ).
- We would like to quickly calculate $\sum_{j=0}^{M-1} f(x_j) e^{-i\xi_k x_j}$ for all k.
- We have $x_j \xi_k = x_0 \xi_k + j\xi_0 \Delta + jk\Delta\zeta$, which essentially reduces the problem to computing $\sum_{j=0}^{M-1} f_j e^{-ijk\Delta\zeta}$, where $f_j = f(x_j) \cdot e^{-ij\xi_0\Delta}$.
- Writing $z = e^{i\Delta\zeta}$, we obtain the expression $\sum_{j=0}^{M-1} f_j z^{-jk}$.
- Note that $z^{-jk} = z^{-j^2/2} z^{(k-j)^2/2} z^{-k^2/2}$, so we must calculate $\sum_{j=0}^{M-1} \widetilde{f}_j z^{(k-j)^2/2}$, where $\widetilde{f}_j = z^{-j^2/2} \cdot f_j$.
- The last sum can be computed using fast discrete convolution.
- The dominant computational cost of this algorithm is that of three applications of FFT to complex arrays of length $2M$.