Boundary-layer transition on broad cones rotating in an imposed axial flow
Convective and absolute instabilities

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INTRODUCTION

BACKGROUND
  Transition within related boundary-layer flows

METHODOLOGY
  Pseudo-routine
  Formulation
  Steady-laminar flow
  Perturbation equations
  Incorporating the basic-flow solutions
  Stability analyses

RESULTS

CONCLUSION
Flow visualizations due to Kohoma, Kobayashi and co-workers.
SLENDER VS. BROAD CONES

- Unstable region:
  - Disks and broad cones ($\psi > 40^\circ$): co-rotating crossflow vortices, related to onset of convective modes of type I & II.
  - Slender cones: pairs of counter-rotating Görtler-type vortices, indicative of centrifugal instability.

- Onset of turbulence:
  - Related to the onset of local AI for $\psi > 40^\circ$ involving modes of type I & III.
  - Occurs far in advance of local AI for slender cones.

- Hypothesized convective-centrifugal mode of instability that dominates on slender cones (Garrett et al. (2009)).

- Linear growth rates of type I and II significantly reduce with reduced $\psi$ (Garrett (2009)).
To be covered...

- Focus on *broad* rotating cones within an enforced axial flow.
- Work extends that of Garrett & Peake (2007) through better solution of basic flows.
- Onset of CI associated with onset of spiral vortices.
- Stationary disturbances assumed (as in practical applications).
- Onset of AI associated with onset of turbulence.
- Incompressible flow and linear methods used with numerical solution.
Pseudo-routine for local CI and AI analyses

- We conduct local analyses at various points on a cone defined by $\psi$.
- Find basic steady flow at a particular location, then:
  1. Perturb the flow at that point.
  2. Consider the response in terms of the spatial growth rates along cone.
  3. Find region of neutral stability in terms of perturbation parameters.
- Repeat at next location, $R_L$, to generate neutral curve.
FORMULATION IN FIXED FRAME OF REFERENCE

Oncoming axial flow
STEADY-LAMINAR FLOW

- Slip condition at edge of boundary layer
  \[ U^* \rightarrow U^*_o(x^*) = C^*x^{*m} \]
- No-slip condition at cone surface \[ V^* = x^*\Omega^* \sin \psi \]
- Non-dimensionalizing velocities with surface velocity, will lead to parameter
  \[ T_s = \frac{C^*x^{*m}}{x^*\Omega^* \sin \psi} = \frac{C^*x^{*m-1}}{\Omega^* \sin \psi} \]
- \( m = 1 \) leads to \( T_s \) independent of position, but \( m \neq 1 \) leads to \( T_s(x^*) \).
- Leads to PDEs in terms of \( z^* \) and \( x^* \) for \( \psi \neq 90^\circ \).
- Use methods similar to Koh & Price (1967).
TRANSFORMATIONS

- Transformation made to define barred quantities as

\[
\bar{x}^* = \frac{1}{l^*^2} \int_0^{x^*} r^*_o^2 dx^*, \quad \bar{z}^* = \frac{r^*_o}{l^*} z^*,
\]

\[
\bar{U}^* = U^*, \quad \bar{V}^* = V^*, \quad \bar{W}^* = \frac{l^*}{r^*_o} \left( W^* + \frac{1}{r^*_o} \frac{dr^*_o}{dx^*} z^* U^* \right),
\]

where \( r^*_o = x^* \sin \psi \) is the local surface radius.

- Which are expressed in terms of the stream-function as

\[
\bar{\psi} = \left( \frac{6 \nu^* \bar{x}^* \bar{U}^*_e}{m + 3} \right)^{1/2} f(s, \eta_1), \quad \bar{U}^* \frac{\partial \bar{\psi}}{\partial \bar{z}^*}, \quad \bar{W}^* = -\frac{\partial \bar{\psi}}{\partial \bar{x}^*}.
\]

- With \( s = \frac{1}{T_s^2} \) and \( \eta_1 = \bar{z}^* \sqrt{\left( \frac{m+3}{6} \frac{\bar{U}^*_o}{\nu^* \bar{x}^*} \right)} \), defining the transformed coordinates for the system.
Some further mathematics leads to the PDEs

\[ f''' + ff'' + \frac{2m}{m + 3} (1 - f'^2) + \frac{2s}{m + 3} \left[ g^2 + 2(1 - m) \left( f'' \frac{\partial f}{\partial s} - f' \frac{\partial f'}{\partial s} \right) \right] = 0, \]

\[ g'' + fg' - \frac{4}{m + 3} f'g + \frac{4(1 - m)s}{m + 3} \left( g' \frac{\partial f}{\partial s} - f' \frac{\partial g}{\partial s} \right) = 0, \]

where a prime denotes differentiation with respect to \( \eta_1 \).

Subject the non-dimensional boundary conditions

\[ f = 0, \quad f' = 0, \quad g = 1 \quad \text{on} \quad \eta_1 = 0, \]

\[ f' \to 1, \quad g \to 0 \quad \text{as} \quad \eta_1 \to \infty. \]

This system is solved for \( f(\eta_1, s; \psi) \) and \( g(\eta_1, s; \psi) \) using NAG routine D03PEF.
Perturbation Equations

- The derivation of the perturbation equations is identical to Garrett & Peake (2007) (axial flow appears in the basic-flow equations only).
- Non-dimensionalize with length, velocity, pressure and time scales of $\delta^*, r_{o,s}^* \Omega^*, \rho^* r_{o,s}^2 \Omega^2$ and $\delta^*/\Omega^* r_{o,s}^*$ respectively.
- Leads to the local Reynolds number $R_L = \frac{x^*_s \Omega^* \delta^* \sin \psi}{\nu^*} = r_{o,s}$.
- Perturb the mean flow at each position with quantities of the form
  \[
  \hat{u}(\eta, x, t; R_L, \psi) = u(\eta; \psi) \exp(i(\alpha x + \beta R_L \theta - \gamma t)).
  \]
- Leads to a system of sixth-order perturbation equations after linearization and use of a parallel-flow-type approximation.
INCORPORATING THE BASIC-FLOW SOLUTIONS

- The steady-velocities in the local perturbation equations are

\[ U(\eta; x_s, \psi) = \frac{U^*}{x_s^* \Omega^* \sin \psi}, \quad V(\eta; x_s, \psi) = \frac{V^*}{x_s^* \Omega^* \sin \psi}, \quad W(\eta; x_s, \psi) = \frac{W^*}{(\nu^* \Omega^*)^{1/2}}, \]

- These are not the solutions in terms of \( f(\eta_1, s; \psi) \) and \( g(\eta_1, s; \psi) \), i.e. not \( \bar{U} \) etc.

- However, the connection is given by

\[ U = s^{-1/2} \frac{\partial f}{\partial \eta_1}(\eta_1, s), \quad V = g(\eta_1, s), \]

which are expressible in terms of \( \eta \) using the coordinate stretching for a fixed axial-flow parameter \( s \)

\[ \eta_1 = \eta \left( \frac{m + 3}{2s^{1/2} \sin \psi} \right)^{1/2}. \]
The full perturbation equations require $W$ in SLC-terms which we cannot easily connect to $f$ and $g$.

Preliminary analysis uses Orr–Sommerfeld equation which avoids $W$

$$\left[ i \left( D^2 - k^2 \right)^2 + R_L \left( \alpha U + \beta V - \gamma \right) \left( D^2 - \gamma^2 \right) - R_L \left( \alpha D^2 U + \beta D^2 V \right) \right] \phi_3 = 0,$$

where $k^2 = \alpha^2 + \beta^2$ is the effective wavenumber of the disturbance; $\phi_i$ are transformed perturbation variables ($i = 1, 2 \ldots 4$).

Imposing the boundary conditions

$$\phi_i = 0, \quad \eta = 0,$$

$$\phi_i \rightarrow 0, \quad \eta \rightarrow \infty,$$

forms an eigenvalue problem.
STABILITY ANALYSES

- Eigenvalue problem is solved to form the dispersion relation

\[ D(\alpha, \beta, \gamma; R_L, \psi) = 0. \]

- For the spatial CI analysis . . .
  - The phase speed of disturbances in the azimuthal direction is fixed via \( c = \gamma/\beta \) and we solve

\[ D(\alpha, \beta; c, R_L, \psi) = 0, \]

for \( \alpha \in \mathbb{C} \) at each \( \beta \in \mathbb{R} \).
  - \( \alpha_i < 0 \) indicates spatial CI.

- For the spatio-temporal AI analysis . . .
  - We solve

\[ D(\alpha, \beta, \gamma; R_L, \psi) = 0 \text{ s.t. } \frac{\partial \gamma}{\partial \alpha} = 0, \]

for \( \alpha, \gamma \in \mathbb{C} \) at each \( \beta \in \mathbb{R} \).
  - \( \gamma_i > 0 \) at a pinch point indicates AI.
CI results

Figure: Orr–Sommerfeld neutral curves for convective instability of stationary vortices for $\psi = 70^\circ$ with $s = \infty, 400, 25, 10 \& 5$, arrow indicates direction of decreasing $s$ (i.e. increasing axial flow).
AI RESULTS

Figure: Orr–Sommerfeld neutral curves for absolute instability of a cone rotating in otherwise still fluid \((s \to \infty)\) for \(\psi = 70^\circ\). Curves are shown for \(s = 10000, 1000, 400, 100\) and 50 from left to right (i.e. increasing axial flow).
OS VS. FULL PERTURBATION EQUATIONS
FURTHER WORK

- Interpretation of results
- Formulation of full model
- Better solution of steady flow (solve PDEs directly)
- High $R_L$-asymptotic comparisons (underway)
- Investigation of growth rates (underway)
- Experimental comparisons (intended joint project with KTH)
- Non-linear and global analyses
- Compressible flow, with heating (about to start)
Any questions?