$\mathbb{C}$–Seminar

Lectures given by Franz Pedit in 1999/2000

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Contents

Chapter 1

Introduction

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- holomorphic line bundles and sheaf theory from the point of view of differential geometry...

- introduction to techniques of algebraic geometry of Riemann surfaces getting more and more important in surface theory

- treatment of classical complex theory as a background for new quaternionic theory (maybe the main motivation) ★

(*"Globale Bemerkungen: Compact 2–dimensional = RS, Wann =, wann ≅, Voraussetzungen immer in den Satz, oder globale "Kapitelvoraussetzungen"?, Testleser? Sprachliche Feinheiten..,"*)

Chapter 2

Complex manifolds

The reader is assumed to be familiar with the basic notions on real and complex manifold theory: We only review briefly the basic definitions without going into details, for further references see for example [KN], [Wa], [FK], [We] and [Mi].

★ nach der neuesten Diskussionslage soll das komplett neu gestaltet werden: insbesondere konforme Klassen einführen, $J \leftrightarrow [g]$ Existenz isothermer Koordinaten, etc.

2.1 Almost complex structures and conformal structures

Recall

Definition 2.1. Let V be a real vector space. $J \in \mathrm{End}_{\mathbb{R}}(V)$ is called a *complex structure* if $J^2 = -Id$.

Clearly, each vector space which admits a complex structure is even dimensional and can be interpreted as complex vector space by defining the multiplication with a complex number $z = a + ib$ via $z \cdot v := av + bJv,\ v \in V$. This way we identify $\mathbb{R}^2$ together with the complex structure $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ with the complex vector space $(\mathbb{C}, i)$. Given two complex vector space (V, J_V) and (W, J_W) a map $A \in \mathrm{Hom}_{\mathbb{R}}(V, W)$ is complex linear if $AJ_V = J_W A$ (antilinear if $AJ_V = -J_W A$). We denote the space of complex linear (antilinear) homomorphims with $\mathrm{Hom}_+(V, W)$ (resp. $\mathrm{Hom}_+(V, W)$). Notice that there is a direct sum decomposition $\mathrm{Hom}(V, W) = \mathrm{Hom}_+(V, W) \oplus \mathrm{Hom}_-(V, W)$, where $A \in \mathrm{Hom}(V, W)$ can be uniquely written as $A = A_+ + A_-$ where $A_\pm = \frac{1}{2}(A \mp J_W A J_V) \in \mathrm{Hom}_\pm(V, W)$.

Consider smooth maps $f : U \overset{\text{open}}{\subset} \mathbb{C} \to V$ where (V, J) is a complex vector space. We have $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y}$ where $x, y : U \to \mathbb{R}$ are the coordinate functions. Introducing the complex coordinate $z = x + iy : U \to \mathbb{C}$ this reads as $df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar z}$, where $\frac{\partial f}{\partial z} := f_z := \frac{1}{2}(\frac{\partial f}{\partial x} - J\frac{\partial f}{\partial y})$ and $\frac{\partial f}{\partial z} := f_{\bar z} := \frac{1}{2}(\frac{\partial f}{\partial x} + J\frac{\partial f}{\partial y})$. On the other hand we can decompose the real linear map $d_p f \in \mathrm{Hom}(\mathbb{C}, V)$ in its complex linear and its complex antilinear part. Using $dz(J) = idz$ resp. $d\bar z(J) = -id\bar z$ one finds $(df)_+ = f_z dz$ and $(df)_- = f_{\bar z} d\bar z$.

Thus, $f : U \subset \mathbb{C} \to V$ is *holomorphic*, i.e. $d_p f \in \mathrm{Hom}(\mathbb{C}, V)$ is complex linear, if and only if $df = (df)_+$, i.e. $f_{\bar z} = 0$.

In particular, if $f : U \subset \mathbb{C} \to \mathbb{C}$, $f(x,y) = u(x,y) + iv(x,y) \equiv \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$, then f is holomorphic if and only if

$$\begin{aligned} f_{\bar z} = 0 \quad &\iff \quad \frac{\partial f}{\partial x} + J\frac{\partial f}{\partial y} = 0 \\ &\iff \quad \begin{pmatrix} u_x \\ v_x \end{pmatrix} = -J\begin{pmatrix} u_y \\ v_y \end{pmatrix} = \begin{pmatrix} v_y \\ -u_y \end{pmatrix} \\ &\iff \quad u_x = v_y,\ v_x = -u_y \text{ (these are the Cauchy–Riemann equations).} \end{aligned}$$

Definition 2.2. A $2m$–dimensional (real) manifolds M which is equipped with an *almost complex structure* J, i.e.:

$$J \in \Gamma(\mathrm{End}(TM)) \text{ with } J^2 = -I.$$

is called an *almost complex manifold*. Furthermore, a map $f : N \to M$ between almost complex manifolds (N, J_N) and (M, J_M) is called *almost complex* if $df \circ J_N = J_M \circ df$.

Example 2.3. 1. As simplest example we clearly have $\mathbb{C}^m$ with the almost complex structure J which is given by the multiplication with i, i.e.

$$J_p X := iX = \begin{pmatrix} i & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \\ 0 & \dots & & i \end{pmatrix} X,\ X \in T_p\mathbb{C}^m \cong \mathbb{C}^m.$$

Now, an almost complex linear map $f : U \subset \mathbb{C}^m \to \mathbb{C}^m$ satisfies $idf = dfi$, whence is a *holomorphic* map.

2. Let (M, g) be a 2–dimensional oriented Riemannian manifold. Let $X \in T_pM$ and define $JX \in T_pM$ by

$$g(X, JX) = 0,\ |X| = |JX|,\ X, JX \text{ are positive oriented if } X \neq 0.$$

This defines an almost complex structure on M.

Notice that if we have a conformal change of metric then we obtain the same complex structure.

Definition 2.4. A *conformal structure* on M is an equivalence class of conformally equivalent Riemannian metrics, where

$$g \overset{\text{conf}}{\sim} \tilde g \iff \text{there exists } u \in C^\infty(M) : \ \tilde g = e^{2u} g.$$

Theorem 2.5.

1. *Let (M, J) be an almost complex manifold. Then there exists a Riemannian metric g on M, such that J is* compatible *with g, i.e. for all $X \in T_pM$:*

$$g(JX, JX) = g(X, X). \tag{2.1}$$

Moreover, if (2.1) *then*

$$\begin{aligned} & g(JX, JY) = g(X, Y),\ g(JX, Y) = -g(X, JY) \\ \textit{and} \quad & g(JX, X) = 0 \ \textit{ for all } X, Y \in T_pM. \end{aligned}$$

2. *If* $\dim M = 2$, *then all such metrics are conformally equivalent.*

Proof.

1. Choose any Riemannian metric h on M (compare Lemma 3.20). Define g by $g(X, Y) := h(X, Y) + h(JX, JY)$. Check the details!

2. Sei $X \in T_pM$, $X \neq 0$. Then X, JX is a basis of T_pM and $g(X, X) = \rho\tilde{g}(X, X)$. Thus $g(JX, JX) = \rho\tilde{g}(JX, JX)$ and $g = \rho\tilde{g}$.

$\square$

Thus, we have a 1:1 correspondence:

$$\left\{ \text{almost complex structures on } M^2 \right\} \leftrightarrow \left\{ \begin{matrix} \text{conformal structures} \\ \text{and orientation} \end{matrix} \quad \text{on } M^2 \right\}$$

2.2 Conformal maps

Definition 2.6. Let M and N be manifolds with conformal structures. A smooth map $f\colon M \to N$ is called *conformal* if for all $p \in M$ and X, $Y \in T_pM \setminus \{0\}$:

$$\frac{|d_pf(X)|}{|X|} = \frac{|d_pf(Y)|}{|Y|}$$

where $|.|$ is defined by any metric of the conformal class.

Remark 2.7. 1. One can easily check that this is well–defined, that means independent of the chosen metric in the conformal class.

2. A smooth map $f\colon M \to N$ is conformal if and only if for all X, $Y \in T_pM \setminus \{0\}$ with $g_M(X, Y) = 0$ and $|X| = |Y|$: $|d_pf(X)| = |d_pf(Y)|$ and $g_N(d_pf(X), d_pf(Y)) = 0$.

   In particular, for $f\colon M \overset{\text{open}}{\subset} \mathbb{R}^2 \cong \mathbb{C} \to \mathbb{C}$ this reads as

$$|f_x| = |f_y|, \quad \langle f_x, f_y \rangle = 0.$$

3. A smooth map $f\colon M \subset \mathbb{C} \to \mathbb{C}$ is conformal if and only if for all $p \in M$ either $f_z(p) = 0$ or $f_{\bar z}(p) = 0$ since

$$\begin{aligned} 4\, f_z\, \overline{f_{\bar z}} &= (f_x - Jf_y)(\overline{f_x} - J\overline{f_y}) = |f_x|^2 - |f_y|^2 - J(f_x\overline{f_y} + f_y\overline{f_x}) \\ &= |f_x|^2 - |f_y|^2 - 2J\,\mathrm{Re}\,(f_x\overline{f_y}) = |f_x|^2 - |f_y|^2 - 2J\langle f_x, f_y\rangle. \end{aligned}$$

Definition 2.8. A map $f\colon M \subset \mathbb{C} \to \mathbb{C}$ is called *antiholomorphic*, if $f_z = 0$.

Lemma 2.9. *Any conformal map $f\colon M \to \mathbb{R}^2$ on $M \subset \mathbb{R}^2$ open and connected is holomorphic or antiholomorphic.*

Proof. First, recall that $f_{z\bar z} = f_{\bar z z} = 4\Delta f$. Thus on each of the open sets

$$\begin{aligned} M_+ &= \{\, p \in M \mid d_p f \neq 0,\ f_{\bar z}(p) = 0 \,\} \\ M_- &= \{\, p \in M \mid d_p f \neq 0,\ f_z(p) = 0 \,\} \\ M_0 &= \{\, p \in M \mid d_q f = 0,\ \text{for all } q \text{ in a neighborhood of } p \,\} \end{aligned}$$

the map $f_{z\bar z}$ is zero. But f is smooth and $M \subset \overline{M_+ \cup M_0 \cup M_-}$ which gives $f_{z\bar z} \equiv 0$. So f_z is holomorphic ($f_{\bar z}$ is antiholomorphic) and we can choose a holormophic function g with $g_z = f_z$ (a antiholomorphic function h with $h_{\bar z} = f_{\bar z}$). Thus $f = g + h$ (modulo constant). If $M_+ \neq \emptyset$. then $f_{\bar z} = h_{\bar z} = 0$ on M_+ but then h is constant on M_+ and since h is antiholomorphic also constant on M. Hence $f = g + \text{const}$, i.e. f is holomorphic. Similarly, f is antiholomorphic if $M_- \neq \emptyset$. ☐

Remark 2.10. We have shown that any conformal map is harmonic. Moreover, any harmonic map can be written as sum of a holomorphic and an antiholomorphic function.

2.3 Complex manifolds

In general, an almost complex manifold can not be made into a complex manifold. Recall,

Definition 2.11. A manifold M is a *complex manifold* if there exists an almost complex structure J on M and an atlas with charts (U, φ), $\varphi : U \to \mathbb{R}^{2m} = \mathbb{C}^m$ which are almost complex i.e. they satisfy

$$d\varphi \circ J = J_0 \circ d\varphi.$$

where the almost complex structures on $\mathbb{C}^m$ are identified by some fixed J_0.

If (M, J) is a complex manifold we call J the *complex structure* of M. A map $f : N \to M$ between complex manifolds is called *holomorphic* if $df(J_M) = J_N df$.

A (complex) one dimensional complex manifold is called a *Riemann surface*.

Remark 2.12. One can check that a manifold is complex if and only if the transition maps $\varphi \circ \psi^{-1}$ are holomorphic on $\mathbb{C}^m$ (where φ, ψ are chart maps).

Example 2.13. We will describe only as an example for a complex manifold the complex projective space $\mathbb{C}P^1$, further and more advanced examples of Riemann surfaces can be found for example in [FK].

Let $M = \mathbb{C}P^1$, i.e. for $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^2 \backslash \{0\}$ we define the equivalence class $[v] = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{C}P^1$ by

$$v \sim \tilde{v} \iff \exists \lambda \in \mathbb{C}^* = \mathbb{C}\backslash\{0\} : v = \tilde{v}\lambda.$$

Furthermore, we call $\begin{bmatrix} 1 \\ 0 \end{bmatrix} =$ "∞", $\begin{bmatrix} 0 \\ 1 \end{bmatrix} =$ "0". We have $\mathbb{C}P^1 = U_0 \cup U_\infty$ where

$$U_0 := \mathbb{C}P^1 \backslash \{\infty\},\ U_\infty := \mathbb{C}P^1 \backslash \{0\}$$

In any equivalence class $[v] \in U_0$ there is a unique representant of the form $v = \begin{pmatrix} v_1 \\ 1 \end{pmatrix} \in \mathbb{C}^2$ (and similarly for U_∞). Thus we can define *affine charts of* $\mathbb{C}P^1$ by

$$\varphi_0 : U_0 \to \mathbb{C}, \begin{bmatrix} z \\ 1 \end{bmatrix} \to z, \quad \varphi_\infty : U_\infty \to \mathbb{C}, \begin{bmatrix} 1 \\ w \end{bmatrix} \to w$$

We have $\varphi_0(U_0 \cap U_\infty) = \varphi_\infty(U_0 \cap U_\infty) = \mathbb{C}^*$. Hence M is a complex manifold since the transition map $\varphi_\infty \circ \varphi_0^{-1} : \mathbb{C}^* \to \mathbb{C}^*, z \to \frac{1}{z}$ is biholomorphic.

Notice that on $M = \mathbb{C}P^1$ there are no distinguished points. In particular, the points 0 and ∞ arise from our construction of $\mathbb{C}P^1$ but are not canonically given on $\mathbb{C}P^1$.

Reversing the construction choose arbitrary points p and q in $\mathbb{C}P^1$. Define $U_1 = \mathbb{C}P^1 \backslash \{p\}, U_2 = \mathbb{C}P^1 \backslash \{q\}$ and local coordinates $z : U_1 \to \mathbb{C}$ and $w : U_2 \to \mathbb{C}$ *centered* at p resp. q i.e. $z(p) = 0$ and $w(q) = 0$. ★

anständig zu Ende führen!

As mentioned above there is a condition on the almost complex structure in order to turn the almost complex manifold in a complex manifold. For manifolds of arbitrary dimension it is given by the well known

Theorem 2.14 ([NN])**.** *An almost complex manifold* (M, J) *is complex if and only if the* Nijenhuis *tensor*

$$N(X, Y) := 2([JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]) \tag{2.2}$$

vanishes.

Proof. A proof of this theorem can be found in Appendix 8 of part II of [KN] in the case that the almost complex structure is real analytic. Kohn's proof, [Ko], in the C^∞–case uses an analogue of the Hodge–Kodeira–decomposition theorem. □

In fact, we will see later in the case of a Riemann surface that the vanishing of the Nijenhuis tensor is exactly the condition that a certain operator gives a holomorphic structure on TM, cf. Examples 4.15. This way we will prove latter using elliptic operator theory, cf. , that in the special case of a two–dimensional manifold any almost complex structure is complex, without using the Newlander–Nirenberg theorem.

★ zitat anbringen

Theorem 2.15. *If $dim_{\mathbb{R}} M = 2$ then*

$$(M, J) \text{ is almost complex if and only if } (M, J) \text{ is complex.}$$

Proof. Using Theorem 2.14 the statement follows directly since

$$\begin{aligned} N(X, JX) &= 2([JX, J^2X] - [X, JX] - J[X, J^2X] - J[JX, JX]) \\ &= 2(\underbrace{-[JX, X] - [X, JX]}_{=0} + J([X, X] - [JX, JX])) = 0. \end{aligned}$$

□

Another way to prove the theorem above is to use Chern's [Ch] elementary proof of the existence of isothermic coordinates for 2–dimensional real manifolds. But this is equivalent to the existence of a complex structure:

Theorem 2.16. *Let M^2 be an oriented manifold. Then the following statements are equivalent*

1. *Any almost complex structure is complex.*
2. *Any Riemannian metric g on M is locally flat i.e. for all $p \in M$ there exists a neighborhood U and a map $u : U \to \mathbb{R}$ such that $\tilde{g} = e^{2u} g$ is flat on U.*
3. *For any Riemannian metric g and $p \in M$ there exists a neighborhood U of p and a conformal diffeomorphism $\varphi \colon U \to V \subset \mathbb{R}^2$.*
4. *For any Riemannian metric g and $p \in M$ there exists a neighborhood U of p, an open set $V \subset \mathbb{R}^2$ and a conformal diffeomorphism $\varphi \colon V \to U$ satisfying $|f_x|^2 = |f_y|^2$, $\langle f_x, f_y \rangle = 0$. φ is called an* isothermic *parametrization of M in p.*

Proof. (1) $\Longrightarrow$ (3): The Riemannian metric g induces by (1) a complex structure i.e. locally a biholomorphic map φ. But either φ or $\bar{\varphi}$ is conformal.

(3) $\Longleftrightarrow$ (4): If φ is a conformal diffeomorphism then φ^{-1} is also conformal.

(3) $\Longrightarrow$ (2): Use the by φ on U induced metric, which is conformally flat, and observe that the induced complex structures coincide which implies the conformal equivalence of the metrics.

(2) $\Longrightarrow$ (1): Use the locally given flat metric to define an isometry which can be chosen (change φ to $\bar{\varphi}$ if necessary) to be holomorphic. $\square$

Chapter 3

Vector bundles

3.1 Vector bundles and transition functions

For a differentiable vector bundle (E, M, π) we will frequently use the short–hand notation $E \to M$. Recall:

Definition 3.1. 1. A *complex vector bundle* of rank r is given by $E \to M$ where the *fibers* $E_p = \pi^{-1}(p)$ are r–dimensional $\mathbb{C}$–vector spaces and the *trivialization maps* are $\mathbb{C}$–linear on the fibers.

2. We denote the *trivial bundle* by $\underline{\mathbb{C}}^r := M \times \mathbb{C}^r$.

3. If $U \subset M$ is an open set, then we define the set of sections on U by

$$\Gamma(E_U) := \{s : U \to E, s \text{ smooth}, \ \pi \circ s = \mathrm{id}_U\}.$$

   In particular, we use the short–hand notation $\Gamma(E) = \Gamma(E_M)$ for the set of *global sections* of E.

4. A *line bundle* is a rank–1–bundle.

Remark 3.2. Clearly, the definition of vector bundles, and the following statements can also be given more generally, for example for an arbitrary field $\mathbb{K}$ or for vector bundles over the quaternions.

Sometimes it is more convenient to define a vector bundle by prescribing its transition functions. We give an outline of this construction.

Recall that a *vector bundle morphism* from the vector bundles (E_1, π_1) to (E_2, π_2) over M is a smooth map $\varphi : E_1 \to E_2$ which takes the fiber over p to the fiber over p and restricts on each fiber to a linear map $\varphi_p : (E_1)_p \to (E_2)_p$.

Definition 3.3. Let $\{U_i\}_{i \in I}$ be a *trivializing open cover* of a complex vector bundle $E \to M$, i.e. there are bundle isomorphisms $\Phi_i : E_{U_i} \to U_i \times \mathbb{C}^r$. The *transition functions* $g_{ij} : U_i \cap U_j \to \mathrm{GL}_r(\mathbb{C})$ are defined by

$$g_{ij}(p)(v) := \mathrm{pr}_2(\Phi_{i,p} \circ \Phi_{j,p}^{-1}(p, v)), \ p \in U_i \cap U_j, \ v \in \mathbb{C}^r,$$

where pr_2 denotes the projection of $U_i \times \mathbb{C}^r$ onto the second factor $\mathbb{C}^r$. We will most of the timee identify $g_{ij} = \Phi_i \Phi_j^{-1}$.

Corollary 3.4. *The transition functions of a complex vector bundle satisfy the cocycle condition*

$$g_{ij} \cdot g_{jk} = g_{ik} \quad \text{on } U_i \cap U_j \cap U_k. \tag{3.1}$$

In particular,

$$g_{ii} = \mathrm{id}, \;\; g_{ji} = g_{ij}^{-1}.$$

More generally, we give the

Definition 3.5. Let $\{U_i\}$ be an open cover of M and

$$g_{ij} : U_i \cap U_j \to \; \mathrm{GL}_r(\mathbb{C}) \text{ smooth functions.}$$

If $\{g_{ij}\}$ satisfy the cocycle condition (3.1) then we call $\{g_{ij}\}$ a *1–cocycle.*

Remark 3.6. We will later see the link to Čech cohomology theory, cf. Chapter B, Example B.7.

We may ask if a bundle is given uniquely by its transition functions.

Let $\tilde{\Phi}$ be another trivialization with respect to the same open cover of M, i.e. $\tilde{\Phi}_i : E_{U_i} \to U_i \times \mathbb{C}^r$. Φ and $\tilde{\Phi}$ induce a 0*–cocycle* $\{b_i\}$, i.e maps $b_i : U_i \to \; \mathrm{GL}_r(\mathbb{C})$, by

$$b_i(p) := \tilde{\Phi}_{i,p} \circ \Phi_{i,p}^{-1}$$

Then the transition functions transform with

$$\tilde{g}_{ij} = b_i g_{ij} b_j^{-1}. \tag{3.2}$$

This motivates to define the following equivalence relation: Two 1–cocycles $\{g_{ij}\}$ and $\{\tilde{g}_{ij}\}$ are equivalent if there exists a 0–cocycle $\{b_i\}, b_i : U_i \to \; \mathrm{GL}_r(\mathbb{C})$ with (3.2).

Theorem 3.7 (Uniqueness)**.** *Let $E, \tilde{E} \to M$ be complex vector bundles, and $\{U_i\}$ an open trivializing cover of E and $\tilde{E}$. Then*

$$E \cong \tilde{E} \iff \{g_{ij}\} \sim \{\tilde{g}_{ij}\}.$$

Proof. The correspondence between the isomorphism $\varphi : E \to \tilde{E}$ and the required 0–cocycle $\{b_i\}$ is given by

$$b_i(p) = \tilde{\Phi}_{i,p} \circ \varphi_p \circ \Phi_{i,p}^{-1},$$

where $\Phi, \tilde{\Phi}$ are the trivialization maps with respect to the cover $\{U_i\}$. □

Theorem 3.8 (Existence). *Let M be a manifold, $\{U_i\}$ an open cover and $\{g_{ij}\}$ a 1–cocycle with values in $GL_r(\mathbb{C})$. Then there exists a complex vector bundle of* rank r *with local trivialization maps such that the transition functions are exactly the g_{ij}'s.*

Proof. Define $E := \left(\bigcup_{i\in I} \{i\} \times (U_i \times \mathbb{C}^r)\right) / \sim$, where the equivalence relation is given by

$$(i, p, \xi) \sim (j, q, \eta) \iff p = q \in U_i \cap U_j,\ \xi = g_{ij}(p)\eta.$$

Define the projection map $\pi : E \to M$ by $\pi([i, p, \xi]) := p$ and a bijection $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^m$, $\psi_i([i, p, \xi]) := (p, \xi)$. There is a unique manifold structure on E which makes each ψ_i into a diffeomorphism. This turns π into a smooth submersion. ☐

Remark 3.9. We introduce some useful notations:

1. Given an open cover $\{U_i\}$ of M a section $s \in \Gamma(E)$ is in local coordinates given by
$$s_{U_i} = \mathrm{pr}_2 \circ \Phi_i \circ s|_{U_i} : U_i \to \mathbb{C}^r.$$
The behavior under change of coordinates is
$$s_{U_j}(p) = g_{ji}(p) s_{U_i}(p) \text{ for } p \in U_i \cap U_j,$$
where the g_{ij}'s are again the transition functions.

   This can be used to define global sections: Assume L to be a line bundle and $\{s_i\}$ fixed local sections which trivialize the line bundle over U_i. We write a global section s on M by
$$s = \{(U_i, f_i)\} \text{ where } f_i : U_i \to \mathbb{C} \text{ and } f_i = \mathrm{pr}_2 \circ \Phi_i \circ s_i$$
(sometimes even dropping the explicit specification of the trivialization in mind). Since s should be a global section observe that the f_i have to satisfy the compatibility condition
$$f_j = g_{ji} f_i.$$
Conversely, given $\{(U_i, f_i)\}$ where $f_i : U_i \to \mathbb{C}$ we like to define a line bundle via the transition functions $\{g_{ij}\}$, where $f_j = g_{ji} f_i$. Clearly this is possible if the $\{g_{ij}\}$'s satisfy the cocycle condition.

2. Let E be a complex vector bundle over M. Given a local *frame* $\underline{s} = (s_1, \ldots, s_r)$, i.e. $s_1, \ldots, s_r : U \to E$ linearly independent sections in $\Gamma(E_U)$, then an isomorphism $\Phi : E_U \to U \times \mathbb{C}^r$ is given by
$$\Phi^{-1}(p, \xi) = \sum_{i=1}^{r} s_i(p)\xi_i =: \underline{s}(p)\xi,\ p \in U,\ \xi \in \mathbb{C}^r.$$

3.2 Connections on vector bundles

Let M be a complex manifold of dimension m and let $E \to M$ be a complex vector bundle of rank r. We denote by $C^\infty(M)$ the set of smooth maps $f : M \to \mathbb{C}$ and by $\Omega^k(E)$ the set of smooth k–forms with values in E.

Definition 3.10. 1. A linear map $\nabla : \Gamma(E) \to \Omega^1(E)$ is a *connection* if it satisfies the Leibniz rule

$$\nabla(\psi f) = (\nabla\psi)f + \psi df, \quad f \in C^\infty(M),\ \psi \in \Gamma(E).$$

2. A connection which satisfies
$$\nabla J = 0$$
is called a *complex connection* (where $(\nabla J)\psi = \nabla(J\psi) - J(\nabla\psi)$ for $\psi \in \Gamma(E)$).

The *absolute exterior differential* $d^\nabla : \Omega^k(E) \longrightarrow \Omega^{k+1}(E)$ with respect to ∇ is defined by

$$(d^\nabla\omega)(X_0,\dots,X_k) := \sum_i (-1)^i \nabla_{X_i}(\omega(X_0,\dots,\hat{X}_i,\dots,X_k)) + \\ + \sum_{i<j} (-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k). \tag{3.3}$$

For $\psi \in \Omega^0(E) = \Gamma(E)$ we have $d^\nabla_X\psi = \nabla_X\psi$, and for $\omega \in \Omega^1(E) = \Gamma(T^*M \otimes E)$ the usual formula,

$$d^\nabla\omega(X,Y) = \nabla_X(\omega(Y)) - \nabla_Y(\omega(X)) - \omega([X,Y]) \tag{3.4}$$

holds. The *curvature tensor* $\mathcal{R}^\nabla \in \Omega^2(\mathrm{End}(E))$ is defined by $\mathcal{R}^\nabla := d^\nabla \circ d^\nabla|_{\Gamma(E)}$. Thus
$$\mathcal{R}^\nabla(X,Y)\psi = \nabla_X\nabla_Y\psi - \nabla_Y\nabla_X\psi - \nabla_{[X,Y]}\psi.$$

★ brauchen wir irgendwo Bianchi?

Examples 3.11. The usual constructions to create new vector spaces apply also on vector bundles: Let E and $\tilde{E}$ be complex vector bundles over M, $\{U_i\}$ an open cover, $\{g_{ij}\}$ resp. $\{\tilde{g}_{ij}\}$ the transition functions, J_E resp. $J_{\tilde{E}}$ almost complex structures and ∇_E resp. $\nabla_{\tilde{E}}$ connections of E resp. $\tilde{E}$. Then

1. $E \oplus \tilde{E}$ has transition functions $\begin{pmatrix} g_{ij} & 0 \\ 0 & \tilde{g}_{ij} \end{pmatrix}$, almost complex structure $J_{E\oplus\tilde{E}} = J_E \oplus J_{\tilde{E}}$ and connection $\nabla_{E\oplus\tilde{E}} = \nabla_E \oplus \nabla_{\tilde{E}}$.

2. $E \otimes \tilde{E}$ has transition functions $g_{ij} \otimes \tilde{g}_{ij}$ and almost complex structure $J_{E\otimes\tilde{E}}(\psi \otimes \varphi) = (J_E\psi) \otimes \varphi = \psi \otimes (J_{\tilde{E}}\varphi)$ (the latter identity is given by the usual construction of the tensor product over $\mathbb{C}$). Requiring a product rule, the connection $\nabla_{E\otimes\tilde{E}}$ on $E \otimes \tilde{E}$ is canonically given by

$$\nabla_{E\otimes\tilde{E}}\psi \otimes \varphi = (\nabla_E\psi) \otimes \varphi + \psi \otimes (\nabla_{\tilde{E}}\varphi), \quad \psi \otimes \varphi \in \Gamma(E \otimes \tilde{E}).$$

3. Denote the non–degenerate pairing between the dual bundle E^* and E by $<, >$, i.e.

$$< \alpha, \psi >:= \alpha(\psi), \;\; \alpha \in E^*, \psi \in E.$$

   The dual bundle E^* has transition functions $(g_{ij}^*)^{-1}$ and almost complex structure $J_{E^*} = J_E^*$ where the dual map is defined by the pairing $<, >$. The canonically induced connection ∇_{E^*} on E^* is given by

$$(\nabla_{E^*}\alpha)(\psi) = d(\alpha\psi) - \alpha\nabla_E\psi$$

   or, using the pairing,

$$d < \alpha, \psi >=< \nabla_{E^*}\alpha, \psi > + < \alpha, \nabla_E\psi >$$

   for any $\alpha \in \Gamma(E^*), \;\; \psi \in \Gamma(E)$.

   Notice, that for a line bundle L the bundle $L^* \otimes L$ is isomorphic to $\underline{\mathbb{C}}$. The identification is given by

$$L^* \otimes L \cong \mathrm{Hom}(L, L) = \mathrm{End}(L), \;\; \alpha \otimes \psi \mapsto (\varphi \mapsto < \alpha, \psi > \varphi)$$

   and

$$\mathrm{End}(L) \cong \underline{\mathbb{C}}, \;\; \mathrm{Id} \mapsto 1.$$

   In particular, the constructions in (2) and (3) coincide for line bundles. ★

stimmt das jetzt endlich?!

4. Using (2) and (3) $\mathrm{Hom}(E, \tilde{E}) = E^* \otimes \tilde{E}$ is a complex vector bundle with almost complex structure $(J_{\mathrm{Hom}(E,\tilde{E})}T)\psi = T(J_E\psi) = J_{\tilde{E}}T\psi$ and connection $(\nabla_{\mathrm{Hom}(E,\tilde{E})}T)\psi = \nabla_{\tilde{E}}(T\psi) - T(\nabla_E\psi)$ where $T \in \Gamma(\mathrm{Hom}(E, \tilde{E})), \; \psi \in \Gamma(E)$.

5. $\Lambda^k(E)$ has transition functions $\wedge^k g_{ij}$. Its complex structure is obtained by $J_{\Lambda^k(E)}(\psi_1 \wedge \ldots \wedge \psi_k) = (J_E\psi_1) \wedge \psi_2 \wedge \ldots \wedge \psi_k$ where $\psi_i \in E$. The connection $\nabla_{\Lambda^k(E)}$ on $\Lambda^k(E)$ is again given by the product rule $\nabla_{\Lambda^k(E)}(\psi_1 \wedge \ldots \wedge \psi_k) = \sum_i \nabla_{\Lambda^k(E)}(\psi_1 \wedge \ldots \wedge (\nabla_E\psi_i) \wedge \ldots \wedge \psi_k)$ for $\psi_i \in \Gamma(E)$.

   In particular, if E is a rank r vector bundle then $\Lambda^r(E)$ has the transition function $\det g_{ij}$. Notice, that $\Lambda^r(E)$ is a line bundle, it is called the *determinant bundle* of E.

   If $E = TM^*$ then we obtain the vector bundle of differentiable k–forms: $\Lambda^k(TM^*)$. The set of sections in this bundle is $\Omega^k(M) = \Gamma(\Lambda^k(TM^*))$. Moreover, $\Omega^k(E) = \Gamma(\Lambda^k(TM^*) \otimes E)$.

3.3 Example: the tautological bundle over $\mathbb{C}P^n$

Now, we define a complex line bundle over M by giving the transition functions

$$\begin{aligned} g_{\infty 0} : U_0 \cap U_\infty \to \ \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*, \qquad g_{\infty 0}(\begin{bmatrix} z \\ 1 \end{bmatrix}) = z, \\ g_{0\infty} : U_0 \cap U_\infty \to \mathbb{C}^*, \qquad g_{0\infty}(\begin{bmatrix} 1 \\ w \end{bmatrix}) = w. \end{aligned}$$

They satisfy the cocycle condition since

$$\begin{aligned} (g_{0\infty} g_{\infty 0})(\begin{bmatrix} z \\ 1 \end{bmatrix}) &= g_{0\infty}(\begin{bmatrix} z \\ 1 \end{bmatrix}) g_{\infty 0}(\begin{bmatrix} z \\ 1 \end{bmatrix}) \\ &= g_{0\infty}(\begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}) g_{\infty 0}(\begin{bmatrix} z \\ 1 \end{bmatrix}) = z^{-1} z = 1 \end{aligned}$$

and give hence a complex line bundle, the so–called *tautological bundle.*

On the other hand we can also give a line bundle over M by

$$\Sigma_{[v]} := ([v], \mathbb{C} \cdot v) \subset \mathbb{C}P^1 \times \mathbb{C}^2, \ v \in \mathbb{C}^2 \backslash \{0\}.$$

This gives also the tautological bundle (and so explains the name): Consider the local sections

$$s_0 : U_0 \to \Sigma_{U_0}, \ s_0(\begin{bmatrix} z \\ 1 \end{bmatrix}) = (\begin{bmatrix} z \\ 1 \end{bmatrix}, \begin{pmatrix} z \\ 1 \end{pmatrix}) \tag{3.5}$$

$$s_\infty : U_\infty \to \Sigma_{U_\infty}, \ s_\infty(\begin{bmatrix} 1 \\ w \end{bmatrix}) = (\begin{bmatrix} 1 \\ w \end{bmatrix}, \begin{pmatrix} 1 \\ w \end{pmatrix}) \tag{3.6}$$

which define, as we have seen before, isomorphism $\Phi_0 : \Sigma_{U_0} \to U_0 \times \mathbb{C}$ and $\Phi_\infty : \Sigma_{U_\infty} \to U_\infty \times \mathbb{C}$. Hence Σ is a complex line bundle. But, the transition function $g_{\infty 0} = \Phi_\infty \Phi_0^{-1} : U_0 \cap U_\infty \to \ \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$ is given by (without identifications)

$$\begin{aligned} g_{\infty 0}(\begin{bmatrix} z \\ 1 \end{bmatrix}) a &= \mathrm{pr}_2(\Phi_\infty \Phi_0^{-1}(\begin{bmatrix} z \\ 1 \end{bmatrix}, a)) = \mathrm{pr}_2(\Phi_\infty(\begin{bmatrix} z \\ 1 \end{bmatrix}, \begin{pmatrix} za \\ a \end{pmatrix})) \\ &= \mathrm{pr}_2(\Phi_\infty(\begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}, \begin{pmatrix} 1 \\ z^{-1} \end{pmatrix} za)) = \mathrm{pr}_2((\begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}, za)) \\ &= za, \ a \in \mathbb{C} \end{aligned}$$

i.e.

$$g_{\infty 0}(\begin{bmatrix} z \\ 1 \end{bmatrix}) = z,$$

★ which is the transition function of the tautological bundle.

geht holomorph!

Remark 3.12. As we will see later, compare Remark 5.19, any line bundle over $\mathbb{C}P^1$ is isomorphic to a line bundle whose transition functions are given by

$$g_{\infty 0} : U_0 \cap U_\infty \to \mathrm{GL}_1(\mathbb{C}), \; g_{\infty 0}(\begin{bmatrix} z \\ 1 \end{bmatrix}) = \frac{1}{z^k}, \; k \in \mathbb{Z}.$$

In fact, we will see that all of these line bundles are holomorphic line bundles.

As an application of the tautological bundle we can identify tangent vectors of $\mathbb{C}P^1$ with certain homomorphisms:

Lemma 3.13. *The map*

$$\hat{} : T_p\mathbb{C}P^1 \to \mathrm{Hom}_{\mathbb{C}}(\Sigma_p, \mathbb{C}^2/\Sigma_p), \quad v \mapsto \hat{v},$$

is a well–defined $\mathbb{R}$*–linear isomorphism, where* $\hat{v}$ *is defined by*

$$\hat{v}(\psi) := \pi_p d_p \tilde{\psi}(v), \; \textit{for } \psi \in \Sigma_p.$$

Here $\tilde{\psi}$ *is any local extension of* ψ *to a local section of* Σ*, and* π_p *the projection of* $\mathbb{C}^2$ *onto* $\mathbb{C}^2/\Sigma_p$*.*

Proof. 1. Let $\tilde{\psi} \in \Gamma(\Sigma\big|_U)$ be a local section with $\tilde{\psi}(p) = 0$ and $\varphi \in \Sigma\big|_U$ a local frame, i.e. $\varphi(q) \neq 0$ for all $q \in U$. Let $\alpha \in \Gamma(\Sigma^*\big|_U)$ such that $< \alpha, \varphi >= 1$. Then $\tilde{\psi} = \varphi\lambda$ where $\lambda :=< \alpha, \tilde{\psi} >\in C^\infty(U, \mathbb{C})$, $\lambda(p) = 0$. Now compute

$$\begin{aligned} \pi_p(d_p\tilde{\psi}(v)) &= \pi_p d_p(\varphi\lambda)(v) = \pi_p(d_p\varphi(v) \underbrace{\lambda(p)}_{=0} + \varphi(p) d_p\lambda(v)) \\ &= \underbrace{(\pi_p(\varphi(p))}_{=0} d_p\lambda(v) = 0. \end{aligned} \tag{3.7}$$

In particular, this shows that $\hat{v}(\psi)$ is defined independently of the local extension of ψ: let $\tilde{\psi}_1, \tilde{\psi}_2$ be two such extensions then $\pi_p(d_p\tilde{\psi}_1(v)) = \pi_p(d_p\tilde{\psi}_2(v))$ if and only if $\pi_p(d_p(\tilde{\psi}_1 - \tilde{\psi}_2)(v)) = 0$. But by assumption $\tilde{\psi} := \tilde{\psi}_1 - \tilde{\psi}_2$ has a zero at p, which gives the claim with (3.7). Furthermore, $\hat{v}$ is clearly a $\mathbb{C}$ linear map from Σ_p to $\mathbb{C}^2/\Sigma_p$, i.e. $\hat{}$ is a well–defined map from $T_p\mathbb{C}P^1$ to $\mathrm{Hom}_{\mathbb{C}}(\Sigma_p, \mathbb{C}^2/\Sigma_p)$.

2. The map $\hat{}$ is $\mathbb{R}$–linear, and also injective: Let $p \in \mathbb{C}P^1$ and let $z : U \to \mathbb{C}$ be an affine chart, i.e. $z^{-1}(w) = \begin{bmatrix} w \\ 1 \end{bmatrix}$. We choose the affine chart so that z is centered at p, i.e. $p \in U$ and $z^{-1}(0) = p$.

Assume that $v \in T_p\mathbb{C}P^1$ with $\hat{v} = 0$. Let $\xi = d_p z(v) \in \mathbb{C}$ and $\psi \in \Gamma(\Sigma\big|_U)$ be the section $\psi(\begin{bmatrix} w \\ 1 \end{bmatrix}) = \begin{pmatrix} w \\ 1 \end{pmatrix}$. Then

$$0 = \hat{v}(\psi_p) = \pi_p d_p \psi(v) = \pi_p d_0(\psi \circ z^{-1})(\xi) = \pi_p \begin{pmatrix} d_0 \operatorname{id}_{\mathbb{C}}(\xi) \\ 0 \end{pmatrix} = \pi_p \begin{pmatrix} \xi \\ 0 \end{pmatrix}.$$

This implies $\xi = 0$ since $\ker \pi_p = \mathbb{C}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, hence $v = 0$. By a dimension argument we are done.

□

Corollary 3.14. *The map* $\hat{}\; : T\mathbb{C}P^1 \to \operatorname{Hom}_{\mathbb{C}}(\Sigma, \underline{\mathbb{C}}^2/\Sigma),\; v \mapsto \hat{v}$ *is a smooth* $\mathbb{R}$*–linear bundle isomorphism.*

Proof. We already know that the above map is a $\mathbb{R}$–linear isomorphism fiberwise. Let U be a *trivializing open set* in $\mathbb{C}P^1$ for $T\mathbb{C}P^1$ as well as for Σ and $\underline{\mathbb{C}}^2/\Sigma$, i.e. all the bundles can be trivialized over U. Let X_1, X_2 be a frame for $T\mathbb{C}P^1\big|_U$, ψ and $\tilde{\psi}$ frames of $\Sigma\big|_U$ resp. $\underline{\mathbb{C}}^2/\Sigma\big|_U$. Then

$$\widehat{X_i(p)}(\psi(p)) = \tilde{\psi}(p)\eta_i(p) \text{ for some map } \eta_i : U \to \mathbb{C}.$$

Let $\alpha \in \Gamma((\underline{\mathbb{C}}^2/\Sigma)^*\big|_U)$ with $<\alpha, \tilde{\psi}>= 1$ then

$$\eta_i(p) =<\alpha_p, \tilde{\psi}(p)\eta_i(p)>=<\alpha_p, \widehat{X_i(p)}(\psi(p))>=<\alpha_p, \pi_p d_p\psi(X_i(p))>.$$

Thus $\eta_i : U \to \mathbb{C}$ is smooth since all maps on the right hand side depend smoothly on p. □

We only explained in detail $\mathbb{C}P^1$ and how to construct the tautological bundle Σ over $\mathbb{C}P^1$ but it is clear that the same construction can be done for arbitrary dimension n, i.e. we have $\Sigma \to \mathbb{C}P^n$ given by $\Sigma_{[p]} = ([p], \operatorname{span}_{\mathbb{C}} p) \subset \{[p]\} \times \mathbb{C}^{n+1}$ where $p \in \mathbb{C}^{n+1}$ and $[p] \in \mathbb{C}P^n$ is its equivalence class. By similar arguments as before, the tangential space of $\mathbb{C}P^n$ is given by $\operatorname{Hom}_{\mathbb{C}}(\Sigma, \underline{\mathbb{C}}^{n+1}/\Sigma)$.

Now, if one is interested in smooth maps f from a Riemann surface M to $\mathbb{C}P^n$, then there is a canonical associated line bundle given by $L := f^*\Sigma \subset \underline{\mathbb{C}}^{n+1}$, i.e. $L_p = (p, \Sigma_{f(p)}) \subset \{p\} \times \mathbb{C}^{n+1}$. Conversely, any line bundle $L \subset \underline{\mathbb{C}}^{n+1}$ over a Riemann surface M determines a smooth map $f : M \to \mathbb{C}P^n$ by $p \mapsto f(p) := \operatorname{pr}_2 L_p$. We sum up:

Proposition 3.15. *Let M be a Riemann surface. There is a one–to–one correspondence between*

$$\textit{maps } f : M \to \mathbb{C}P^n \qquad \textit{and} \qquad \textit{line bundles } L \subset \underline{\mathbb{C}}^{n+1}.$$

Remark 3.16. Again this generalizes to: a map from a Riemann surface into a Grassmannian corresponds to a vector subbundle in a trivial bundle.

Definition 3.17. Let $L \subset \underline{\mathbb{C}}^{n+1}$ be a line subbundle over M. We define the *derivative* δ *of* L in $\underline{\mathbb{C}}^{n+1}$

$$\delta_p(v)(\psi_p) := \pi_p(d_p\psi(v)), \ p \in M, \ v \in T_pM, \ \psi \in \Gamma(L),$$

where $\pi_p : \underline{\mathbb{C}}^{n+1} \to \underline{\mathbb{C}}^{n+1}/L_p$.

As for the map $v \mapsto \hat{v}$ one can show that $\delta \in \Gamma(\mathrm{Hom}_{\mathbb{R}}(TM, \mathrm{Hom}_{\mathbb{C}}(L, \underline{\mathbb{C}}^{n+1}/L)))$. In this sense, δ measures the movement of the bundle L in $\underline{\mathbb{C}}^{n+1}$. One easily verifies

Lemma 3.18. *Let $f : M \to \mathbb{C}P^n$ and let $L = f^*\Sigma$. Then*

1. $f^*T\mathbb{C}P^n \cong f^*\mathrm{Hom}_{\mathbb{C}}(\Sigma, \mathbb{C}P^n \times \mathbb{C}^{n+1}/\Sigma) = \mathrm{Hom}_{\mathbb{C}}(L, \underline{\mathbb{C}}^{n+1}/L)$ *(where $\underline{\mathbb{C}}^{n+1}$ is the trivial bundle over M!)*

2. $df \in \Gamma(\mathrm{Hom}_{\mathbb{R}}(TM, f^*(T\mathbb{C}P^n))) = \Gamma(\mathrm{Hom}_{\mathbb{R}}(TM, \mathrm{Hom}_{\mathbb{C}}(L, \underline{\mathbb{C}}^{n+1}/L)))$

3. *Under the map* $\widehat{\ }$ *the differential* df *corresponds to* δ, *i.e.*

$$\widehat{d_pf(v)} = \delta_p(v).$$

Proof. (1) and (2) are clear. Let $\psi \in \Gamma(L)$ then $\psi = \varphi \circ f$ for some $\varphi \in \Sigma$ and

$$\begin{aligned} \widehat{d_pf(v)}\psi(p) &= \widehat{d_pf(v)}\varphi(f(p)) = \pi_{f(p)}d_{f(p)}\varphi(d_pf(v)) \\ &= \pi_{f(p)}d_p(\varphi \circ f)(v) = \delta_p(v)(\psi)(p). \end{aligned}$$

□

3.4 Further tools: Partition of unity and transversality

For further statements on vector bundles we do need results on the existence of global non–vanishing sections and on the zeros of "generic" sections. An important ingredient to prove the existence of non–trivial sections in a vector bundle and similar problems as the existence of connections, hermitian metrics, is the following standard tool of topology:

Lemma and Definition 3.19. Let M be manifold and $\{U_i\}_{i\in I}$ an open cover of M. Then there exists *a partition of unity* subordinated to this cover i.e. maps $g_i : M \to \mathbb{R}, \ g_i \geq 0$, such that

1. $\operatorname{supp} g_i = \overline{\{p \in M \mid g_i(p) \neq 0\}} \subset U_i$,
2. for all $p \in M$ there exists a neighborhood $W_p \subset M$ so that
$$\#\{i \in I \mid \operatorname{supp} g_i \cap W_p \neq \emptyset\} < \infty$$
and
3. $\sum_{i \in I} g_i(p) = 1$ for all $p \in M$.

Note that this implies $\bigcup_{i \in I} \operatorname{supp} g_i = M$.

It is clear that this tool enables us to glue local data, which is obtained by the local trivialization of the vector bundle, together to global ones:

Lemma 3.20. *1. On every complex vector bundle $E \to M$ there exists a non–trivial section $\psi \in \Gamma(E)$.*

2. On every complex vector bundle $E \to M$ there exists a (complex) connection.

3. On every complex vector bundle $E \to M$ there exists an almost complex structure.

4. On every complex vector bundle $E \to M$ there exists a hermitian metric, *i.e. a smooth bilinear bundle map $<,>: E \times E \to \mathbb{C}$ which satisfies on each fiber*

(a) $< \psi\lambda, \varphi\mu >_p = \bar{\lambda} < \psi, \varphi >_p \mu$, for all $p \in M$, $\psi, \varphi \in E_p$ and $\lambda, \mu \in \mathbb{C}$.

(b) $< \psi, \varphi >_p = \overline{< \varphi, \psi >_p}$, for all $p \in M$, $\psi, \varphi \in E_p$.

(c) $< \psi, \psi >_p \geq 0$ and $< \psi, \psi >_p = 0 \iff \psi = 0$, for all $p \in M$, $\psi \in E_p$.

Strategy of proof. Use a trivialization of E to pull back the desired objects from $U \times \mathbb{C}^r$ to $E\big|_U$. The so obtained local objects can be glued together with a subordinated partition of unity. The details of the proof are left as excercise. $\square$

Moreover, if we prescribe local data of the vector bundle, in order to have global data, we need certain compatibility conditions. Clearly, given local sections on the open sets of an open cover there exists a global section extending those if the given sections coincide on the overlaps (use the partition of unity). Now, we ask if we can give local sections on the overlaps:

Lemma 3.21. *Let $\{U_i\}_{i\in I}$ be an open cover of M and $E \to M$ a complex vector bundle over M. Assume that there are given local section $\psi_{ij} : U_i \cap U_j \to E$. If the ψ_{ij}'s satisfy the cocycle condition*

$$\psi_{ij} + \psi_{jk} + \psi_{ki} = 0 \text{ on } U_i \cap U_j \cap U_k \tag{3.8}$$

then there exist sections $\varphi_i \in \Gamma(U_i, E)$ such that

$$\psi_{ij} = \varphi_i - \varphi_j \text{ on } U_i \cap U_j.$$

Remark 3.22. In our spirit of avoiding sheaf theory we have to formulate and prove statements directly which experts of sheaf cohomology clearly identify as special cases of their theory. For example this lemma is a special case of the statement that every fine sheaf has vanishing first cohomology, compare Appendix B.22.

Proof. Again, details are left as excercise: define φ_i by glueing ψ_{ik} together with a subordinated partition of unity, i.e. $\varphi_i = \sum g_k \psi_{ik}$, and use the cocycle condition. □

Furthermore, we are interested in the zeros of sections in a vector bundle. Differential topology provides us with results on the zeros of "generic" sections. For details on transversality compare for example [GP] and [Hi].

Definition 3.23. Let $f : M \to \tilde{N}$ differentiable and $N \subset \tilde{N}$ a submanifold. Then

1. f intersects N *transversally* if

$$d_p f(T_p M) + T_{f(p)} N = T_{f(p)} \tilde{N} \text{ for all } p \in f^{-1}(N).$$

2. If one considers the inclusion map incl : $M \to \tilde{N}$, one obtains as special case:

   Two submanifolds $M, N \subset \tilde{N}$ are *transverse* if

$$T_p M + T_p N = T_p \tilde{N} \text{ for all } \; p \in M \cap N.$$

3. Again this specializes to: Two sections $\psi, \tilde{\psi} \in \Gamma(E)$ in a vector bundle $E \to M$ are *transverse*, if for all $z = \psi(p) = \tilde{\psi}(p), p \in M$:

$$d_p \psi(T_p M) + d_p \tilde{\psi}(T_p M) = T_z E.$$

   (In particular: if two sections don't intersect at all then they are transverse!)

Theorem 3.24 (Special case of the transversality theorem)**.** *Given a section $\psi \in \Gamma(E)$, where E is a vector bundle over M, then any section $\varphi \in \Gamma(E)$ can be disturbed such that ψ and φ are transverse.*

Corollary 3.25. *Let M be a 2–dimensional compact manifold, $L \to M$ a complex line bundle. If $\psi \in \Gamma(L)$ intersects the* zero–section $0 \in \Gamma(L)$ *transversally then the zero–set $\psi^{-1}(0) = \{p \in M \mid \psi(p) = 0\}$ is finite.*

Proof. It suffices to show that the zero–set is discrete then it is finite by compactness of M.

Let $p_0 \in M$ with $\psi(p_0) = 0$ and $\Phi : E\big|_U \to U \times \mathbb{C}$ a local trivialization around p_0. This defines a map $f : M \to \mathbb{C}$ with $f(p_0) = 0$ by $\Phi \circ \psi(p) =: (p, f(p))$. By transversality

$$d_{p_0}\psi(T_{p_0}M) + d_{p_0}0(T_{p_0}M) = T_{0_{p_0}}L.$$

Thus the map df is surjective hence f is a local diffeomorphism around p_0. In particular, $f(q) \neq 0$ in a neighborhood of p_0 since $f(p_0) = 0$. □

Corollary 3.26. *If $E \to M$ is a vector bundle over a 2–dimensional manifold M of $r = \operatorname{rank} E \geq 2$ then there exists a nowhere vanishing section.*

Proof. By the transversality theorem we can assume that there exists a section $\psi \in \Gamma(E)$ which is transverse to the zero–section $0 \in \Gamma(E)$. Since $\dim d\psi_p(T_pM) = \dim d0_p(T_pM) = 2$ and $\dim T_pE = 2 + 2r > 4$ it is impossible that 0 and ψ intersect. □

Corollary 3.27. *If $E \to M$ is a complex vector bundle of* $\operatorname{rank} E \geq 2$ *then there exists a line bundle L such that*

$$E \cong_{C^\infty} \underline{\mathbb{C}}^{r-1} \oplus L.$$

Proof. Let ψ be a nowhere vanishing section of E. Define a line bundle $\tilde{L} := \operatorname{span}\psi$ which is isomorphic to $\underline{\mathbb{C}}$. Choose a hermitian form on E and define $\tilde{E}$ as the orthogonal complement of $\tilde{L}$ with respect to this hermitian form. Thus $E = \tilde{L} \oplus \tilde{E} \cong \underline{\mathbb{C}} \oplus \tilde{E}$ where $\tilde{E}$ is a complex vector bundle of rank $= r - 1$. This can be done inductively as long as $\operatorname{rank} \tilde{E} \geq 2$. Splitting the last factor we obtain a line bundle L. □

Corollary 3.28. *Let M be a 2–dimensional compact manifold and $E \to M$ a complex vector bundle over M of* $\operatorname{rank} E = r$. *If $E = \underline{\mathbb{C}}^{r-1} \oplus L$ then $L \cong \Lambda^r E$.*

Proof. Let $\psi_1, \ldots, \psi_{r-1}$ globally linearly independent sections framing $\underline{\mathbb{C}}^{r-1}$. Define $L \to \Lambda^r E$ by $v \in L_p \mapsto \psi_1(p) \wedge \ldots \wedge \psi_{r-1}(p) \wedge v \in (\Lambda^r E)_p$. This defines a bundle isomorphism. □

Chapter 4

Holomorphic vector bundles and holomorphic structures

We will consider holomorphic vector bundles, in particular we are interested in line bundles over Riemann surfaces. Hereby, we prefer to understand a holomorphic vector bundle as a vector bundle equipped with a certain operator, the $\bar{\partial}$–operator. In this chapter we explain how to construct this operator from holomorphic charts and vice versa.

4.1 Holomorphic vector bundles

We want to consider vector bundles which carry the structure of a complex manifold. We define

Definition 4.1. 1. A *holomorphic vector bundle* is a triple (E, M, π) with

   (a) E, M are complex manifolds.

   (b) $\pi : E \to M$ is a submersion such that the fiber $E_p = \pi^{-1}(p)$ is a $\mathbb{C}$–vector space.

   (c) For all $p_0 \in M$ there exist an open neighborhood $U \subset M$ and a local trivialization map, i.e. a holomorphic diffeomorphism

$$\begin{array}{ccc} \Phi : E|_U & \longrightarrow & U \times \mathbb{C}^r \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & U & \end{array}$$

   such that $\Phi_p : E_p \to \{p\} \times \mathbb{C}^r$ is $\mathbb{C}$–linear.

2. If $U \subset M$ is an open set, then we define the set of holomorphic sections on U by

$$H^0(E_U) := \{s : U \to E, s \text{ holomorphic}, \ \pi \circ s = \mathrm{id}_U\}.$$

Again we use the short–hand notation $H^0(E) = H^0(E_M)$ for the global holomorphic sections of E.

We construct vector bundles out of transition functions. We need a condition on the transition functions to see when this bundle is holomorphic:

Theorem 4.2. *Let $E \to M$ be a $\mathbb{C}$–vector bundle over a complex manifold M and $\{U_i\}$ an open cover. Then:*

$E \to M$ is a holomorphic vector bundle if and only if the transition functions $g_{ij} : U_i \cap U_j \to GL_r(\mathbb{C})$ are holomorphic.

Remark 4.3.

1. Assume L to be a holomorphic line bundle and $\{s_i\}$ fixed local holomorphic sections which trivialize the line bundle over U_i. Recall that if s is a global section on M we have, Remark 3.9, $s = \{(U_i, f_i)\}$ where $f_i : U_i \to \mathbb{C}$ and $f_i = \mathrm{pr}_2 \circ \Phi_i \circ s_i$.

   Thus s is holomorphic if and only if the f_i's are holomorphic.

   Conversely, given $\{(U_i, f_i)\}$ where $f_i : U_i \to \mathbb{C}$ are holomorphic we obtain a line bundle via $\{g_{ij}\}$, where $f_j = g_{ji} f_i$, if the $\{g_{ij}\}$'s satisfy the cocycle condition.

2. Let E be a holomorphic vector bundle over M. Given a local frame $\underline{s} = (s_1, \dots, s_r)$ then an isomorphism $\Phi : E_U \to U \times \mathbb{C}^r$ is given by

$$\Phi^{-1}(p, \xi) = \sum_{i=1}^{r} s_i(p)\xi_i =: \underline{s}(p)\xi, \; p \in U, \; \xi \in \mathbb{C}^r.$$

   The frame $\underline{s}$ is holomorphic if and only Φ is holomorphic.

Example 4.4 (The complex projective line)**.** Clearly, the tautological bundle Σ is a holomorphic line bundle over $\mathbb{C}P^1$: its transition functions are given by, cf. 3.3,

$$g_{\infty 0} : U_0 \cap U_\infty \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*, \qquad g_{\infty 0}(\begin{bmatrix} z \\ 1 \end{bmatrix}) = z,$$
$$g_{0\infty} : U_0 \cap U_\infty \to \mathbb{C}^*, \qquad g_{0\infty}(\begin{bmatrix} 1 \\ w \end{bmatrix}) = w.$$

which are holomorphic maps. Similarly, the tautological bundle over $\mathbb{C}P^n$ is a holomorphic bundle.

4.2 The canonical bundle

Let M be a n–dimensional complex manifold with complex structure J and $\pi : E \to M$ be a complex vector bundle over M. We define for a 1–form $\omega \in \Omega^1(E)$:

$$* \, \omega(X) := \omega(JX). \tag{4.1}$$

Definition 4.5. The bundles $\Lambda^{(p,q)}(M)$ are defined by:

1. For $(p,q) = (1,0)$ resp. $(p,q) = (0,1)$ the fibers of $\Lambda^{(p,q)}(M)$ are given by

$$\begin{aligned} \Lambda^{(1,0)}_{\tilde{p}}(M) &= \{\omega : T_{\tilde{p}}M \to \mathbb{C} \ \mathbb{R}\text{–linear}, *\omega = i\omega\} \\ \Lambda^{(0,1)}_{\tilde{p}}(M) &= \{\eta : T_{\tilde{p}}M \to \mathbb{C} \ \ \mathbb{R}\text{–linear}, *\eta = -i\eta\}, \end{aligned} \tag{4.2}$$

   where $\tilde{p} \in M$. We call

$$\begin{aligned} \omega \in \Gamma(\Lambda^{(1,0)}(M)) =: \Omega^{(1,0)}(M) \text{ a } (1,0)\textit{–form} \\ \eta \in \Gamma(\Lambda^{(0,1)}(M)) =: \Omega^{(0,1)}(M) \text{ a } (0,1)\textit{–form}. \end{aligned}$$

2. The (p,q)*–forms* are defined by:

$$\omega \in \Omega^{(p,q)}(M) \text{ if and only if locally } \omega = \sum_j \alpha_j \wedge \beta_j$$

   where $\alpha_j \in \Lambda^p(\Omega^{(1,0)}(M))$, $\beta_j \in \Lambda^q(\Omega^{(0,1)}(M))$.

3. The *canonical bundle* resp. *anti–canonical bundle* are given by

$$K := \Lambda^{(n,0)}(M) \text{ resp. } \bar{K} := \Lambda^{(0,n)}(M).$$

For the rest of this chapter we assume M to be a Riemann surface with complex structure J.

Remark 4.6. We restrict our considerations to Riemann surfaces only for convenience; notice, that most of the statements can be made similarly for complex manifolds with arbitrary dimension — in particular, a $\bar{\partial}$–operator can be defined, also.

The $*$–operator defined by (4.1) over a Riemann surface is then the negative of the usual HODGE star operator (which we will denote by $\star$), compare Definition 6.8.

Remark that now

$$K_p = \{\omega : T_pM \to \mathbb{C} \text{ linear}, *\omega = i\omega\},\ \bar{K}_p = \{\eta : T_pM \to \mathbb{C} \text{ linear}, *\eta = -i\eta\}.$$

In particular, any $(1,0)$–form can be written locally as $\omega = fdz$ (resp. a $(0,1)$–form as $\eta = fd\bar{z}$) for some C^∞ function f which implies that $\Lambda^{(2,0)} = \Lambda^{(0,2)} = \{0\}$ (compare "Type argument", cf. Corollary 4.7).

Now, we define the complex bundles KE and $\bar{K}E$ by the fibers

$$\begin{aligned} KE_p &:= \{\omega : T_pM \to E, *\omega = J_{\Lambda^1(E)}\omega\}, \\ \text{resp. } \bar{K}E_p &:= \{\eta : T_pM \to E, *\eta = -J_{\Lambda^1(E)}\eta\}. \end{aligned}$$

where the complex structure $J_{\Lambda^1(E)}$ on $\Lambda^1(E)$ is defined via

$$(J_{\Lambda^1(E)}\tilde{\omega})(X) := J_E(\tilde{\omega}(X)),\ \tilde{\omega} \in \Lambda^1(E).$$

The bundles KE and $\bar{K}E$ are complex vector bundles with almost complex structures $J_{KE} := J_{\Lambda^1(E)}|_{KE},\ J_{\bar{K}E} := J_{\Lambda^1(E)}|_{\bar{K}E}$. We see this by identifying KE with the tensor product $K \otimes E$, via $\tilde{\omega}(X) := (\omega \otimes \psi)(X) := \omega(X)\psi$.

In the following we will identify the appearing complex structures in the following sense ($\omega(X)$ is complex linear):

$$\begin{aligned} \omega(X)(J_E\psi) &= J_E(\omega(X)\psi) \\ &= (J_{KE}(\tilde{\omega}))(X) \\ &= (*\tilde{\omega})(X) \\ &= (\omega \otimes \psi)(JX) \\ &= (\omega(JX))\psi \\ &= (i\omega(X))\psi \\ &= ((i\omega) \otimes \psi)(X). \end{aligned}$$

Corollary 4.7 ("Type argument")**.** *Let $E, \tilde{E}$ be vector bundles over M with a fiberwise bilinear map $\cdot : E \times \tilde{E} \to \mathbb{C}$. For any $\omega \in \Gamma(KE),\ \tilde{\omega} \in \Gamma(K\tilde{E}),\ \eta \in \Gamma(\bar{K}E),\ \tilde{\eta} \in \Gamma(\bar{K}\tilde{E})$:*

$$\omega \wedge \tilde{\omega} = 0, \quad \eta \wedge \tilde{\eta} = 0,$$

where

$$\omega \wedge \tilde{\omega}(X, Y) = \omega(X) \cdot \tilde{\omega}(Y) - \omega(Y) \cdot \tilde{\omega}(X).$$

Furthermore, if $\cdot$ is non–degenerate, then $\omega \wedge \tilde{\eta} = 0$ if and only if $\omega = 0$ or $\tilde{\eta} = 0$.

Proof.

$$\begin{aligned}\omega \wedge \tilde{\omega}(X, JX) &= \omega(X) \cdot \tilde{\omega}(JX) - \omega(JX) \cdot \tilde{\omega}(X) \\ &= J\omega(X) \cdot \tilde{\omega}(X) - J\omega(X) \cdot \tilde{\omega}(X) = 0\end{aligned}$$

Similarly, $\omega \wedge \tilde{\eta}(X, JX) = -2J\omega(X) \cdot \tilde{\eta}(X)$. □

Remark 4.8. In general, $\Lambda^n(TM^*, \mathbb{C}) = \bigoplus_{p+q=n} \Lambda^{(p,q)}(M)$. Using this in the case of a Riemann surface M where by type $\Lambda^{(2,0)}(M) = \Lambda^{(0,2)}(M) = \{0\}$ we identify

$$\bar{K}K \cong \Lambda^2(TM^*, \mathbb{C}).$$

via the isomorphism

$$\omega \otimes \eta \mapsto \omega \wedge \eta. \tag{4.3}$$

4.3 Holomorphic structures

Our aim is now to equip a holomorphic vector bundle with an $\bar{\partial}$–operator such that $H^0(E) = \ker \bar{\partial}\big|_{\Gamma(E)}$. Vice versa, given such an $\bar{\partial}$–operator on a complex vector bundle E we will turn E into a holomorphic vector bundle.

First, recall the definition of the $\bar{\partial}$–operator for complex–valued functions:

$$\bar{\partial} : C^\infty(M, \mathbb{C}) \to \Gamma(\bar{K}),\ \bar{\partial} f := \frac{1}{2}(df + i * df),$$

where M is a Riemann surface. Of course there is a natural extension to maps $g : M \to \mathrm{GL}_r(\mathbb{C})$.

Now, any 1–form splits into a K– and a $\bar{K}$–component; computing the $\bar{K}$–component of *df* we obtain exactly $\bar{\partial} f$. The K–component gives also an operator $\partial : C^\infty(M, \mathbb{C}) \to \Gamma(K)$.

The significance of the $\bar{\partial}$–operator in our considerations is clearly the following property: f is a holomorphic function if and only if $\bar{\partial} f = 0$.

Now, let $E \to M$ be an holomorphic vector bundle. We like to define an operator $\bar{\partial} : \Gamma(E) \to \Gamma(\bar{K}E)$ which again characterizes holomorphic sections of the bundle. For a holomorphic frame $\underline{s} = (s_1, \ldots, s_r)$ on $E_U, U \subset M$ open, $s_i : U \to E$, any section $\psi \in \Gamma(E_U)$ is given by

$$\psi = \underline{s}\xi, \text{ where } \xi : U \to \mathbb{C}^r.$$

Define

$$\bar{\partial}\psi := \underline{s}(\bar{\partial}\xi).$$

$\bar{\partial}$ is well defined, i.e. it does not depend on the choice of the frame $\underline{s}$: If $\underline{\tilde{s}}$ is another holomorphic frame, then there exists a holomorphic map $g : U \to$

$\mathrm{GL}_r(\mathbb{C})$ with $\underline{\tilde{s}} = \underline{s}g$, i.e. $\tilde{s}_i = \sum_{j=1}^n s_j g_{ji}$. Notice that $*dg = idg$ again is equivalent to $\bar{\partial}g = 0$.

Now, if $\psi = \underline{s}\xi$ is an arbitrary section, then

$$\psi = \underline{\tilde{s}}\tilde{\xi}, \text{ where } \tilde{\xi} = g^{-1}\xi.$$

Hence

$$\underline{\tilde{s}}(\bar{\partial}\tilde{\xi}) = \underline{s}g\bar{\partial}(g^{-1}\xi) = \underline{s}g((\bar{\partial}g^{-1})\xi + g^{-1}\bar{\partial}\xi) = \underline{s}(\bar{\partial}\xi).$$

The $\bar{\partial}$–operator defined above satisfies the equation

$$\bar{\partial}(\psi\lambda) = (\bar{\partial}\psi)\lambda + \psi\bar{\partial}\lambda,$$

where $\lambda : U \to \mathbb{C}, U \subset M$ open. Furthermore, $\ker \bar{\partial}|_{\Gamma(E_U)} = H^0(E_U)$.

In general, we give the following

Definition 4.9. A *holomorphic structure* on a complex vector bundle $E \to M$ is given by a *$\bar{\partial}$–operator*, i.e. a map

$$\bar{\partial} : \Gamma(E) \to \Gamma(\bar{K}E),$$

satisfying

$$\bar{\partial}(\psi\lambda) = (\bar{\partial}\psi)\lambda + \psi(\bar{\partial}\lambda),$$

for all $\lambda : M \to \mathbb{C}$.

Remark 4.10. Both, the operator $\bar{\partial}$ on functions $f : M \to \mathbb{C}$ (considering f as a section in the trivial bundle) and the operator $\bar{\partial}$ on sections of a holomorphic vector bundle $E \to M$, are $\bar{\partial}$–operators. Note, that for a given complex vector bundle with $\bar{\partial}$–operator we can define further $\bar{\partial}$–operators by $\bar{\partial} + \omega$ with $\omega \in \Gamma(\bar{K}\operatorname{End}(E))$ (check that this defines in fact an $\bar{\partial}$–operator!). Applying this to the trivial bundle $\underline{\mathbb{C}}$ and the constant map $1 : M \to \mathbb{C}$, $p \mapsto 1$ we see that $(\bar{\partial} + \omega)1 = \omega 1 = \omega \neq 0$ for $\omega \in \Gamma(\bar{K}\underline{\mathbb{C}}) = \Gamma(\bar{K})$, $\omega \neq 0$. In particular, for this $\bar{\partial}$–operator, "holomorphic sections", i.e. maps $f : M \to \mathbb{C}$ with $(\bar{\partial} + \omega)f = 0$, are no more holomorphic functions in the usual sense. That means, the trivial bundle, and so any holomorphic vector bundle, could be equipped with different holomorphic structures depending on the chosen $\bar{\partial}$–operator. In general, unless otherwise stated, we will equip the trivial bundle always with the holomorphic structure defined by the usual $\bar{\partial}$–operator on functions.

Note also, that any two $\bar{\partial}$–operators on E differ by $\omega \in \Gamma(\bar{K}\operatorname{End}(E))$.

Definition 4.11. Let $E \to M$ be a complex vector bundle over a Riemann surface with holomorphic structure $\bar{\partial}$. Define the DOLBEAULT *cohomology groups*

$$H^0_{\bar{\partial}}(E) := \ker \bar{\partial}\big|_{\Gamma(E)},$$

and

$$H^1_{\bar{\partial}}(E) := \Gamma(\bar{K}E)/\bar{\partial}\Gamma(E).$$

Our aim is to turn each complex vector bundle E with $\bar{\partial}$–operator into a holomorphic vector bundle such that we can recover $H^0_{\bar{\partial}}(E)$ as the set of holomorphic sections of E, i.e. $H^0_{\bar{\partial}}(E) = H^0(E)$ while $H^1_{\bar{\partial}}(E)$ will be the dual space of $H^0(KE)$ (Serre duality theorem, compare 6.24). We skip the subscript $\bar{\partial}$ if it is clear which $\bar{\partial}$–operator we have in mind.

First, we give a further example:

Example 4.12. Let ∇ be a complex connection on the complex vector bundle (E, J). We can decompose ∇ into $\nabla = \nabla' + \nabla''$ where:

$$\nabla' = \frac{1}{2}(\nabla - J * \nabla),\ \nabla'' = \frac{1}{2}(\nabla + J * \nabla).$$

Notice that $\nabla' : \Gamma(E) \to \Gamma(KE)$ and $\nabla'' : \Gamma(E) \to \Gamma(\bar{K}E)$. In fact, this corresponds to a decomposition of $\Omega^1(E)$ into

$$\Omega^1(E) = \Gamma(KE) \oplus \Gamma(\bar{K}E).$$

Now, ∇'' is a $\bar{\partial}$–operator:

$$*\nabla'' = \frac{1}{2}(*\nabla - J\nabla) = -\frac{i}{2}(J * \nabla + \nabla) = -J\nabla''$$

and

$$\begin{aligned} 2\nabla''(\psi\lambda) &= \nabla(\psi\lambda) + J * \nabla(\psi\lambda) \\ &= (\nabla\psi)\lambda + \psi d\lambda + J(*\nabla\psi)\lambda + J\psi * d\lambda \\ &= 2(\nabla''\psi)\lambda + 2\psi\bar{\partial}\lambda. \end{aligned}$$

Remark 4.13. In fact, any $\bar{\partial}$–operator can be completed to a complex connection ∇ such that $\nabla'' = \bar{\partial}$ (later we will see, that for line bundles even flat connections can be obtained). Observe also that if we equip E with the complex structure $-J$ then ∇' gives a holomorphic structure on $(E, -J)$.

We take this as occasion to define

Definition 4.14. 1. Let E be a complex vector bundle over M with complex structure J. Then we denote by $\bar{E}$ the bundle E equipped with the complex structure $-J$.

2. An operator $\partial : \Gamma(E) \to \Gamma(KE)$ gives an *anti–holomorphic structure* on E if ∂ is an holomorphic structure on $\bar{E}$.

We proceed in collecting those examples of holomorphic structures we will use later frequently:

Examples 4.15. 1. Recall the identification $\bar{K}K = \Omega^2(M, \mathbb{C})$ via $\eta \otimes \omega \mapsto \eta \wedge \omega$. We define a $\bar{\partial}$–operator on K, i.e. $\bar{\partial} : \Gamma(K) \to \Gamma(\bar{K}K) = \Omega^2(M, \mathbb{C})$ by

$$\bar{\partial}\omega = d\omega.$$

This gives in fact a holomorphic structure:

$$\begin{aligned}\bar\partial(\omega\lambda) &= d(\omega\lambda) = (d\omega)\lambda - \omega\wedge d\lambda = (d\omega)\lambda - \omega\wedge(\bar\partial\lambda + \partial\lambda)\\ &\overset{\omega\in\Gamma(K)}{=} (d\omega)\lambda - \omega\wedge(\bar\partial\lambda) = (\bar\partial\omega)\lambda + \bar\partial\lambda\otimes\omega.\end{aligned}$$

Written in local coordinates we obtain for $\omega = f dz$:

$$\bar\partial(\omega) = d(f dz) = f_{\bar z} d\bar z \wedge dz,$$

i.e. ω is a holomorphic section in $\Gamma(K)$ if and only if $f_{\bar z} = 0$. This gives

$$H^0(K) = \{\textit{holomorphic 1–forms} \text{ on } M\}. \tag{4.4}$$

2. Let E_1 and E_2 be a complex vector bundles over M with holomorphic structures $\bar\partial_1$ and $\bar\partial_2$. Define an operator $\bar\partial$ on $E = E_1 \otimes E_2$ by

$$\bar\partial(\psi\otimes\varphi) = \bar\partial_1\psi\otimes\varphi + \psi\otimes\bar\partial_2\varphi.$$

First, $\bar\partial(\psi\otimes\varphi) \in \Gamma(\bar K E)$ for all $\psi\otimes\varphi \in \Gamma(E)$. Clearly, $\bar\partial$ is $\mathbb{C}$ linear and

$$\begin{aligned}\bar\partial((\psi\otimes\varphi)\lambda) &= \bar\partial(\psi\otimes(\varphi\lambda)) = (\bar\partial_1\psi)\otimes\varphi\lambda + \psi\otimes((\bar\partial_2\varphi)\lambda + \varphi\bar\partial\lambda)\\ &= (\bar\partial(\psi\otimes\varphi))\lambda + (\psi\otimes\varphi)\bar\partial\lambda.\end{aligned}$$

Hence $\bar\partial$ gives a holomorphic structure on E.

★ check

3. Again (2) gives a holomorphic structure on $\mathrm{Hom}(E,\tilde E) = E^*\otimes\tilde E$ by

$$(\bar\partial T)\psi = \bar\partial(T\psi) - T(\bar\partial\psi),\ T\in\Gamma(\mathrm{Hom}(E,\tilde E)),\ \psi\in\Gamma(E).$$

4. Given $\bar\partial$ on the complex vector bundle E there is a unique $\bar\partial$ operator on E^* such that

$$\bar\partial <\alpha,\psi> = <\bar\partial\alpha,\psi> + <\alpha,\bar\partial\psi>, \text{ where } \alpha\in\Gamma(E^*), \psi\in\Gamma(E). \tag{4.5}$$

(Here $<\alpha,\psi>$: $M\to\mathbb{C}$ and $\bar\partial$ is the usual holomorphic structure on functions).

To define $\bar\partial\alpha$ by (4.5) for $\alpha\in\Gamma(E^*)$ one has to check that (4.5) is tensorial in ψ. It is also easy to verify that $\bar\partial$ has values in $\Gamma(\bar K E^*)$ and satisfies the required product rule.

Again, consider a holomorphic line bundle $L\to M$ and consider the trivial bundle $\underline{\mathbb{C}}$ together with the usual holomorphic structure $\bar\partial\varphi = \frac{1}{2}(d\varphi + i*d\varphi)$, $\varphi : M\to\mathbb{C}$. Hence (4.5) is nothing else but the condition that the $\bar\partial$ operators on L and L^* give the canonical holomorphic structure on $\underline{\mathbb{C}}$ by the tensor product construction in (2).

5. As a consequence of (3) there is a unique $\bar\partial$ operator on K^* where K is the canoncial bundle over a Riemann surface M equipped with the holomorphic structure $\bar\partial\alpha = d\alpha,\ \alpha \in \Gamma(K)$.

   Since $<,>$: $TM \times K \to \mathbb{C}, (X,\alpha) \mapsto \alpha(X)$ is a non degenerate pairing between TM and K we can identify TM with the complex dual bundle of K, i.e. $TM = K^*$.

   For $\omega \in \Gamma(K)$ define $\eta(X,Y) = \frac{1}{2}(d\omega(X,Y) + id\omega(JX,Y))$ for $X, Y \in \Gamma(TM)$. Check, that $\eta \in \Gamma(\bar K K)$ (use $d\omega(JX,Y) = -d\omega(Y,JX)$ which comes from the fact that $d\omega$ is a multiple of the volume form and the corresponding equation for the volume form).

   Consider now the induced two–form

$$\begin{aligned}\eta(X,JX) &= \frac{1}{2}(d\omega(X,JX) + id\omega(JX,JX)) - \frac{1}{2}(d\omega(JX,X) + id\omega(J^2X,X))\\ &= d\omega(X,JX).\end{aligned}$$

   But $d\omega$ is the two–form which is induced by $\bar\partial\omega$ thus $\eta = \bar\partial\omega$. Since

$$\bar\partial_X < Y,\omega > - < \bar\partial_X Y,\omega > = < Y, \bar\partial_X\omega >$$

   and

$$\begin{aligned}< Y,\bar\partial_X\omega > &= \bar\partial\omega(X,Y) = \eta(X,Y) = \frac{1}{2}(d\omega(X,Y) + id\omega(JX,Y))\\ &= \frac{1}{2}(X\omega(Y) - Y\omega(X) - \omega([X,Y])\\ &\qquad + i(JX)\omega(Y) - iY\omega(JX) - i\omega([JX,Y]))\\ &= \frac{1}{2}(X\omega(Y) - Y\omega(X) - \omega([X,Y])\\ &\qquad + i(JX)\omega(Y) - i^2Y\omega(X) - \omega(J[JX,Y]))\\ &= \frac{1}{2}(X\omega(Y) + i(JX)\omega(Y)) - \frac{1}{2}(\omega([X,Y]) - J[JX,Y]))\\ &= \bar\partial_X(\omega(Y)) - < \frac{1}{2}([X,Y] + J[JX,Y]),\omega >\\ &= \bar\partial_X < Y,\omega > - < \frac{1}{2}([X,Y] + J[JX,Y]),\omega >\end{aligned}$$

   one gets

$$\bar\partial_X Y = \frac{1}{2}([X,Y] + J[JX,Y]). \tag{4.6}$$

   Notice, that with (4.6) the Nijenhuis tensor of M (cf. 2.2) becomes

$$\begin{aligned}-\frac{1}{4}N(X,Y) &= -\frac{1}{2}([JX,JY] - [X,Y] - J[X,JY] - J[JX,Y])\\ &= \bar\partial_X Y - \bar\partial_{JX} JY.\end{aligned}$$

   In particular, $N = 0$ if and only if $\bar\partial$ is a holomorphic structure.

6. Let E be a complex vector bundle with holomorphic structure $\bar{\partial}$. Assume that $L \subset E$ is a $\bar{\partial}$ *stable* complex subbundle, i.e. $\bar{\partial}\psi \in \Gamma(\bar{K}L)$ for all $\psi \in \Gamma(L)$. Then there is an induced holomorphic structure on E/L given by
$$\bar{\partial}(\psi \operatorname{mod} L) := (\bar{\partial}\psi) \operatorname{mod} L.$$

7. Let $f : M \to N$ be a holomorphic map between complex manifolds M and N, E a complex vector bundle over N with holomorphic structure $\bar{\partial}$ and $F = f^*E$ be the pullback of E. We define the holomorphic structure on F by
$$(\bar{\partial}_X \psi)_p = (\bar{\partial}_{df(X)}\psi)_{f(p)},\ p \in M, X \in T_pM, \psi \in \Gamma(F).$$

8. Let M be a Riemann surface, $f : M \to \mathbb{C}P^n$ a holomorphic map and $L = f^*\Sigma$ the associated line bundle. Then there is a unique holomorphic structure on $L^{-1} = (f^*\Sigma)^{-1}$ such that $\alpha\big|_L \in H^0(L^{-1})$ for all $\alpha \in (\mathbb{C}^{n+1})^*$.

★ baue alles zusammen. Wann ist f holomorphe Abbildung?

Let $\alpha_i \in (\mathbb{C}^{n+1})^*$ the coordinate maps.

★ erläutern

4.4 Existence of (local) holomorphic sections

Now, our aim is to turn any complex vector bundle E with holomorphic structure $\bar{\partial}$ into a holomorphic vector bundle such that $\ker \bar{\partial}_U = H^0(E_U)$.

For this purpose we first claim:

★ Das muß noch alles geglättet werden: ein Job für Franz!

Proposition 4.16. *Given a holomorphic structure $\bar{\partial} : \Gamma(E) \to \Gamma(KE)$ there exists locally a flat complex connection ∇ on E such that*
$$\nabla'' = \bar{\partial}.$$

Proof. Given any connection ∇ and a local frame $\underline{s}$ we have
$$\nabla\underline{s} = \underline{s}\omega,\ \text{where } \omega \in \Omega^1(U, \mathfrak{gl}_r(\mathbb{C})) \text{ is the } \textit{(local) connection form.}$$

Now, ∇ is flat connection if and only if
$$\mathcal{R}^\nabla = 0 \iff d\omega + \omega \wedge \omega = 0.$$

Since ω decomposes into $\omega = \alpha + \beta$ with $\alpha \in K\mathfrak{gl}_r(\mathbb{C}), \beta \in \bar{K}\mathfrak{gl}_r(\mathbb{C})$ we obtain $\nabla''\underline{s} = \underline{s}\beta$. Rewriting the zero–curvature condition on ω in terms of α and β we have
$$d\alpha + d\beta + \alpha \wedge \beta + \beta \wedge \alpha = 0,$$
and, since $d\alpha = \bar{\partial}\alpha,\ d\beta = \partial\beta$:
$$\bar{\partial}\alpha + [\alpha \wedge \beta] + \partial\beta = 0,$$

where

$$[\alpha \wedge \beta](X,Y) := [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)].$$

Locally, this equation looks like

$$A_{\bar{z}} - [A,B] - B_z = 0$$

where

$$\alpha = Adz,\ \beta = Bd\bar{z},\ A, B : U \to \mathfrak{gl}_r(\mathbb{C}).$$

Hence we have to solve a so called $\bar{\partial}$–problem for A with given $B : U \to \mathfrak{gl}_r(\mathbb{C})$.

In the case of a holomorphic line bundle E, we only have to show that for any $g\colon U \to \mathbb{C}$ and any $p \in U$ there is a neighborhood V of p and a map $f\colon V \to \mathbb{C}$ such that

$$f_{\bar{z}} = g \quad \text{on } V.$$

This can be proven with methods of complex analysis, cf. Appendix F.

For the general case we refer to [BP], where one finds an elementary proof of the existence of an holomorphic frame even in the case of a quaternionic vector bundle with generalized $\bar{\partial}$–operator. The proof given there works literally for complex vector bundles with $\bar{\partial}$–operator.

$\square$

Notice that the local existence of the flat connection $\nabla = \bar{\partial} + \nabla'$ garantuees that $\bar{\partial}^2\psi = 0$ for all $\psi \in \Gamma(E)$ (type argument and $(d^\nabla)^2 = 0$).

Now, we turn to

Theorem 4.17. *A complex vector bundle E together with a holomorphic structure $\bar{\partial}$ can uniquely be turned into a holomorphic vector bundle such that*

$$\ker \bar{\partial}\big|_{\Gamma(E)} = H^0(E).$$

Proof. Locally, choose a flat complex connection such that $\nabla'' = \bar{\partial}$. Given again an arbitrary local frame $\underline{s}$ and $\nabla\underline{s} = \underline{s}\omega$ recall that ★

$$\mathcal{R}^\nabla = \prime \iff \lceil \omega + \omega \wedge \omega = \prime \overset{\text{Maurer–Cartan–Lemma!}}{\iff} \exists\, \} : \mathcal{U} \to \mathrm{GL}_\nabla(\mathbb{C}) : \}^{-\infty} \lceil \} = \omega.$$

irgendwer muß irgendwann mal M-C erklären

So we obtain a local ∇–parallel frame by $\underline{\tilde{s}} := \underline{s}\, g^{-1}$. In particular,

$$\bar{\partial}\underline{\tilde{s}} = \nabla''\underline{\tilde{s}} = 0.$$

Hence we have a local trivialization $\Phi : E_U \to U \times \mathbb{C}^r$ of the bundle induced by the local ∇–parallel frames $\underline{\tilde{s}}$ which satisfy in particular $\bar{\partial}\underline{\tilde{s}} = 0$.

Given two ∇–parallel frames $\underline{s}$ and $\underline{\tilde{s}} = \underline{s}g^{-1}$, the transition map

$$\Phi\tilde{\Phi}^{-1} : U \cap \tilde{U} \to \mathrm{GL}_r(\mathbb{C}),$$

is exactly g. In particular, g is holomorphic since

$$\bar{\partial}\underline{s} = (\bar{\partial}\underline{\tilde{s}})g + \underline{\tilde{s}}(\bar{\partial}g).$$

$\square$

Chapter 5

Classification of complex line bundles over a Riemann surface

We turn now towards a classification of holomorphic line bundles. The first step in that direction is the classification of all *complex* line bundles over a Riemann surface by their degree.

In the first part of this chapter we define the degree of vector bundles over Riemann surfaces. We do not introduce here Chern classes (compare Appendix E for this general concept) but define the degree of a vector bundle by the integral over the trace of the curvature tensor of any complex connection. Deriving then the degree formula for complex line bundles over Riemann surfaces we see that the degree is defined independently of the complex connection and delivers in fact an integer. Moreover, we will see that the degree gives an isomorphism between the set of all complex line bundles over a Riemann surface and the integers.

5.1 The degree of vector bundles

Definition 5.1. Let M^2 be a closed oriented 2-dimensional manifold. The *degree* of a complex line bundle L is defined by

$$\deg(L) := \frac{i}{2\pi} \int_M \mathcal{R}^\nabla, \tag{5.1}$$

where $\mathcal{R}^\nabla$ is the curvature tensor of any complex connection ∇ on M.

We will see in a moment that the degree of a line bundle is the number of zeros (counted with orientation) of transverse sections so it is defined independently of the chosen complex connection and is in fact an integer.

Example 5.2. A compact oriented Riemannian manifold (M^2, g) has a canonical compatible complex structure $J \in \Gamma(\operatorname{End} TM)$ i.e. $g(JX, JY) = g(X, Y)$. Let ∇

be the LEVI–CIVITA connection of g then the curvature tensor is given by

$$\mathcal{R}(X,Y) = K(< Y,. > X- < X,. > Y),$$

where K is the Gaussian curvature of M. It is an easy exercise to check $\mathcal{R}(X,Y) = -JK\omega_g(X,Y)$ where ω_g is the induced volume form.

Thus, the GAUSS–BONNET theorem yields that the degree of the canonical bundle $K_M = TM^*$ satisfies

$$\deg(K_M) = 2g - 2 \tag{5.2}$$

where g is the genus of M. In particular we have

$$\deg(K_{S^2}) = -2 \quad \text{and} \quad \deg(K_{T^2}) = 0. \tag{5.3}$$

Before proving the degree formula for line bundles we give

Definition 5.3. Let L be a holomorphic line bundle over a Riemann surface, $\psi \in H^0(L)$ and $p \in M$. Let $\psi = \varphi g$ for some non–vanishing local holomorphic section φ of L. The order of ψ at p is defined as the multiplicity of the zero of g at p, i.e.

$$\operatorname{ord}_p \psi = \operatorname{ord}_p g.$$

Remark 5.4. Check that this is well–defined, i.e. does not depend on the trivialization given by φ.

Theorem 5.5 (Degree formula)**.** *Let L be a complex line bundle over a compact Riemann surface. For any section $\psi \in \Gamma(L)$ which is transverse to the zero–section, we have*

$$\deg(L) = \sum_{\text{zeros of } \psi} \pm 1 \text{ (depending on the orientation).}$$

If M is a compact Riemann surface and L is a holomorphic line bundle, then for any holomorphic section $\psi \in H^0(L)$ we have

$$\deg(L) = \sum_{p \text{ zero of } \psi} \operatorname{ord}_p \psi.$$

Proof. We prove both degree formulas simultaneously. Since ψ is transverse to the zero–section or a holomorphic section, its zeros are isolated points $p_1, \ldots, p_k$. For each p_j we take neighborhoods B_j diffeomorphic to a closed disc such that $B_1, \ldots, B_k$ are disjoint.

On the B_j's we take non-vanishing sections ψ_j in L, which can be supposed to be holomorphic in case of a holomorphic bundle. Define $g_j \colon B_j \to \mathbb{C}$ by

$$\psi = \psi_j \cdot g_j.$$

Then the only zero of g_j in B_j is p_j. In the case of a holomorphic bundle L and holomorphic ψ and ψ_j, g_j must be holomorphic, too. Now we take a complex connection ∇. In the holomorphic case we can choose a connection with $\nabla'' = \bar{\partial}$. The connection forms $\omega \in \Omega^1(M\backslash\{p_1, \dots, p_k\}, \mathbb{C})$ and $\omega_j \in \Omega^1(B_j, \mathbb{C})$ are defined by

$$\begin{aligned} \nabla\psi &= \psi \cdot \omega \qquad \text{on } M\backslash\{p_1, \dots, p_k\}, \\ \nabla\psi_j &= \psi_j \cdot \omega_j \qquad \text{on } B_j. \end{aligned}$$

By taking the covariant differential of $\psi = \psi_j \cdot g_j$ we obtain

$$\omega = \omega_j + dg_j \cdot g_j^{-1} \qquad \text{on } B_j\backslash\{p_j\}. \tag{5.4}$$

A short computation shows $\mathcal{R}^\nabla\,\psi = \psi d\omega$ and $\mathcal{R}^\nabla\,\psi_j = \psi_j d\omega_j$ on $M\backslash\{p_1, \dots, p_k\}$ and B_j, respectively. Using (5.4) we get

$$\mathcal{R}^\nabla = d\omega = d\omega_j \qquad \text{on } B_j\backslash\{p_j\}.$$

For the degree of L we obtain by Stokes theorem

$$\begin{aligned} \deg(L) &= \frac{i}{2\pi}\int_M \mathcal{R}^\nabla \\ &= \frac{i}{2\pi}\int_{M\backslash(B_1\cup\ldots\cup B_k)} \mathcal{R}^\nabla + \frac{i}{2\pi}\sum_j \int_{B_j} \mathcal{R}^\nabla \\ &= \frac{i}{2\pi}\int_{M\backslash(B_1\cup\ldots\cup B_k)} d\omega + \frac{i}{2\pi}\sum_j \int_{B_j} d\omega_j \\ &= \frac{i}{2\pi}\sum_j \int_{\partial B_j} \omega_j - \omega \overset{(5.4)}{=} \frac{1}{2\pi i}\sum_j \int_{\partial B_j} d(\log(g_j)). \end{aligned}$$

The proof is finished once we understand the integrals $\frac{1}{2\pi i}\int_{\partial B_j} d(\log(g_j))$.

In the case of holomorphic L and ψ, we have already seen that if we choose a holomorphic ψ_j, then g_j is also holomorphic and has a zero of the same order $m = \operatorname{ord}_{p_j}\psi$ as ψ has at p_j. Let $g_j = a_m z^m + a_{m+1} z^{m+1} \dots$, $m \in \mathbb{N}$ at 0 in a holomorphic chart $z\colon U \to M$ centered at $p \in M$, then the Laurent series of $d(\log(g_j)) = dg_j g_j^{-1}$ is $\frac{dg_j}{g\ j} = \frac{m a_m z^{m-1} + \dots}{a_m z^m + \dots} dz = \left(\frac{m}{z} + \text{some holomorphic rest}\right) dz$. Thus $d(\log(g_j))$ has residue m i.e. $\frac{1}{2\pi i}\int_{\partial B_j} d(\log(g_j)) = m$.

In the case of a transverse section ψ in an arbitrary complex line bundle L, define γ to be the closed curve obtained by taking the restriction of g_j to ∂B_j. By the integral transformation formula, we obtain

$$\frac{1}{2\pi i}\int_{\partial B_j} dg_j g_j^{-1} = \frac{1}{2\pi i}\int_\gamma \frac{1}{z} dz,$$

i.e. the winding number of γ. Since ψ is a transverse section, g_j is a local diffeomorphism at p_j and, without loss of generality, we can suppose that g_j is a diffeomorphism on B_j. This implies that the winding number of γ has to be ± 1 depending on the orientation of the zero p_j of ψ. □

Remark 5.6. In fact, the same proof works also for meromorphic sections instead of holomorphic sections of a holomorphic line bundle. However, we will prove the degree formula for meromorphic section later using the point bundles to enlighten the mechanism of constructing holomorphic sections in a suitable vector bundle out of meromorphic sections.

Corollary 5.7. *Let $L \to M$ be a line bundle over a 2–dimensional compact manifold M. Then* $\deg L \in \mathbb{Z}$.

Proof. Recall, that there exist a section which is transverse to the zero–section! (cf. Theorem 3.24) □

Corollary 5.8. *Let $L \to M$ be a holomorphic line bundle over a Riemann surface M. If* $\deg L < 0$ *then L has no (global) holomorphic sections.*

Now, its time to consider more generally complex vector bundles over a compact Riemann surface. We define

Definition 5.9. The *degree* of a complex vector bundle E of rank r over M is defined by

$$\deg E := \frac{1}{2\pi} \int_M \operatorname{tr} \mathcal{R}^\nabla .$$

where $\mathcal{R}^\nabla \in \Omega^2(\operatorname{End}(E))$ is the curvature tensor of a complex connection on E.

Remark 5.10. If E is a complex line bundle, then $\operatorname{End}(E) \cong \underline{\mathbb{C}}$, thus $\operatorname{tr} R^\nabla \cong R^\nabla$ and both definitions of the degree coincide for line bundles.

The next lemma will show that the degree of a vector bundle is the degree of its determinant bundle. Thus, the degree is in fact an integer and does not depend on the chosen complex connection.

We state some properties of the degree of a complex vector bundle:

Lemma 5.11. *Let E and $\tilde{E}$ be complex vector bundles over a compact Riemann surface M.*

1. *Isomorphic vector bundles have the same degree.*
2. $\deg(E \oplus \tilde{E}) = \deg(E) + \deg(\tilde{E})$.
3. *The degree of the trivial bundle is* $\deg \underline{\mathbb{C}}^r = 0$.

4. *The degree of a complex vector bundle is the degree of its determinant bundle (cf. 3.11,(5)), i.e. if E is a rank r bundle then*
$$\deg E = \deg \Lambda^r E.$$
5. $\deg(\Sigma) = -1$ *for the tautological bundle* $\Sigma \to \mathbb{C}P^1$.
6. $\deg(E/\tilde{E}) = \deg(E) - \deg(\tilde{E})$ *for a complex subbundle* $\tilde{E}$ *of* E.
7. $\deg(E \otimes \tilde{E}) = \operatorname{rank}(E)\deg(\tilde{E}) + \operatorname{rank}(\tilde{E})\deg(E)$.
8. *The degree of the dual bundle* E^* *is* $\deg(E^*) = -\deg(E)$.

Proof. 1. Suppose we have an isomorphism $T\colon E \to \tilde{E}$ of complex vector bundles. Then a complex connection ∇_E on E induces a connection $\nabla_{\tilde{E}} = T\nabla_E T^{-1}$ on $\tilde{E}$. For this connection we have $\mathcal{R}^{\nabla_{\tilde{E}}} = T\mathcal{R}^{\nabla_E}T^{-1}$ and the statement follows by Ad-invariance of the trace.

2. If we take as the connection on $E \oplus \tilde{E}$ the direct sum $\nabla_E \oplus \nabla_{\tilde{E}}$ of the two complex connections on E and $\tilde{E}$, then
$$\mathcal{R}^{\nabla_E \oplus \nabla_{\tilde{E}}} = \mathcal{R}^{\nabla_E} \oplus \mathcal{R}^{\nabla_{\tilde{E}}} \in \Omega^2(\operatorname{End}(E \oplus \tilde{E}))$$
thus $\operatorname{tr}(\mathcal{R}^{\nabla_E \oplus \nabla_{\tilde{e}}}) = \operatorname{tr}(\mathcal{R}^{\nabla_E} \oplus \mathcal{R}^{\nabla_{\tilde{E}}})$.

3. There is a flat connection ∇ on the trivial rank r bundle $\underline{\mathbb{C}}^r$. But $\mathcal{R}^{\nabla} = 0$ implies $\deg(\underline{\mathbb{C}}^r) = 0$.

4. $E = \underline{\mathbb{C}}^{r-1} \oplus \Lambda^r E$ thus with (2) and (3) $\deg E = \deg \underline{\mathbb{C}}^{r-1} + \deg \Lambda^r E = \deg \Lambda^r E$.

5. See Remark 5.19.

6. $E/\tilde{E}$ is isomorphic to the orthogonal complement of $\tilde{E}$ for any hermitian product on E. Therefore, E and $\tilde{E} \oplus E/\tilde{E}$ are isomorphic as complex vector bundles and (3) yields the formula.

7. Let ∇_E and $\nabla_{\tilde{E}}$ be complex connections on E and $\tilde{E}$. The tensor product connection $\nabla_E \otimes \nabla_{\tilde{E}}$ satisfies
$$\mathcal{R}^{\nabla_E \otimes \nabla_{\tilde{E}}} = \mathcal{R}^{\nabla_E} \otimes \operatorname{Id}_{\tilde{E}} + \operatorname{Id}_E \otimes \mathcal{R}^{\nabla_{\tilde{E}}}.$$
Therefore
$$\operatorname{tr}(\mathcal{R}^{\nabla_E \otimes \nabla_{\tilde{E}}}) = \operatorname{tr}(\mathcal{R}^{\nabla_E}) \operatorname{rank}_{\tilde{E}} + \operatorname{rank}_E \operatorname{tr}(\mathcal{R}^{\nabla_{\tilde{E}}}).$$

8. Let ∇_E and ∇_{E^*} be the complex connections on E and E^* defined by
$$d < \alpha, \psi > = < \nabla_{E^*}\alpha, \psi > + < \alpha, \nabla_E \psi > .$$
Then $\mathcal{R}^{\nabla_{E^*}} = -(\mathcal{R}^{\nabla_E})^*$ and $\deg E^* = -\deg E$.

$\square$

5.2 Riemann–Hurwitz formula

★ neu

We use the degree formula to compute the branch order of a holomorphic map by its degree:

Definition 5.12. Let M, N be compact Riemann surfaces and $f : M \to N$ be holomorphic. The *degree* of f is given by

$$\deg f = \#\text{sheets} = \frac{1}{\operatorname{vol} N} \int_M f^* \omega_N$$

where ω_N is the volume form of N.

★ f const?

Lemma 5.13. *Let M, N be compact Riemann surfaces, $f : M \to N$ be holomorphic and $L \to N$ be a line bundle. Then*

$$\deg f^* L = \deg f \deg L.$$

★ proof?

In particular:

Lemma 5.14. *Let $f : M \to \mathbb{C}P^1$ a holomorphic function and $L = f^*\Sigma$ the associated line bundle. Then*

$$\deg L = -\deg f.$$

Example 5.15. Let $f : M \to \mathbb{C}P^1$ a map of $\deg f > 0$. Then the pullback of the tautological bundle $L = f^*\Sigma$ has no holomorphic sections since $\deg L = -\deg f < 0$. In particular, it is more natural to assign to f the line bundle $L = f^*\Sigma^{-1}$ which has the same degree as f.

★ f holomorph, embedding

Definition 5.16. Let M, N be Riemann surfaces.

1. Let ω be a holomorphic 1–form. Let z be a coordinate around p with $z(p) = 0$ and write $\omega = g(z)dz$ (thus g is a holomorphic function in z, i.e. $g_{\bar{z}} = 0$). Then the *order* of ω at p is defined by

$$\operatorname{ord}_p(\omega) = \operatorname{ord}_p g.$$

2. The *branch order* of a holomorphic map $f : M \to N$ at p is defined by

$$b_p(f) := \operatorname{ord}_p(df).$$

The *branch order* of f is defined by

$$b(f) := \sum_{p \in M} b_p(f).$$

Remark 5.17. Recall the holomorphic 1–forms are exactly the holomorphic sections in the canonical bundle equipped with the holomorphic structure d, cf. (4.4). The definition of the order of a holomorphic 1–form coincides with the given one for holomorphic sections. In particular, it does not depend on the choice of the local coordinate.

Theorem 5.18 (Riemann–Hurwitz formula)**.** *Let $f : M \to N$ be a holomorphic map between two compact Riemann surfaces. Then*

$$b(f) = 2(g_M - 1 - \deg f(g_N - 1)).$$

Proof. Consider df as map $df : TM \to f^*TN$. We have $*df = J_N df$ and so $df \in \Gamma(K_M(f^*TN))$. There are canonical holomorphic structures on K, TN and f^*TN hence $K_M(f^*(TN))$ is a holomorphic vector bundle. In fact, df is a holomorphic section in this bundle: It suffices to show that $\bar\partial df(Y) = 0$ where $df(Y) \in \Gamma(f^*TN)$ and $Y \in \Gamma(TM)$ is a holomorphic section, i.e. $\bar\partial Y = 0$. But

$$\begin{aligned}
\bar\partial_X df(Y) &= \frac{1}{2}([df(X), df(Y)] + J[Jdf(X), df(Y)]) \\
&= \frac{1}{2}df([X,Y] + J[JX,Y]) = df(\bar\partial_X Y) = 0.
\end{aligned}$$

Since $df \in H^0(K_M(f^*(TN)))$ the degree formula yields $\deg(K_M(f^*(TN)) = \sum_{p\in M} \operatorname{ord}_p df = b(f)$. On the other hand we can compute

$$\deg(K_M(f^*(TN)) = \deg K_M + \deg(f^*TN).$$

Since $\deg(f^*TN) = \deg f \deg TN$ and $\deg TN = \deg K_N^* = -\deg K_N$ we obtain the result, cf. (5.2). $\square$

5.3 Classification

Let M be a compact oriented manifold of dimension 2. The *group of line bundles*

$$\mathcal{L} = \{L \to M : \text{complex line bundle}\}/\text{Isomorphism}$$

is an abelian group with respect to $\otimes$. The identity element is the trivial bundle $\underline{\mathbb{C}}$, and the inverse of a line bundle L is its dual bundle, $L^* =: L^{-1}$.

Using (8) of Lemma 5.11 we have $\deg(L_1 \otimes L_2) = \deg(L_1) + \deg(L_2)$ and

$$\deg\colon \mathcal{L} \longrightarrow \mathbb{Z}$$

is a group homomorphism. Our aim is to show that it is an isomorphism.

Consider first the following example

Example 5.19 (Line bundles over $\mathbb{C}P^1$)**.** As in Example 2.13 we take the open cover $\{U_0, U_\infty\}$ of $\mathbb{C}P^1$. For $k \in \mathbb{Z}$, a holomorphic line bundle L^{-k} is given by

$$g_{\infty 0} \colon U_0 \cap U_\infty \to \mathbb{C}^*, \qquad \begin{bmatrix} z \\ 1 \end{bmatrix} \mapsto z^k.$$

We like to define a section s_∞^{-k} by $\{(U_0, 1), (U_\infty, w^{-k})\}$ where once again we understand $f_0 : U_0 \to \mathbb{C}$ by $f_0(\begin{bmatrix} z \\ 1 \end{bmatrix}) = 1$ and $f_\infty : U_\infty \to \mathbb{C}$ by $f_\infty(\begin{bmatrix} 1 \\ w \end{bmatrix}) = w^{-k}$ (the notation will be explained later!) Then s_∞^{-k} is globally defined since on $U_0 \cap U_\infty$

$$f_\infty(\begin{bmatrix} z \\ 1 \end{bmatrix}) = f_\infty(\begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}) = z^k = g_{\infty 0}(\begin{bmatrix} z \\ 1 \end{bmatrix}) f_0(\begin{bmatrix} z \\ 1 \end{bmatrix}).$$

Now, if $k < 0$ then s_∞^{-k} is holomorphic and has one single zero of order $-k$ at ∞ and the degree formula implies that $\deg(L^{-k}) = -k$.

The pairing $L^{-k} \times L^k \to L^{-k} \otimes L^k = \underline{\mathbb{C}}$ given by $(s_\infty^{-k}, s_\infty^k) \mapsto s_\infty^{-k} \otimes s_\infty^k$ is non–degenerate, i.e. $L^{-k} = (L^k)^{-1}$. Thus the bundle L^k with transition function

$$g_{\infty 0} \colon U_0 \cap U_\infty \to \mathbb{C}^*, \qquad \begin{bmatrix} z \\ 1 \end{bmatrix} \mapsto \frac{1}{z^k}$$

has $\deg L^k = -\deg L^{-k} = k$.

In particular, this proves that the tautological bundle Σ with transition function $g_{0\infty}(z) = \frac{1}{z}$ has $\deg(\Sigma) = -1$ (compare Lemma 5.11, (5)).

★ gefällt mir nicht! - Katrin

Definition 5.20. A section $\psi \in \Gamma(E|_{M \setminus \{p_1, \dots, p_r\}})$ is called a *meromorphic section* of a holomorphic vector bundle $E \to M$, if it is meromorphic in every local holomorphic trivialization of E. The collection of meromorphic sections of E is denoted $\mathcal{M}(E)$. We will write $\psi \colon M \to E$ meromorphic, $\psi \in \Gamma(E)$ meromorphic and so on. In particular, a *meromorphic 1–form* ω is a section of K which can be written locally as $\omega = f dz$ with meromorphic function f.

Write the meromorphic section ψ as $\psi = \varphi g$ for some non–vanishing local holomorphic section φ of L and let $g(z) = a_k z^k + a_{k+1} z^{k+1} + \dots$ the Laurent expansion of g in a local coordinate centered at p. Then the *order of the meromorphic section* ψ at p is defined by

$$\operatorname{ord}_p \psi := k = \operatorname{ord}_p g$$

Remark 5.21. If $\psi \in H^0(E)$ then this coincides with Definition 5.3.

Generalizing the strategy of constructing line bundles over $\mathbb{C}P^1$ we obtain for any $k \in \mathbb{Z}$ a holomorphic line bundle of degree k:

Definition 5.22. For a fixed $p \in M$ and $k \in \mathbb{Z}$ we define the *point bundle* $L(kp)$ as follows:

Take a local chart $z\colon U \to \mathbb{C}$ with $z(p) = 0$ and an open cover $\{U_1, U_2\}$ of M defined by $U_1 = U$ and $U_2 = M \backslash \{p\}$. By $L(kp)$ denote the holomorphic line bundle with transition function $g_{12} = z^k\colon U \backslash \{p\} \to \mathbb{C}^*$.

We define the ("famous section") s_p^k of $L(kp)$ by $\{(U_1, z^k), (U_2, 1)\}$. This section has one zero of order k (if $k \geq 0$) or a pole of order $-k$ (if $k < 0$) at p and is holomorphic on $M \backslash \{p\}$. ★ unique up to...

Lemma 5.23. *The degree of a point bundle is given by* $\deg L(kp) = k$.

Proof. If $k > 0$ take the famous section s_p^k which is holomorphic and has a single zero of order k. The degree formula shows $\deg L(kp) = k$.

Again, the map given by $s_p^{-k} \times s_p^k \mapsto s_p^{-k} \otimes s_p^k$ is non–degenerate, and $L(-kp)$ is the dual of $L(kp)$. In particular, $\deg L(-kp) = -\deg L(kp) = -k$. □

The main theorem of this section classifies all C^∞ line bundles over a two–dimensional manifold:

Theorem 5.24. *Let M be a compact oriented two–dimensional manifold. Then $\deg\colon \mathcal{L} \longrightarrow \mathbb{Z}$ is an isomorphism of groups, i.e. for any $k \in \mathbb{Z}$ there exists a unique (up to isomorphism) line bundle L over M with* $\deg L = k$.

Proof. For any $k \in \mathbb{Z}$ there is a line bundle of $\deg k$, namely the point bundle $L(kp)$ for arbitrary $p \in M$. In order to prove the injectivity of $\deg\colon \mathcal{L} \longrightarrow \mathbb{Z}$ we show that L admits a non vanishing section if it has $\deg(L) = 0$. We start with a transverse section ψ. Let $z\colon U \tilde{\to} \{w \in \mathbb{C} : |w| < 4\}$ be a chart and let $U_1 = \{z : |z| < 1\}$. Without loss of generality we can assume that all zeroes of ψ lie inside of U_1. (This can be arranged as follows. First, one constructs a diffeomorphism which moves all zeros inside the disc U_1, for example, taking the time-1 map of the flow of a suitable vector field. Then one takes the pullback of L and ψ with respect to this diffeomorphism.) We take a non vanishing section φ on U. Then there is a function $g\colon U \to \mathbb{C}$ such that $\psi = \varphi \cdot g$. A similar argument as in the proof of the degree formula shows that $\deg(L) = 0$ implies

$$\int_\gamma \frac{dg}{g} = 0$$

for any closed curve γ in $U \backslash U_1$. Therefore we find a function $f\colon U \backslash U_1 \to \mathbb{C}$ satisfying $e^f = g$. Let $\sigma\colon \mathbb{R} \to \mathbb{R}$ be a C^∞ function with $\sigma(x) = 0$ for $x < 2$ and $\sigma(x) = 1$ for $x > 3$. Define $\tilde{f}\colon U \to \mathbb{C}$ by $\tilde{f}(z) = \sigma(|z|) f(z)$. Then

$$\tilde{\psi} := \begin{cases} \varphi \cdot e^{\tilde{f}} & \text{on } U \\ \psi & \text{on } M \backslash U \end{cases}$$

is a smooth section of L without zeros. □

Corollary 5.25 (Degree formula for meromorphic sections)**.** *If M is a compact Riemann surface and L is a holomorphic line bundle over M, then for any meromorphic section $\psi \in \mathcal{M}(L)$ we have*

$$\deg(L) = \sum_{p \text{ zero/pole of } \psi} \operatorname{ord}_p \psi.$$

Proof. Assume that ψ has a single pole at $p \in M$ of order k. Consider the tensor product $L \otimes L(kp)$. The degree is

$$\deg(L \otimes L(kp)) = \deg L + \deg L(kp) = \deg L + k.$$

Let s_p^k be again the famous holomorphic section of $L(kp)$ with a single zero of order k at p. The section $\psi \otimes s_p^k \in \Gamma(L \otimes L(kp))$ is clearly meromorphic, but in p the pole of ψ and the zero of s_p^k cancel. Thus $\psi \otimes s_p^k \in H^0(L \otimes L(kp))$ and $\psi \otimes s_p^k$ has the same zeros as ψ has. Applying the degree formula we have

$$\deg L + k = \deg(L \otimes L(kp)) = \sum_{p \text{ zero of } \psi \otimes s_p^k} \operatorname{ord}_p \psi \otimes s_p^k = \sum_{p \text{ zero of } \psi} \operatorname{ord}_p \psi$$

which gives

$$\deg L = -k + \sum_{p \text{ zero of } \psi} \operatorname{ord}_p \psi = \sum_{p \text{ zero/pole of } \psi} \operatorname{ord}_p \psi.$$

Now, if ψ has further poles one considers the tensor product of L with the corresponding point bundles at the poles. $\square$

Chapter 6

Elliptic operators, Hodge decomposition theorem and Serre duality theorem

We give a short introduction to the theory of elliptic operators. Far away of proving anything (this would require longer excursions to the theory of Sobolev spaces) we formulate a fundamental theorem for elliptic operators which garantuees solutions of certain differential equations. We will see how then the Hodge decomposition theorem and the Serre duality theorem follow from this general statement.

6.1 Elliptic operators

We start with the "local" definition of elliptic differential operators on open sets in $\mathbb{R}^n$.

Definition 6.1 (Local)**.** Let $U \subset \mathbb{R}^m$ be open and let V, W be finite dimensional vector spaces over the field $\mathbb{C}$ or $\mathbb{R}$.

1. A *linear differential operator of degree* r is a map

$$T : C^\infty(U, V) \to C^\infty(U, W)$$

of the form

$$T = \sum_{|t| \le r} a_t \partial_t,$$

where $a_t : U \to \mathrm{Hom}(V, W)$ is a C^∞–map for all multi–indices $t = (t_1, \ldots, t_m)$, $|t| = \sum_i t_i$ and $\partial_t = \frac{\partial^{|t|}}{\partial x_1^{t_1} \ldots \partial x_m^{t_m}}$.

2. The *symbol* of T at $x \in U$ is the map $\sigma(T,x)\colon \mathbb{R}^m \to \mathrm{Hom}(V,W)$ defined by
$$\sigma(T,x)(\xi) := \sum_{|t|=r} a_t(x)\xi^t,$$
where $\xi \in \mathbb{R}^m$ and $\xi^t = \xi_1^{t_1} \cdot \ldots \cdot \xi_m^{t_m}$.

3. The operator T is called *elliptic* at $x \in U$ if $\sigma(T,x)(\xi) \in \mathrm{Isom}(V,W)$ for all $\xi \in \mathbb{R}^m \backslash \{0\}$. T is called elliptic if it is elliptic for all $x \in U$.

Remark 6.2. Note that even in the case of complex spaces V and W one only has to plug in real vectors $\xi \in \mathbb{R}^m\backslash\{0\}$ in order to check ellipticity.

Example 6.3.

a) The symbol of the (classical) LAPLACE operator on functions $\Delta = \sum_{i=1}^m \frac{\partial^2}{(\partial x^i)^2}$ is $\sigma(\Delta, x)(\xi) = \sum_{i=1}^m \xi_i^2$. The operator is elliptic as its symbol is never zero for all $\xi \in \mathbb{R}^m\backslash\{0\}$. The quadric $\sum_{i=1}^m \xi_i^2 = 1$ is an ellipse.

b) The operator $\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ has the symbol $\sigma(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}, x)(\xi) = \xi_1^2 - \xi_2^2$, which is zero on the so called characteristic directions $\xi_1 = \pm\xi_2$. The operator is called an hyperbolic operator and the quadric $\xi_1^2 - \xi_2^2 = 1$ is an hyperbola.

c) Similarly, the operator $\frac{\partial}{\partial x} - \frac{\partial^2}{\partial y^2}$ has the symbol $\xi_1 - \xi_2^2$ and is called a parabolic operator.

d) The operator $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ has the symbol $\sigma(\frac{\partial}{\partial \bar{z}})(\xi_1,\xi_2) = \frac{1}{2}(\xi_1 + i\xi_2)$ and is elliptic.

The local definition of linear differential operators on open sets in $\mathbb{R}^n$ can be generalized to linear differential operators on vector bundles over manifolds as follows.

Definition 6.4 (Global)**.** Let M be a C^∞–manifold of dimension m and let $E \to M$ and $F \to M$ be vector bundles over the field $\mathbb{R}$ or $\mathbb{C}$.

1. A *linear differential operator of degree* r is a linear map
$$T : \Gamma(E) \to \Gamma(F)$$
such that with respect to an open cover $\{U_i\}$ of M by chart neighborhoods for M trivializing E and F it is a linear differential operator of degree r in the sense of the local definition above. That means, if $E|_{U_i} \cong U_i \times V$ and $F|_{U_i} \cong U_i \times W$ with finite dimensional vector spaces V and W, the restriction
$$T|_{U_i}\colon \underbrace{C^\infty(U_i,V)}_{\Gamma(E|_{U_i})} \to \underbrace{C^\infty(U_i,W)}_{\Gamma(F|_{U_i})}$$

can locally be written as

$$T|_{U_i} = \sum_{|t|\leq r} a_t^i \partial_t.$$

2. Let $SM^* = \{\alpha \in T^*M \mid \alpha(p) \neq 0 \text{ for all } p \in M\}$. The pull back of E and F with respect to the projection $\pi : SM^* \to M$ defines the bundles π^*E and π^*F on SM^*, e.g.

$$\pi^*E = \{(\alpha, e) \in SM^* \times E \quad | \quad \pi_{T^*M}(\alpha) = \pi_E(e)\} \subset SM^* \times E.$$

The *symbol* of T is the vector bundle homomorphism $\sigma(T) : \pi^*E \to \pi^*F$ given by

$$\sigma(T)(\alpha_p, \psi_p) := (\alpha_p, \frac{1}{r!} T(f^r\psi)(p))$$

where $f \in C^\infty(\mathbb{R})$ with $f(p) = 0$ and $df_p = \alpha_p$ and $\psi \in \Gamma(E)$ with $\psi(p) = \psi_p$.

3. A linear differential operator $T : \Gamma(E) \to \Gamma(F)$ is called *elliptic* if the symbol $\sigma(T) : \pi^*E \to \pi^*F$ is an isomorphism of vector bundles.

Let U_i be a chart neighborhood with coordinates $(x_1, \ldots, x_m)$ trivializing E and F such that T is of the local form

$$T|_{U_i} = \sum_{|t|\leq r} a_t^i \partial_t.$$

Using the generalized Leibniz rule $(ab)^{(k)} = \sum_j \begin{pmatrix} k \\ j \end{pmatrix} a^{(j)} b^{(k-j)}$ one easily sees that

$$\partial_t(f^r\psi)_p = \begin{cases} 0 & |t| < r \\ r!(\frac{\partial f}{\partial x_1})_p^{t_1} \ldots (\frac{\partial f}{\partial x_m})_p^{t_m} \psi_p & |t| = r \end{cases}$$

for any $f \in C^\infty(U_i)$ with $f(p) = 0$ and $\psi \in \Gamma(E|_{U_i})$. Thus, the symbol does not depend on the choice of f and ψ but only on $df_p = \alpha_p = \xi_1 dx_1 + \ldots + \xi_m dx_m$ and ψ_p and the local and the global definition of the symbol coincide. Thus, an operator T locally given by $T|_{U_i} = \sum_{|t|\leq r} a_t \partial_t$ is elliptic (on U_i) if for all $\xi \in \mathbb{R}^m \setminus \{0\}$ and all $p \in U_i$

$$\sum_{|t|=r} a_t(x)\xi^t \in \mathrm{Isom}(V, W).$$

Example 6.5. 1. The Laplace operator

$$\Delta = -\mathrm{div\ grad} \colon C^\infty(M) \to C^\infty(M)$$

on functions over a semi–Riemannian manifold $(M^{(n,k)}, g)$ has the symbol

$$\sigma(\Delta)(\alpha_p, h_p) = -g(\alpha_p, \alpha_p)h_p.$$

This operator is elliptic in the case of Riemannian manifolds ($k = 0$). It is not elliptic in the case of an indefinite metric g, i.e. if $k \neq 0, n$.

2. Any linear first order differential operator
$$D\colon \Gamma(E) \to \Gamma(F)$$
between vector bundles E and F on a manifold M satisfies a product rule
$$D(f\psi) = fD(\psi) + A(df \otimes \psi)$$
with $A \in \Gamma(\mathrm{Hom}(T^*M \otimes E, F))$. Its symbol is therefore
$$\sigma(D)(\alpha_p, \psi_p) = A(\alpha_p \otimes \psi_p).$$
The next three examples are applications of this one.
3. The symbol of a connection $\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ on a bundle E is
$$\sigma(\nabla)(\alpha_p, \psi_p) = \alpha_p \otimes \psi_p.$$
Thus, a connection is only elliptic in case of 1–dimensional manifolds.
4. The symbol of $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ is
$$\sigma(d)(\alpha_p, \psi_p) = \alpha_p \wedge \psi_p.$$
So, d is only elliptic in case of 1–dimensional manifolds and $k = 0$.
5. The symbol of a holomorphic structure $\bar\partial\colon \Gamma(E) \to \Gamma(\Lambda^{(0,1)} \otimes E)$ on a complex vector bundle E over an almost complex manifold (M, J) is
$$\sigma(\bar\partial)(\alpha_p, \psi_p) = \mathrm{pr}_{\Lambda^{(0,1)}}(\alpha_p) \otimes \psi_p.$$
Thus, $\bar\partial$ is elliptic in case of a complex vector bundle over a Riemann surface, because for any $\alpha \in T^*M$ we have $\mathrm{pr}_{\Lambda^{(0,1)}}(\alpha) = \frac{1}{2}(\alpha + i * \alpha) \neq 0$.
In particular, two different different $\bar\partial$–operators on complex vector bundle have the same symbol (the symbol only depends on the complex structure of M not on the holomorphic structure).

6.2 Fundamental theorem

("Fundamentaler Satz, den man ein für alle mal lernen sollte!")

It shows that elliptic equations on closed manifolds behave where much like finite dimensional linear equations. The interesting point about this theorem is that its statement contains nothing but C^∞–sections of vector bundles. It doesn't need distributions, Sobolev spaces etc. For a slightly more elegant formulation we recall that for a bounded linear operator with $\dim\ker T < \infty$ and $\dim W/\mathrm{Im}\, T < \infty$ it makes sense to define $\mathrm{coker}\, T := W/\mathrm{Im}\, T$ and the *index* of T, i.e. $\mathrm{index}(T) := \dim\ker T - \dim\mathrm{coker}\, T$.

★ ich weiß nicht so recht, ob das tatsächlich Sinn macht, das so zu definieren, müßte man wohl funkanal. checken. Jedenfalls darf man

Theorem 6.6 (Fundamental Theorem)**.** *Let M be a compact orientable C^∞ manifold, E, F vector bundles over $\mathbb{R}$ or $\mathbb{C}$ on M. Let $T\colon \Gamma(E) \to \Gamma(F)$ be an elliptic linear differential operator. If we choose a Riemannian metric g on M and fiber metrics $<,>_E$ and $<,>_F$ on E and F, the following holds:*

1. *The formally adjoint operator T^* of T defined by*

$$\int_M < T^*\varphi, \psi >_E dg = \int_M < \varphi, T\psi >_F dg$$

   for all $\psi \in \Gamma(E)$ and $\varphi \in \Gamma(F)$ is again an elliptic linear differential operator.

2. $\ker(T)$ *and* $\ker(T^*)$ *are finite dimensional.*

3. *The following direct sum decomposition is orthogonal with respect to the L^2 scalar products on $\Gamma(E)$ and $\Gamma(F)$:*

$$\begin{aligned} \Gamma(F) &= \mathrm{Im}\ (T) \oplus \ker(T^*) \\ \Gamma(E) &= \mathrm{Im}\ (T^*) \oplus \ker(T). \end{aligned}$$

4. *The linear equation $Te = f$ has a solution if and only if $f \in (\ker(T^*))^\perp$.*

5. *The index of T depends only on the (homotopy part of the) symbol.*

We don't prove this theorem here. The reason is that even if the statement doesn't contain distributions or Sobolev spaces, we don't know any way to prove it without the aid of these standard tools from the theory of partial differential equations. This theorem is included as an exercise on the last page of [Wa]. A proof is sketched in Corollary 2.5 of [Ka]. An honest proof (using pseudodifferential operators) can by found for example in [Gi] or [We].

There is one disadvantage in the way the Fundamental Theorem is stated above. The choice of metrics on M and on the bundles E and F is not really natural. Even if in Riemannian geometry for example one normally has canonical metrics, in conformal or complex geometry one normally hasn't. That's why we give a reformulation of this theorem which doesn't contain any artifical choice of metrics.

The metrics g on M and $< . >_E$ and $< . >_F$ define isomorphisms

$$E \cong \mathcal{D}E^* \qquad \psi \mapsto < \psi, . >_E dg$$

and

$$F \cong \mathcal{D}F^* \qquad \varphi \mapsto < \varphi, . >_F dg$$

where $\mathcal{D} = \Lambda^m TM^*$ is the *density bundle* of M and $dg \in \Gamma(\mathcal{D})$ is the volume form of (M, g). If we don't have these metrics, we cannot use these identifications. But we can use the bundles $\mathcal{D}E^*$ and $\mathcal{D}F^*$ instead.

★ Beweis der ultimaten Version?

Theorem 6.7. ("Ultimate Version of the Fundamental Theorem") *Let M be a compact orientable C^∞ manifold, E, F vector bundles over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ on M. Let $T\colon \Gamma(E) \to \Gamma(F)$ be an elliptic linear differential operator. Then the following holds:*

1. *The formally adjoint operator $T^*\colon \mathcal{D}F^* \to \mathcal{D}E^*$ of T defined by*

$$\int_M < T^*\omega, \psi >= \int_M < \omega, T\psi > .$$

   for all $\psi \in \Gamma(E)$ and $\omega \in \Gamma(\mathcal{D}F^*)$ is again an elliptic linear differential operator.

2. $\ker(T)$ *and* $\ker(T^*)$ *are finite dimensional.*

3. *The pairing*

$$\ker(T^*) \times \operatorname{coker} T \to \mathbb{K}$$

   induced by

$$\Gamma(\mathcal{D}F^*) \times \Gamma(F) \to \mathbb{K} \qquad (\omega, \varphi) \mapsto \int_M < \omega, \varphi >$$

   is non–degenerate.

   The same holds for the pairing

$$\operatorname{coker} T^* \times \ker(T) \to \mathbb{K}$$

   induced by

$$\Gamma(\mathcal{D}E^*) \times \Gamma(E) \to \mathbb{K} \qquad (\omega, \psi) \mapsto \int_M < \omega, \psi > .$$

4. *The linear equation $Te = f$ has a solution if and only if*

$$\int_M < \omega, f >= 0$$

   for all $\omega \in \ker T^*$.

5. *The index of T only depends on the (homotopy part of the) symbol.*

6.3 Laplace–Beltrami operator on hermitian vector bundles

As an demonstration of the universality of this theorem, we prove the HODGE decomposition Theorem 6.16 for differential forms on a Riemannian manifold (M, g) with values in a real or complex vector bundle E equipped with a metric $< . >_E$ and a metric connection ∇. We recall just the basic definitions, for details to the HODGE theory we refer essentially to [Wa] and [We].

First, let V be a $\mathbb{K}$–vector space , $\mathbb{K} = \mathbb{R}, \mathbb{C}$, with dim $V = n$ and inner product $<,>$. The inner product on V can be extended to $\Lambda(V)$ by setting the inner product to zero for elements which are homogeneous of different degrees and otherwise by extending bilinearly $< v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_k >= \det(< v_i, w_j >)$. Hence any orthonormal basis $e_1, \ldots, e_n$ of V induces an orthonormal basis of $\Lambda(V)$. An *orientation* of V is given by a *volume form* $\omega \in \Lambda^n(V)$, i.e. a form with $< \omega, \omega >= 1$ (notice that $\Lambda^n(V)$ is one–dimensional, hence $\Lambda^n(V)\backslash\{0\}$ has two components, and there are exactly two possible choices of an orientation). If $\omega(e_1, \ldots, e_n) =< \omega, e_1 \wedge \ldots \wedge e_n >= 1$ then an orthonormal basis $e_1, \ldots, e_n$ of V is called *positive oriented.*

Definition 6.8. Let $(V, <, >, \omega)$ be an oriented vector space. The HODGE *star operator* $\star : \Lambda V \to \Lambda V$ is defined by

$$\star 1 = \epsilon\ e_1 \wedge \ldots \wedge e_n,\ \star(e_1 \wedge \ldots \wedge e_n) = \epsilon\ 1$$
$$\star(e_1 \wedge \ldots \wedge e_k) = \epsilon\ (e_{k+1} \wedge \ldots \wedge e_n),$$

for any orthonormal basis $e_1, \ldots, e_n$ of V where $\epsilon = \omega(e_1, \ldots, e_n)$.

Direct consequences of the definition are

Lemma 6.9.

1. $\star : \Lambda^k(V) \to \Lambda^{n-k}(V)$
2. $\star \star \eta = (-1)^{k(n-k)}\eta,\ \eta \in \Lambda^k(V)$
3. $\langle v, w\rangle = \star(w \wedge \star v),\ w \wedge \star v = \langle v, w\rangle\omega,\ v, w \in \Lambda^k V.$

Remark 6.10. Clearly, we can extend the definition of both, the orientation and the HODGE star operator, to Riemannian manifolds by considering $V = TM^*$. (M, g) is said to be *orientable* if there exists a global non–vanishing n–form $\omega \in \Omega^k(M)$. The choice of a volume form ω_g on M gives an orientation. The $\star$–operator operates on smooth k-forms on a Riemannian manifold M in an obvious manner.

We now show that on a Riemann surface M the Hodge star operator $\star$ is the negative of the star operator $*$ defined in Definition 4.1. Let $z = x + iy$ be a complex coordinate, then by definition $*dz = idz$. On the other hand we have $\star dx = dy$ and $\star dy = -dx$ and thus $\star dz = dy - idx = -idz$.

From now on we assume that (M, g) is a n–dimensional compact and oriented Riemannian manifold with volume form ω_g. Let E be a $\mathbb{K}$–vector bundle over M, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, with Euclidian or hermitian form $h : E \times E \to \mathbb{C}$.

Definition 6.11. Let $\alpha = \tilde{\alpha}\psi, \beta = \tilde{\beta}\varphi \in \Omega^k(E),\ \tilde{\alpha}, \tilde{\beta} \in \Omega^k(M),\ \ \psi, \varphi \in \Gamma(E)$ and define h on $\Omega^k(E)$ by extending

$$h(\alpha, \beta)_p := < \tilde{\alpha}, \tilde{\beta} >_p \ \ h(\psi, \varphi)_p.$$

Furthermore, we define $h(\alpha \wedge \beta)$ by the usual formula using h as product on E, compare Corollary 4.7. In particular,

$$h(\alpha \wedge \beta) = \tilde{\alpha} \wedge \tilde{\beta}\ h(\psi, \varphi).$$

If we define $\star\alpha := (\star\tilde{\alpha})\psi$ we have $h(\alpha \wedge \star\beta) = < \tilde{\alpha}, \tilde{\beta} > \omega_g h(\psi, \varphi) = h(\alpha, \beta)\omega_g$. Hence, we can equip $\Omega^k(E)$ with an inner product

$$(\alpha, \beta) := \int_M h(\alpha \wedge \star\beta).$$

Let now ∇ be a with h compatible connection on E, i.e. $\nabla h = 0$. Then there is an induced $d^\nabla : \Omega^k(E) \to \Omega^{k+1}(E)$, compare (3.3), and we can define

★ Eindeutigkeitsfragen?

Definition 6.12.
1. The *codifferential operator* $\delta^\nabla : \Omega^k(E) \to \Omega^{k-1}(E)$ is defined by
$$\delta^\nabla := (-1)^{n(k+1)+1} \star d^\nabla \star .$$
2. The *Laplacian* resp. the LAPLACE–BELTRAMI *operator* $\Delta : \Omega^k(E) \to \Omega^k(E)$ is defined by
$$\Delta := d^\nabla \delta^\nabla + \delta^\nabla d^\nabla.$$
3. The *harmonic k–forms* $\mathrm{Harm}^k(E)$ are defined by
$$\mathrm{Harm}^k(E) := \{\omega \in \Omega^k(E) \mid \Delta\omega = 0\}.$$

Example 6.13. Let $E = M \times \mathbb{C}$ and h the usual multiplication on $\mathbb{C}$ valued forms. Then $\Omega^k(E) = \Omega^k(TM^* \otimes \mathbb{C})$ are the complex valued k forms and $(,)$ is the usual inner product on $\Omega^k(E)$

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta.$$

The exterior differentiation gives a flat compatible connection and we obtain the usual LAPLACE–BELTRAMI operator on complex valued forms.

Some easy statements for the Laplacian and the codifferential operator are given in

Proposition 6.14.

1. *$(d^\nabla\alpha, \beta) = (\alpha, \delta^\nabla\beta)$ for $\alpha \in \Omega^k(E), \beta \in \Omega^{k+1}(E)$.*

2. *$\alpha \in \mathrm{Harm}^k(E) \iff d^\nabla\alpha = 0$ and $d^\nabla \star \alpha = 0$*

3. *$\star\Delta = \Delta\star$*

4. *$(\Delta\alpha, \beta) = (\alpha, \Delta\beta)$.*

5. *Δ is elliptic.*

Proof. 1. Since $d^\nabla \star \beta \in \Omega^{n-k}(E)$ we have

$$\begin{aligned} d^\nabla \star \beta &= (-1)^{(n-k)k} \star \star d^\nabla \star \beta = (-1)^{(n-k)k}(-1)^{n(k+2)+1} \star \delta^\nabla \beta \\ &= (-1)^{-k^2+1} \star \delta^\nabla \beta \end{aligned}$$

thus

$$\begin{aligned} (d^\nabla\alpha, \beta) &= \int_M h(d^\nabla\alpha \wedge \star\beta) = \int_M dh(\alpha \wedge \star\beta) - (-1)^k h(\alpha \wedge d^\nabla \star \beta) \\ &= \int_M (-1)^{k-1-k^2+1} h(\alpha \wedge \star\delta^\nabla\beta) = (\alpha, \delta^\nabla\beta). \end{aligned}$$

2. $(\Delta\alpha, \alpha) = (\delta^\nabla\alpha, \delta^\nabla\alpha) + (d^\nabla\alpha, d^\nabla\alpha)$, which gives the assertion by the positive definiteness of $(,)$.

3./4. Simple calculation.

4. As in the case of the Laplace operator on functions the symbol can be shown to be

$$\sigma(\Delta)(\alpha_p, \psi_p) = -g(\alpha_p, \alpha_p)\psi_p.$$

Therefore, Δ is elliptic.

∎

Corollary 6.15. *If M is connected and $f \in \mathrm{Harm}^0(M) = \mathrm{Harm}^0(\underline{\mathbb{C}})$ then f is constant.*

Proof. If $\Delta f = 0$ then $df = 0$. Since M is connected we obtain $f \equiv$ const. ∎

6.4 Hodge decomposition theorem and Poincaré duality

The fundamental theorem in this subsection will be the HODGE decomposition theorem. In particular, this theorem guarantees solutions ω of the differential equation $\Delta\omega = \alpha$ for given k–forms α orthogonal to the harmonic forms. Furthermore, we obtain the isomorphism between the k^{th} DE RHAM cohomology group and the harmonic forms.

Theorem 6.16 (HODGE decomposition theorem)**.** *Let M be a compact and oriented manifold and E a hermitian vector bundle over M with connection ∇. Then* $\mathrm{Harm}^k(E)$ *is for each $0 \leq k \leq n$ finite dimensional and one has the following orthogonal direct sum decomposition:*

$$\Omega^k(E) = \mathrm{Im}\ \Delta \oplus \mathrm{Harm}^k(E).$$

Consequently, the equation $\Delta\omega = \alpha$ has a solution $\omega \in \Omega^k(E)$ if and only if the k-form α is orthogonal to the space of harmonic k-forms.

Proof. The LAPLACE–BELTRAMI operator is a (formally) self–adjoint elliptic operator

$$\Delta \colon \Omega^k(E) \to \Omega^k(E)$$

for all $k \in \mathbb{N}$. Therefore, the fundamental theorem implies

$$\dim \mathrm{Harm}^k(E) = \dim \ker(\Delta \colon \Omega^k(E) \to \Omega^k(E)) < \infty$$

and

$$\Omega^k(E) = \mathrm{Im}\ \Delta \oplus \ker \Delta$$

and this direct sum decomposition is orthogonal. $\square$

Corollary 6.17. *Let ∇ be a flat connection. Then there are the following orthogonal direct sum decompositions:*

1. $\Omega^k(E) = \mathrm{Im}\ d^\nabla \delta^\nabla \oplus \mathrm{Im}\ \delta^\nabla d^\nabla \oplus \mathrm{Harm}^k(E)$

2. $\Omega^k(E) = \mathrm{Im}\ d^\nabla \oplus \mathrm{Im}\ \delta^\nabla \oplus \mathrm{Harm}^k(E)$

Proof. First,

$$\mathrm{Im}\ \Delta \quad\subset\quad \mathrm{Im}\ d^\nabla \delta^\nabla + \mathrm{Im}\ \delta^\nabla d^\nabla \subset \mathrm{Im}\ d^\nabla + \mathrm{Im}\ \delta^\nabla.$$

1. From Proposition 6.14 we obtain $\mathrm{Im}\ d^\nabla \delta^\nabla, \mathrm{Im}\ \delta^\nabla d^\nabla \perp \mathrm{Harm}^k(E)$ and

$$(\delta^\nabla d^\nabla \alpha, d^\nabla \delta^\nabla \beta) = (\delta^\nabla \alpha, (d^\nabla)^2 \delta^\nabla \beta) = 0$$

since ∇ is a flat connection.

2. Similarly $\mathrm{Im}\ d^\nabla, \mathrm{Im}\ \delta^\nabla d^\nabla \perp \mathrm{Harm}^k(E)$ and $\mathrm{Im}\ d^\nabla \perp \mathrm{Im}\ \delta^\nabla d^\nabla$.

□

Theorem 6.18. *Let M be a compact and oriented manifold and E a Euclidean/ hermitian vector bundle over M with flat connection ∇. The k'th (complex)* de Rham *cohomology group $H^k_{\mathrm{dR}}(E) = \frac{\{closed\ k\text{–}forms\}}{\{exact\ k\text{–}forms\}}$ is isomorphic to* $\mathrm{Harm}^k(E)$.

In particular, for all $\omega \in H^k_{\mathrm{dR}}(E)$ there exists a unique $\eta \in \mathrm{Harm}^k(E)$ with $[\eta] = \omega$.

Proof. Consider the map $\rho : H^k_{\mathrm{dR}}(E) \to \mathrm{Harm}^k(E)$ defined by $[\alpha] \mapsto \alpha_0$ where $\alpha = \alpha_0 + \alpha_1$, $\alpha_0 \in \mathrm{Harm}^k(E)$, $\alpha_1 \in \mathrm{Im}\ \Delta$, is the unique decomposition of $\alpha \in \Omega^k(E)$ given by the Hodge decomposition theorem. Observe that the map ρ is well defined and gives the sought isomorphism. □

Definition 6.19. The (finite) dimension $b_k(E)$ of the k^{th} de Rham cohomology class is called the k^{th} Betti *number*, i.e.

$$b_k(E) := \dim H^k_{\mathrm{dR}}(E).$$

Theorem 6.20 (Poincaré duality)**.** *Let M be a compact and oriented manifold and E a Euclidian/hermitian vector bundle over M with connection ∇. There is an isomorphism*

$$H^{n-k}_{\mathrm{dR}}(E) \cong H^k_{\mathrm{dR}}(E)^*.$$

Proof. Define the bilinear pairing $(,)_{\mathrm{dR}} : H^k_{\mathrm{dR}}(E) \times H^{n-k}_{\mathrm{dR}}(E) \to \mathbb{K}$

$$([\alpha],[\beta])_{\mathrm{dR}} = \int_M h(\alpha \wedge \beta).$$

We have to show that the pairing $(,)_{\mathrm{dR}}$ is non singular then $H^k_{\mathrm{dR}}(E)$ is isomorphic to the dual space of $H^{n-k}_{\mathrm{dR}}(E)$.

Let $0 \neq [\alpha] \in H^k_{\mathrm{dR}}(E)$ and choose a harmonic representant α. But then $\star\alpha$ is harmonic, too (recall $\Delta \star \alpha = \star\Delta\alpha = 0$), in particular $\star\alpha$ is a d^∇ closed form. Thus $[\star\alpha] \in H^{n-k}_{\mathrm{dR}}(E)$ and $(\alpha, \star\alpha)_{\mathrm{dR}} = \int \alpha \wedge \star\alpha \neq 0$. □

Corollary 6.21. *Let M be connected and E a $\mathbb{K}$ vector bundle over M.*

1. $b_k(E) = b_{n-k}(E)$.

2. $H^n_{\mathrm{dR}}(T^*M, \mathbb{R}) \cong H^0_{\mathrm{dR}}(T^*M, \mathbb{R}) = \mathbb{R}$.

3. *For given* $\beta \in H^k_{\mathrm{dR}}(TM^*, \mathbb{R})$ *there exists a solution of* $\Delta\alpha = \beta$ *if and only if for all* $\gamma \in \mathrm{Harm}^k(TM^*, \mathbb{R})$*:* $(\beta, \gamma) = 0$*.*

   In particular, for given $h \in C^\infty(M)$ *there exists a solution of* $\Delta f = h$ *if and only if* $\int_M h(*1) = \int_M h\omega_g = 0$*.*

Definition 6.22. Let M be a compact Rieamnn surface. The *genus* g of M is given by

$$g = \frac{1}{2}\dim_{\mathbb{R}} H^1_{\mathrm{dR}}(M, \mathbb{R}) = \frac{b_1(M)}{2}.$$

A further important application of the Hodge decomposition theorem is

Lemma 6.23. *Let* M *be a Riemann surface. For the canonical bundle we have* $\dim_{\mathbb{C}} H^0(K) = g$*.*

Proof. Let $\omega = \alpha + i\beta$ be a complex valued 1–form, and α and β its decomposition into its real and imaginary part. ω lies in K if and only if $*\alpha = -\beta$ ($*$ is the negative of the Hodge star operator $\star$). Thus the elements of K are the 1–forms $\omega = \alpha - i * \alpha$, where α is a real valued 1–Form. ω is holomorphic if and only if $d\omega = 0$, that means if and only if $d\alpha = 0$ and $d * \alpha = 0$. Thus the holomorphic sections of K are of the form $\omega = \alpha - i * \alpha$ with a real valued harmonic 1–form α.

By Theorem 6.18 we know that the real dimension of the space of real valued harmonic 1–forms on a compact Riemann surface of genus g is equal to $2g$. Thus

$$\dim_{\mathbb{C}} H^0(K) = \dim_{\mathbb{C}} \mathrm{Harm}^1(M, \mathbb{R}) = \frac{1}{2}\dim_{\mathbb{R}} \mathrm{Harm}^1(M, \mathbb{R}) = g.$$

□

6.5 Serre duality theorem

We used the Fundamental theorem to prove Poincare duality by using the Laplacian as elliptic operator. Exactly the same can be done if we consider $\bar\partial$ as the elliptic operator, and we obtain for the Dolbeault cohomology group defined by $\bar\partial$:

Theorem 6.24 (Serre Duality Theorem)**.** *Let* M *be a compact Riemann surface,* $E \to M$ *a complex vector bundle with holomorphic structure given by* $\bar\partial$*. Then*

1. $H^0(E)$ *is finite dimensional.*

2. $(H^1(E))^* \cong H^0(KE^*)$*.*

3. *There is a solution* $\psi \in \Gamma(E)$ *of the equation* $\bar\partial\psi = \varphi$ *if and only if* φ *is a section in* $\Gamma(\bar K E)$ *satisfying*

$$\int_M <\varphi, \omega>= 0 \textit{ for all } \omega \in H^0(KE^*).$$

4. *Two* $\bar\partial$*–operators on a complex vector bundle* E *have the same index.*

Proof. We already know that $\bar\partial$ is elliptic. Thus the ultimate version of the Fundamental Theorem 6.7, (2), implies that $\ker(\bar\partial)$ is finite dimensional and Theorem 6.7, (3), that the pairing $\ker\bar\partial^* \times \operatorname{coker}\bar\partial \to \mathbb{C}$ is non–degenerate.

Again, the dual operator $\bar\partial^* \colon \Gamma(\underbrace{\mathcal{D}(\bar K E)^*}_{=K\bar K\bar K^{-1}E^*=KE^*}) \to \Gamma(\underbrace{\mathcal{D}E^*}_{=\bar K K E^*})$ is defined by

$$\int_M <\bar\partial^*\omega, \psi>= \int_M <\omega, \bar\partial_E\psi>$$

for all $\omega \in \Gamma(KE^*)$ and $\psi \in \Gamma(E)$. On the other hand consider the induced holomorphic structure $\bar\partial_{KE^*}$ on KE^*: The holomorphic structure on E^* is given by $\bar\partial <\alpha, \psi>=<\bar\partial_{E^*}\alpha, \psi> + <\alpha, \bar\partial_E\psi>$ where $\bar\partial$ is the canonical holomorphic structure on $\mathbb{C}$ valued functions. Tensoring $\underline{\mathbb{C}}$ with K, which is equipped with the holomorphic structure d, we obtain the equation

$$d <\omega, \psi>=<\bar\partial_{KE^*}\omega, \psi> + <\omega, \bar\partial_E\psi> .$$

Integrating this equation Stokes theorem implies

$$\bar\partial_E^* = -\bar\partial_{KE^*}.$$

The ultimate version of the Fundamental Theorem 6.7, (3), proves (2) while Theorem 6.7, (4), implies (3). The last statement follows from the fact that the index of an elliptic operator only depends on its symbol. □

Chapter 7

The Riemann–Roch theorem

The last statement of SERRE's duality theorem will be used to calculate the dimension of the space of holomorphic sections of a holomorphic vector bundle $E \to M$. The RIEMANN–ROCH theorem relates the dimensions of the spaces of holomorphic sections of L and KL^{-1} with the degree and the rank of the vector bundle E and the genus of M.

The RIEMANN–ROCH theorem gives estimates on the dimension of the space of holomorphic sections in a given holomorphic vector bundle.

7.1 The residue pairing

Recall that a meromorphic 1–form ω is locally of the form $\omega = f dz$ where f is a meromorphic function.

Definition 7.1. The *residue of a meromorphic 1–form* ω on a Riemann surface M is defined as

$$\operatorname{Res}_p \omega = \frac{1}{2\pi i} \int_\gamma \omega,$$

where γ is a circle (in some coordinate) around p, small enough to ensure that ω is holomorphic inside γ besides p.

Remark 7.2.

1. Since ω is a holomorphic 1–form beside finitely many points, i.e. $\omega \in H^0(K|_{M\setminus\{\text{ poles of } \omega\}})$ we have $d\omega = 0$ on $M \setminus \{\text{ poles of } \omega\}$ and the definition of $\operatorname{Res}_p \omega$ is independent of the choice of γ, by STOKES theorem.

2. Choose non–intersecting discs U_i around the poles $p_1, \ldots, p_n$ of ω and let $\tilde{M} = M \setminus (U_1 \cup \ldots \cup U_n)$. Then by STOKES theorem

$$2\pi i \sum_{i=1}^n \operatorname{Res}_{p_i} \omega = \sum_{i=1}^n \int_{\partial U_i} \omega = -\int_{\partial \tilde{M}} \omega = -\int_{\tilde{M}} d\omega = 0.$$

3. Let $f\colon M \to \mathbb{C}$ be meromorphic and define the meromorphic 1–form

$$\omega := \frac{df}{f}.$$

If f has the LAURENT expansion $f = a_k z^k + a_{k+1} z^{k+1} \ldots$, $k \in \mathbb{Z}$ at 0 in a holomorphic chart $z\colon U \to M$ centered at $p \in M$, then

$$\omega = \frac{df}{f} = \frac{k a_k z^{k-1} + \ldots}{a_k z^k + \ldots} dz = \left(\frac{k}{z} + \text{some holomorphic rest}\right) dz.$$

Thus

$$\operatorname{Res}_p \omega = \operatorname{ord}_p f.$$

Proposition 7.3 (Residue pairing)**.** *Let M be a Riemann surface and $L(p)$ be the point bundle at p. Then there is a canonical isomorphism*

$$\operatorname{Res} : (KL(p))_p \to \mathbb{C}, \;\; \alpha_p \mapsto \operatorname{Res}(\alpha_p) := \operatorname{Res}_p(\alpha s_p^{-1}).$$

where $\alpha \in H^0(KL(p)\big|_U)$ with $\alpha(p) = \alpha_p$, $\;p \in U$, U open, and s_p^{-1} is the famous section with pole of order 1 at p.

Proof. Let $\alpha \in H^0(KL(p)\big|_U)$ with $\alpha(p) = \alpha_p$. Thus αs_p^{-1} is a local meromorphic 1–form with at most one single pole at p and we can define $\operatorname{Res}(\alpha_p) = \operatorname{Res}_p(\alpha s_p^{-1})$. Notice that this is independent of the chosen α since $\omega s_p^{-1} \in H^0(K)$ for any $\omega \in H^0(KL(p)\big|_U)$ with $\omega_p = 0$ and hence $\operatorname{Res}(\omega s_p^{-1}) = 0$.

The so defined map is an isomorphism: $\operatorname{Res}(\alpha_p) = 0$ if and only if $\alpha s_p^{-1} \in H^0(K)$ if and only if $\alpha_p = 0$.

(The idea of the proof is clear: Given $\alpha_p \in KL(p)$, use a local trivialization to identify α_p with $a_0 \in \mathbb{C}$. Any holomorphic map with $f(p) = a_0$ is of the form $f(z) = a_0 + a_1 z + \ldots$, thus $\frac{f}{z} = \frac{a_0}{z} + a_1 + \ldots$ and $\operatorname{Res}(\frac{f}{z}) = a_0$).

$\square$

Definition 7.4. Let L be a holomorphic line bundle over the Riemann surface M. The *connecting homomorphism* $\delta : L_p \to H^0(KL^{-1}L(p))^*$ is defined by

$$\delta(\psi_p)(\alpha) := \operatorname{Res}(\psi_p \alpha_p), \text{ where } \alpha \in H^0(KL^{-1}L(p)).$$

Remark 7.5. Clearly, this definition comes from the more general frame work of cohomology theory, compare Appendix B, Snake Lemma B.16. In general, the connecting homomorphism turns a short exact sequence into a long exact sequence on cohomology. Recall: An *exact sequence* is a sequence of groups (vector spaces, vector bundles, presheaves, sheaves, ...) of the form

$$V_1 \xrightarrow{\alpha_1} V_2 \xrightarrow{\alpha_2} V_3 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{k-1}} V_k$$

where $\alpha_k : V_k \to V_{k+1}$ are homomorphism with $\ker \alpha_k = \mathrm{Im}\ \alpha_{k-1}$. To remind the ideas of exact sequences we explain in more detail the *short exact sequences*: We investigate under which conditions the sequence

$$0 \xrightarrow{\alpha} U \xrightarrow{\beta} V \xrightarrow{\gamma} W \xrightarrow{\delta} 0.$$

is an exact sequence: Exactness at U requires that $0 = \mathrm{Im}\ \alpha = \ker \beta$, i.e. the map β has to be injective. Hence, we can interpret U as subgroup of V by considering $\beta = \mathrm{incl}$ as the inclusion map of U into V.

At V we need that $U = \mathrm{Im}\ \mathrm{incl} = \mathrm{Im}\ \beta = \ker \gamma$. Hence γ can be understood as a projection map from V to $\mathrm{Im}\ \gamma = V/U \subset W$. By exactness at W the map γ is surjective since $\mathrm{Im}\ \gamma = \ker \delta = W$. Hence, any short exact sequence can be seen as a sequence

$$0 \xrightarrow{\alpha} U \overset{\mathrm{incl}}{\hookrightarrow} V \xrightarrow{\mathrm{proj}} V/U \to 0.$$

Return to the Dolbeault cohomology groups and the connecting homomorphism. We will show that the connecting homomorphism gives a long exact sequence on Dolbeault cohomology. Consider first the maps

$$s_p \quad : \quad H^0(LL(-p)) \to H^0(L), \ \varphi \mapsto \varphi \otimes s_p^1$$

(again s_p^1 is "the famous section" with zero of order 1 at p) and the *evaluation map*

$$\mathrm{ev}_p \quad : \quad \Gamma(L) \to L_p, \ \psi \mapsto \psi(p).$$

Consider the sequence (later we will see that this gives the skyscraper sequence on the level of sheaves, cf. Appendix A, Example A.19, 4):

$$0 \longrightarrow H^0(LL(-p)) \xrightarrow{s_p} H^0(L) \xrightarrow{\mathrm{ev}_p} L_p$$

Since $\mathrm{ev}_p(\varphi \otimes s_p^1) = 0$ we have $\mathrm{Im}\ s_p \subset \ker \mathrm{ev}_p$. Furthermore, for $\psi \in \ker \mathrm{ev}_p$ define $\varphi = \psi \otimes s_p^{-1}$. Then $\varphi \in H^0(LL(-p))$ since $\psi(p) = 0$, and $s_p(\varphi) = \psi$, thus the sequence is exact.

Similarly, there are maps

$$s_p : H^0(KL^{-1}) \to H^0(KL^{-1}L(p)) \text{ and } \mathrm{ev}_p : H^0(KL^{-1}L(p)) \to KL^{-1}L(p)\big|_p.$$

They induce maps on the dual bundles and again we have an exact sequence

$$(KL^{-1}L(p)\big|_p)^* \xrightarrow{\mathrm{ev}_p^*} H^0(KL^{-1}L(p))^* \xrightarrow{s_p^*} (H^0(KL^{-1}))^* \longrightarrow 0$$

Use the Serre duality theorem, which gives an isomorphism $H^1(LL(-p)) = (H^0(K(LL(-p))^{-1})^* = (H^0(KL^{-1}L(p)))^*$, to define a map $s_p : H^1(LL(-p)) \to$

$H^1(L)$. Identifying $(KL^{-1}L(p)\big|_p)^*$ with L_p via the pairing

$$\begin{tikzcd} KL^{-1}L(p)\big|_p \times L_p \arrow[r] \arrow[d, "\text{contraction}"] & \mathbb{C} \\ KL(p)\big|_p \arrow[ur, "\text{Res}"'] & \end{tikzcd}$$

we see that the induced map $\mathrm{ev}_p^* : L_p \to H^1_{\bar\partial}(LL(-p))$ is in fact the connecting homomorphism δ (use again SERRE duality for $H^1(LL(-p)) \cong H^0(KL^{-1}L(p))^*$).

In particular, we have again an exact sequence:

Lemma 7.6. *The connecting homomorphism gives a long exact sequence on cohomology*

$$0 \longrightarrow H^0(LL(-p)) \xrightarrow{s_p} H^0(L) \xrightarrow{\mathrm{ev}_p} L_p \xrightarrow{\delta=\mathrm{ev}_p^*} H^1(LL(-p)) \xrightarrow{s_p^*} H^1(L) \longrightarrow 0$$

Proof. The remaining bit is to show the exactness at L_p. First, for $\psi \in H^0(L)$, $\alpha \in H^0(KL^{-1}L(p))$ we have

$$(\delta \circ \mathrm{ev}_p(\psi))(\alpha) = \delta(\psi_p)(\alpha) = \mathrm{Res}(\psi\alpha s_p^{-1}) = 0$$

since $\psi\alpha s_p^{-1}$ is a meromorphic 1–form with at most one pole at $p \in M$ (cf. Remark 7.2, (2)).

Now, let $\psi_p \in \ker\delta$ and choose $\tilde\psi \in \Gamma(L)$ which is holomorphic on some neighborhood U of p and extends ψ_p. We claim that

$$\bar\partial(\tilde\psi - \varphi \otimes s_p^1) = 0 \tag{7.1}$$

has a solution $\varphi \in \Gamma(LL(-p))$. In fact, (7.1) has a solution if and only if
★ $(\bar\partial\tilde\psi)s_p^{-1} = \bar\partial\varphi$ has a solution. The SERRE duality theorem states that this
richtige Paarung? is equivalent to $(\bar\partial\tilde\psi)s_p^{-1} \perp H^0(KL^{-1}L(p))$. But for $\alpha \in H^0(KL^{-1}L(p))$ we have

$$\begin{aligned} \int_M (\bar\partial\tilde\psi)s_p^{-1}\alpha &= \int_{M\setminus U} d(\tilde\psi s_p^{-1}\alpha) + \int_U \underbrace{(\bar\partial\tilde\psi)}_{=0} s_p^{-1}\alpha \\ &= -\int_{\partial U} \tilde\psi s_p^{-1}\alpha = -2\pi i \,\mathrm{Res}_p(\tilde\psi_p\alpha_p) \\ &= -2\pi i\delta(\psi_p)\alpha = 0 \end{aligned}$$

by the assumption $\psi_p \in \ker\delta$. Thus there is a solution of (7.1), in particular $\psi := (\tilde\psi - \varphi \otimes s_p^1) \in H^0(L)$ with $\mathrm{ev}_p(\psi) = \psi_p$. ☐

7.2 The Riemann–Roch theorem

We introduce the following short–hand notation: For a holomorphic vector bundle $E \to M$ we define

$$h^0(E) := \dim H^0(E).$$

The RIEMANN–ROCH theorem gives an equation on the dimension of the set of holomorphic sections:

Theorem 7.7 (RIEMANN–ROCH theorem)**.** *Let E be a holomorphic vector bundle over a compact Riemann surface M. Then*

$$h^0(E) - h^0(KE^*) = \deg E - (g-1)\operatorname{rank} E.$$

Proof. We will prove the RIEMANN–ROCH theorem in three steps.

1. Let L be a holomorphic line bundle of $\deg L = 0$ with $\bar\partial$–operator $\bar\partial_L$. Then $L \cong \underline{\mathbb{C}}$. On $\underline{\mathbb{C}}$ there is the usual $\bar\partial$–operator on functions if we identify sections in the trivial bundle with maps to $\mathbb{C}$. In particular, $\operatorname{index}(\bar\partial_L) = \operatorname{index}(\bar\partial)$. The only holomorphic functions on a compact Riemann surfaces are constant hence $\dim H^0_{\bar\partial}(\underline{\mathbb{C}}) = 1$. Furthermore $\dim H^0_{\bar\partial}(KL^{-1}) = \dim H^0_{\bar\partial}(K) = g$. Thus using SERRE duality

$$\begin{aligned}\dim H^0_{\bar\partial_L}(L) - \dim H^0_{\bar\partial_{KL^{-1}}}(KL^{-1}) &= \dim\ker\bar\partial_L - \dim\ker\bar\partial_{KL^{-1}} \\ &= \operatorname{index}(\bar\partial_L) = \operatorname{index}(\bar\partial) \\ &= \dim H^0_{\bar\partial}(\underline{\mathbb{C}}) - \dim H^0_{\bar\partial}(K) \\ &= 1-g \\ &= \deg L - (g-1)\operatorname{rank} L.\end{aligned}$$

2. Let L be a holomorphic line bundle with $\deg L = k$. We prove the RIEMANN–ROCH theorem by induction on the degree of L. We consider first the case $k \geq 0$. For $k = 0$ the RIEMANN–ROCH theorem holds by (1). Let L be a holomorphic line bundle of degree k. Then $\deg LL(-p) = \deg L - 1 = k-1$, hence we can assume by induction that the RIEMANN–ROCH theorem holds for $LL(-p)$.

   Consider the long exact sequence from Lemma 7.6. Using a linear algebra argument we see that the alternating sum of the dimensions has to vanish, i.e.

$$\begin{aligned}0 &= \dim H^0(LL(-p)) - \dim H^0(L) + \dim L_p \\ &\quad - \dim H^0(KL^{-1}L(p))^* + \dim H^0(KL^{-1})^* \\ &= h^0(L) - \deg LL(-p)) + g - 1 + 1 - h^0(KL^{-1})\end{aligned}$$

thus

$$h^0(L) - h^0(KL^{-1}) = \deg L - (g-1)\,\mathrm{rank}\,L.$$

Now, if L has negative degree k, then by induction we can assume that the Riemann–Roch theorem holds for $LL(p)$. Consider again the long exact sequence from Lemma 7.6 but for the line bundle $LL(p)$. The same arguments as before give

$$\begin{aligned} 0 &= \dim H^0(L) - \dim H^0(LL(p)) + \dim(LL(p))_p \\ &\quad - \dim H^0(KL^{-1})^* + \dim H^0(KL^{-1}L(-p))^* \end{aligned}$$

and thus the assertion for any line bundle.

3. Let now E be a holomorphic rank r vector bundle of arbitrary degree with $\bar\partial$–operator $\bar\partial_E$. We know that $E \cong_{C^\infty} \underline{\mathbb{C}}^{r-1} \oplus \Lambda^r E$ where we remind that the determinant bundle $\Lambda^r E$ is the canonical line bundle associated to E with $\deg E = \deg \Lambda^r E$. Now, $\bar\partial_{\underline{\mathbb{C}}^{r-1}} \oplus \bar\partial_{\Lambda^r E}$ gives a holomorphic structure on $\underline{\mathbb{C}}^{r-1} \oplus \Lambda^r E$ and thus $\mathrm{index}\,\bar\partial_E = \mathrm{index}(\bar\partial_{\underline{\mathbb{C}}^{r-1}} \oplus \bar\partial_{\Lambda^r E})$. We obtain

$$\begin{aligned} h^0(E) - h^0(KE^*) &= \mathrm{index}\,\bar\partial_E = \mathrm{index}(\bar\partial_{\underline{\mathbb{C}}^{r-1}} \oplus \bar\partial_{\Lambda^r E}) \\ &= \mathrm{index}(\bar\partial_{\underline{\mathbb{C}}^{r-1}}) + \mathrm{index}(\bar\partial_{\Lambda^r E}) \\ &= h^0(\underline{\mathbb{C}}^{r-1}) - h^0(K(\underline{\mathbb{C}}^{r-1})^{-1}) + h^0(\Lambda^r E) - h^0(K(\Lambda^r E)^{-1}) \\ &= (r-1) - g(r-1) + \deg \Lambda^r E - (g-1)\,\mathrm{rank}\,\Lambda^r E \\ &= (1-g)(r-1) + \deg E - (g-1) \\ &= (1-g)\,\mathrm{rank}\,E + \deg E \end{aligned}$$

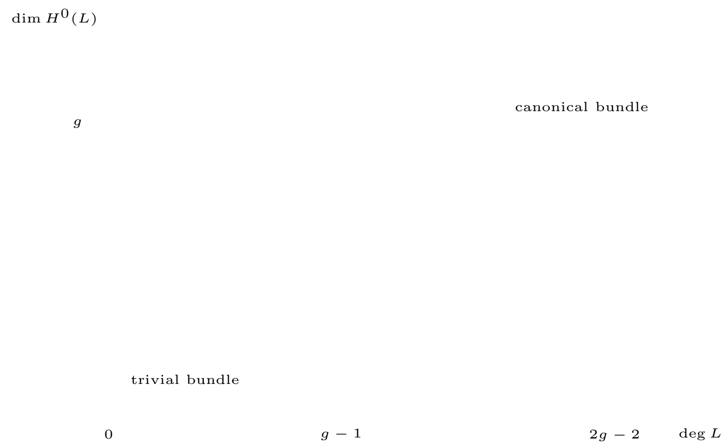
(*"Das war elegant. Da ist die Nachwelt dankbar!"*) $\square$

Example 7.8 (Degree of the canonical bundle)**.** Applying the Riemann–Roch theorem to $L = K$ implies $h^0(K) - h^0(\underline{\mathbb{C}}) = \deg K - g + 1$, since $KL^{-1} = KK^{-1} = \underline{\mathbb{C}}$. Thus

$$\deg K = 2g - 2.$$

(as we have seen before, using the Gauss–Bonnet theorem).

Corollary 7.9. *Let $L \to M$ be a holomorphic line bundle with $\deg L = d$. Upper and lower bounds on $h^0(L)$ are given by the following diagram:*



More precisely,

1. $h^0(L) = 0$ *for* $d < 0$.
2. $0 \leq h^0(L) \leq d+1$
3. $0 \leq d-g+1 \leq h^0(L) \leq g$ *for* $d \geq g-1$.
4. $h^0(L) = d-g+1$ *for* $d > 2g-2$

Proof. 1. Degree formula.

2. Let $n = h^0(L) \geq 1$ and $p \in M$. Then there exists a holomorphic section ψ with a zero at p of order at least $n-1$: the space of holomorphic sections which vanish at p is at least a hyperplane. Inside this hyperplane the space of holomorphic sections which vanish at p of second order is again at least a hyperplane, and so on.

   Thus, the degree of L can be estimated by the order of zeros of ψ, hence $d \geq n-1$.

3. If $d \geq g-1$ then the Riemann–Roch theorem gives
$$h^0(L) = h^0(KL^{-1}) + d - g + 1 \geq d - g + 1 \geq 0.$$
   On the other hand, if $d \geq g$ then $\tilde{n} = h^0(KL^{-1}) \geq 1$. Applying the argument of (2) to KL^{-1} we have
$$2g-2-d = \deg KL^{-1} \geq \tilde{n} - 1 = h^0(L) - d + g - 2.$$

4. If $d > 2g-2$ then $\deg KL^{-1} < 0$ hence $h^0(KL^{-1}) = 0$ and the Riemann–Roch theorem gives the result.

$\square$

Remark 7.10. There is a better upper bound for $h^0(L)$: In fact the holomorphic line bundles are in the grey shaded area for $0 \leq d \leq 2g-2$, i.e.

$$h^0(L) \leq \frac{d}{2} + 1.$$

★
Paul

("The RIEMANN–ROCH *theorem holds literally for quaternionic vector bundles over compact Riemann surfaces. ")*

Chapter 8

Grothendieck splitting and Kodaira embedding

In this chapter we give some applications of the RIEMANN–ROCH theorem . In particular, every holomorphic vector bundle over a sphere splits into holomorphic line subbundles, cf. Grothendieck splitting, Theorem 8.9, and any compact Riemann surfaces can be embedded into a suitable projective space, KODAIRA embedding theorem 8.25, and is hence algebraic, cf. Chows Theorem 8.27.

8.1 Grothendieck splitting

We ask if we can split a holomorphic vector bundle over a Riemann surface into holomorphic line subbundles. For this, we need first a holomorphic line subbundle of E which will be given if we know: ★

Allg.: Gegenbeispiel?

Theorem 8.1 (Existence of a non–trivial meromorphic section)**.** *Let $E \to M$ be a holomorphic vector bundle over a compact Riemann surface M. Then there exists a non trivial meromorphic section of E.*

Proof. Consider the bundle $EL(np)$. We estimate the dimension of the set of holomorphic sections with the RIEMANN–ROCH theorem :

$$\begin{aligned} h^0(EL(np) &\geq \deg EL(np) - (g-1)\operatorname{rank}(EL(np)) \\ &= n \operatorname{rank} E + \deg E - (g-1)\operatorname{rank} E. \end{aligned}$$

Hence, if n is sufficently large, then $EL(np)$ has a holomorphic section ψ and $\varphi = \psi \otimes s_p^{-n}$ is a meromorphic section of E with at most a pole of order n at p. □

This enables us to find a holomorphic subbundle, and in fact a holomorphic flag:

Corollary 8.2. *1. Every holomorphic vector bundle E over a compact Riemann surface has a holomorphic line subbundle L.*

2. A holomorphic vector bundle E of rank r over a compact Riemann surface has a flag *of holomorphic subbundles*

$$0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow E_3 \hookrightarrow \ldots \hookrightarrow E_{r-1} \hookrightarrow E,$$

such that $\operatorname{rank}(E_k) = k$.

Proof. 1. Let $\varphi_i \in H^0(E\big|_{U_i})$ be a set of local holomorphic sections on an open cover $\mathcal{U} = \{U_i\}$ of M . These φ_i define a holomorphic line subbundle L of E, if $\operatorname{span}(\varphi_i) = \operatorname{span}(\varphi_j)$ on $U_i \cap U_j$.

Let $\psi \in \mathcal{M}(E)$ be a meromorphic section of E and $z_i \colon U_i \to \mathbb{C}$ holomorphic charts on open neighbourhoods U_i of the poles $\{p_1, \ldots, p_r\}$ of ψ. If we add $U_0 = M \setminus \{p_1, \ldots, p_r\}$ then $\{U_i\}_{0\le i\le r}$ is a cover of M. Let $n_i := \operatorname{ord}_{p_i} \psi$. Then $\varphi_0 := \psi\big|_{U_0}$ and

$$\begin{array}{rcl} \varphi_i \colon U_i & \longrightarrow & E \\ p & \longmapsto & z_i(p)^{n_i}\psi(p) \end{array}$$

are local holomorphic sections, satisfying $\operatorname{span}(\varphi_i) = \operatorname{span}(\varphi_j)$ on $U_i \cap U_j$.

2. If $L \subset E$ is a holomorphic subbundle of E then E/L is a holomorphic bundle. Thus (1) provides a holomorphic line bundle $\tilde{L} \subset E/L$, whose preimage under the canonical projection $\pi\colon E \to E/L$ is a holomorphic subbundle of E, $\operatorname{rank}(E) = \operatorname{rank}(L) + 1$.

$\square$

Lemma 8.3. *Let*

$$0 \longrightarrow E_1 \xhookrightarrow{\text{incl}} E_2 \xrightarrow{\pi} E_3 \longrightarrow 0 \tag{8.1}$$

be an exact sequence of holomorphic vector bundles over a compact Riemann surface M. Then there exists a connecting homomorphism δ *such that the long sequence on cohomology*

$$\begin{tikzcd} 0 \arrow[r] & H^0(E_1) \arrow[r, hook] & H^0(E_2) \arrow[r] & H^0(E_3) \arrow[dll, "\delta"] & \\ & H^1(E_1) \arrow[r] & H^1(E_2) \arrow[r] & H^1(E_3) \arrow[r] & 0 \end{tikzcd}$$

is exact.

Proof. We start by defining the connecting homomorphism:

Let $\psi \in H^0(E_3)$. Then there exists a $\varphi \in \Gamma(E_2)$ with $\pi\varphi = \psi$. Consider $\alpha = \bar{\partial}\varphi \in \Gamma(\bar{K}E_2)$. But $\pi\alpha = \bar{\partial}\pi\varphi = \bar{\partial}\psi = 0$, hence $\alpha \in \Gamma(\bar{K}E_1)$. If $\tilde{\varphi}$ projects also to ψ, i.e. $\pi(\varphi - \tilde{\varphi}) = 0$ then $\varphi - \tilde{\varphi} \in \Gamma(E_1)$ by exactness of the sequence. Thus $\bar{\partial}(\varphi - \tilde{\varphi}) \in \bar{\partial}\Gamma(E_1)$ and α can be interpreted as $\alpha \in \frac{\Gamma(\bar{K}E_1)}{\bar{\partial}\Gamma(E_1)} = H^1(E_1)$. Hence, define δ by: $\delta(\psi) = \bar{\partial}\varphi$.

Next, we show that the sequence on cohomology is exact:

Clearly, the sequence (8.1) induces a sequence

$$0 \longrightarrow H^0(E_1) \hookrightarrow H^0(E_2) \longrightarrow H^0(E_3)$$

since the maps incl and π are holomorphic. But this sequence is also exact by the exactness of (8.1).

For the exactness of the sequence

$$H^1(E_1) \hookrightarrow H^1(E_2) \longrightarrow H^1(E_3) \longrightarrow 0\,. \tag{8.2}$$

we use the Serre duality theorem:

The exact sequence (8.1) gives an exact sequence

$$0 \longrightarrow E_3^* \xhookrightarrow{\pi^*} E_2^* \xrightarrow{\text{incl}^*} E_1^* \longrightarrow 0\,\cdot$$

The same argument as above gives the exact sequence

$$0 \longrightarrow H^0(E_3^*) \hookrightarrow H^0(E_2^*) \longrightarrow H^0(E_1^*)$$

which gives also the exactness of

$$0 \longrightarrow H^0(KE_3^*) \hookrightarrow H^0(KE_2^*) \longrightarrow H^0(KE_1^*)$$

Dualizing this sequence again and using $H^0(KE^*) = H^1(E)^*$ we obtain the exact sequence (8.2).

Remains to check exactness at $H^0(E_3)$. Clearly, if $\psi \in \operatorname{Im}\pi$ then $\psi = \pi\varphi$ for $\varphi \in H^0(E_2)$, and hence $\delta\psi = \bar{\partial}\varphi = 0$. On the other hand, if $\delta\psi = 0$ then there exists $\varphi \in \Gamma(E_2)$ with $\bar{\partial}\varphi \in \bar{\partial}\Gamma(E_1)$ and $\pi\varphi = \psi$. Let $\varphi_1 \in \Gamma(E_1)$ with $\bar{\partial}\varphi_1 = \bar{\partial}\varphi$. Then the difference $\varphi - \varphi_1$ projects to ψ and is holomorphic. $\square$

Now, we need a condition when a vector bundle E splits holomorphically into a holomorphic line subbundle L and a holomorphic *subbundle* $Q := E/L$.

Lemma 8.4. *Let $L \subset E$ be a holomorphic line subbundle, $Q = E/L$. The sequence*

$$0 \longrightarrow Q^{-1}L \xhookrightarrow{\pi^*} E^{-1}L \xrightarrow{\text{incl}^*} \underline{\mathbb{C}} \longrightarrow 0 \tag{8.3}$$

is a short exact sequence of holomorphic vector bundles. Furthermore, $E = Q \oplus L$ splits holomorphically, if $\delta(1) = 0$ for the connecting homomorphism of the induced long exact sequence on cohomology.

Proof. First, $0 \longrightarrow L \xhookrightarrow{\text{incl}} E \xrightarrow{\pi} Q = E/L \longrightarrow 0$ is a short exact sequence. Dualizing and tensoring with L we obtain the exact sequence (8.3). The induced connecting homomorphism is a map $\delta : \mathbb{C} = H^0(\underline{\mathbb{C}}) \to H^1(Q^{-1}L)$. Since $\delta(1) = 0$, by exactness of the long exact sequence on cohomology there exists $\psi \in H^0(E^{-1}L)$ which maps to 1, in particular, ψ vanishes nowhere. Since the sequence on vector bundles is exact we also know that $\psi_p \notin (Q^{-1}L)_p$. Thus $EL^{-1} = Q^{-1}L \oplus \mathbb{C}\psi$ holomorphically. Since $\mathbb{C}\psi \cong \underline{\mathbb{C}}$ we obtain, again by tensoring L and dualizing, $E = Q \oplus L$. ☐

Corollary 8.5. *Let $L \subset E$ be a holomorphic subbundle and $Q = E/L$. If $H^1(Q^{-1}L) = 0$ then E splits into the holomorphic subbundles $L \oplus Q$.*

Lemma 8.6. *Let $f : L \to E$ a holomorphic map between to holomorphic vector bundles L and E over M. If $f \not\equiv 0$ then*

$$\deg L \leq \deg(fL)$$

where we denote by (fL) the line bundle induced by the image of L under f.

Proof. The zeros of f are isolated thus f defines a line bundle. Since any holomorphic section of L defines a holomorphic secton of (fL) the degree can only increase. ☐

★ Zitat:Wu, genauer erklaeren. Muß L Linienbündel sein?

Lemma 8.7. *Let $E \to M$ be a holomorphic vector bundle. Then*

$$\{\deg L \mid L \subset E \text{ holomorphic line subbundle}\}$$

has an upper bound.

★ example

Remark 8.8. But there is no lower bound.

Proof. We will prove the statement by induction on the rank of E. Then rank 1 case is clear.

Let $H \subset E$ be a fixed holomorphic subbundle of codim $= 1$. Let $L \subset E$ a holomorphic line subbundle. If L is a holomorphic subbundle of H then by assumption $\deg L$ is bounded from above. If not, consider the induced holomorphic map $f : L \hookrightarrow E/H$ given by $L \to E \xrightarrow{\pi} E/H$ where E/H is a line bundle. Then $f \not\equiv 0$, so the induced line bundle is $(fL) = E/H$ and hence $\deg L \leq \deg E/H$. ☐

Now, we are able to prove

Theorem 8.9 (Grothendieck splitting)**.** *Let $E \to \mathbb{C}P^1$ be a holomorphic vector bundle of* rank r. *Then there is a splitting*

$$E = \bigoplus_{i=1}^{r} L_i, \ \deg L_1 \leq \ldots \leq \deg L_r,$$

where $L_i \to \mathbb{C}P^1$ are holomorphic line bundles. The sequence

$$\deg L_1, \ldots, \deg L_r$$

is a holomorphic invariant of E.

Remark 8.10. Note that $d = \deg E = \sum \deg L_i$.

Proof. Choose a holomorphic line subbundle of $E \to \mathbb{C}P^1$ with maximal degree, say $\deg L^N = N$. (Recall, if L is a line bundle over $\mathbb{C}P^1$ then holomorphically $L = L(kp) = L^k$ where $k = \deg L$). By the RIEMANN–ROCH theorem we know that $h^0(L^k) - h^0(KL^{-k}) = k + 1$ for any $k \in \mathbb{Z}$. Since $\deg KL^{-k} = -2 - k \leq 0$ for $k \geq 0$ we know that $h^0(KL^{-k}) = 0$ hence

$$h^0(L^k) = k + 1 \text{ for } k \geq 0.$$

It suffices to show that E splits holomorphically into $E = L^N \oplus Q$ ($Q = E/L^N$). For that we claim $H^1(Q^{-1}L^N) = 0$.

Assume that $H^0(KQL^{-N}) \neq 0$ (SERRE duality!). We proceed by induction over the rank of E, thus we can assume that Q splits holomorphically into line subbundles $Q = \bigoplus_{i=1}^{r-1} L^{a_i}$ where $a_{r-1} \geq a_{r-2} \geq \ldots \geq a_1$. Hence

$$0 \neq H^0(KQL^{-N}) = H^0(KL^{-N} \bigoplus_{i=1}^{r-1} L^{a_i}) = \bigoplus_{i=1}^{r-1} H^0(KL^{-N+a_i})$$

and there exists $j \in \{1, \ldots, r-1\}$ with $H^0(KL^{-N+a_j}) \neq 0$. Therefore $-2 - N + a_j = \deg KL^{-N+a_j} \geq 0$. Consider the exact sequence

$$0 \longrightarrow L^N \hookrightarrow E \longrightarrow Q \longrightarrow 0\,.$$

It induces a short exact sequence

$$0 \longrightarrow L^{-1} \hookrightarrow EL^{-N-1} \longrightarrow QL^{-N-1} \longrightarrow 0$$

and thus the long exact sequence on cohomology

$$0 \longrightarrow H^0(L^{-1}) \hookrightarrow H^0(EL^{-N-1}) \longrightarrow H^0(QL^{-N-1}) \longrightarrow H^1(L^{-1}).$$

Since $\deg L^{-1} < 0$ we have $H^0(L^{-1}) = 0$ and by SERRE duality and the RIEMANN–ROCH theorem $-h^1(L^{-1}) = -h^0(KL) = h^0(L^{-1}) - h^0(KL) = \deg L^{-1} + 1 = 0$. Therefore, the long exact sequence on cohomology gives

$$H^0(EL^{-N-1}) \cong H^0(QL^{-N-1}).$$

Since there exists a $j \in \{1, \ldots, r-1\}$ with $-2 - N + a_j \geq 0$ and hence $h^0(L^{a_j-N-1}) = a_j - N \geq 2$ we obtain

$$H^0(EL^{-N-1}) \cong H^0(QL^{-N-1}) = \bigoplus_{i=1}^{r-1} H^0(L^{a_i-N-1}) \neq 0.$$

This leads to the contradiction since any non–vanishing holomorphic section of $H^0(EL^{-N-1})$ induces a non–trivial holomorphic map $f : L^{N+1} \to E$ and thus $\deg(fL^{N+1}) \geq \deg L^{N+1} = N + 1$ which contradicts the maximality of L^N.

It remains to show that the sequence of the degrees is a holomorphic invariant. Define a map

$$\lambda(n) := h^0(EL^{-n}).$$

If $E = \bigoplus_{i=1}^{r} L^{a_i}$ is a holomorphic splitting then by introducing the map

$$H(x) := \begin{cases} 1; & x \geq 0 \\ 0; & x < 0 \end{cases}$$

we see

$$\lambda(n) = \sum h^0(L^{a_i - n}) = \sum H(a_i - n)(a_i - n + 1)$$

Since $\lambda(a_j) - \lambda(a_j + 1) = r - j + 1$ the map λ defines a_j but is itself defined independently of the splitting.

★ das stimmt nicht. Es wird definitiv nur $\min\{k \mid a_k = a_j\}$ ausgespuckt. Außerdem kann ich a_j daraus nicht rekonstruieren ... man sollte mal in [BT] reingucken, die scheinen das zu machen

$\square$

8.2 Birkhoff factoriziation of loops

We show how the Grothendieck splitting can be used in the context of factorizing loop groups.

Theorem 8.11 (Birkhoff factorization)**.** *Let $g : \mathbb{C}^* \to GL(n, \mathbb{C})$ a holomorphic map. Then there exist holomorphic $A, B : \mathbb{C} \to GL(n, \mathbb{C})$ and integers $a_1 \leq \ldots \leq a_n$ such that*

$$g(z) = A(z) \begin{pmatrix} z^{a_1} & 0 & \ldots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \\ 0 & \ldots & & z^{a_n} \end{pmatrix} B(\frac{1}{z}).$$

Proof. The map g defines a rank n vector bundle E over $\mathbb{C}P^1$ if we interpret g as transition function. Let U_0 and U_∞ be the usual neighborhoods of 0 and ∞, and ψ_0, ψ_∞ holomorphic sections of the vector bundle E over U_0 resp. U_∞ with $\psi_0 = \psi_\infty g$. Now, the vector bundle E splits holomorphically into line subbundles, i.e. $E = \bigoplus L^{a_i}$. Thus, there exists a frame of local sections $\tilde{\psi}_0$, $\tilde{\psi}_\infty$ such that $\tilde{\psi}_0 = \tilde{\psi}_\infty \begin{pmatrix} z^{a_1} & & \\ & \ddots & \\ & & z^{a_n} \end{pmatrix}$. The two frames can be transformed into each other, i.e. there exist holomorphic maps $A, B : \mathbb{C} \to \mathrm{GL}(n, \mathbb{C})$ with $\psi_0 B = \tilde{\psi}_0$, $\psi_\infty A = \tilde{\psi}_\infty$. Hence g is of the sought form. $\square$

8.3 Holomorphic line bundles and divisors

As a further application of the existence of a meromorphic section we show the existence of an isomorphism between the set of isomorphism classes of holomorphic line bundles, the so–called Picard group, and the set of equivalence classes of divisors. Hereby,

Definition 8.12. Let M be a compact Riemann surface. Then the formal sum $D = \sum_{p \in M} n_p p$ is called a *divisor* on M, if $n_p \in \mathbb{Z}$ and $n_p = 0$ besides a finite number of points $p \in M$. The *degree of a divisor* D is defined as

$$\deg D = \sum_{p \in M} n_p \, .$$

Definition 8.13. Let f be a meromorphic function on a compact Riemann surface M. Then we associate to f the divisor

$$(f) = \sum_{p \in M} (\operatorname{ord}_p f) p.$$

Lemma 8.14. *If* $f \in \mathcal{M}(M)$ *then* $\deg(f) = 0$.

Proof. Remark 7.2, (2). ☐

Definition 8.15. The *divisor group* of a compact Riemann surface M is defined as

$$\operatorname{Div}(M) = \{\, D \mid D \text{ is a Divisor} \,\}/_{\sim}$$

where $D_1 \sim D_2$ if and only if there exists a meromorphic function f on M, such that $D_2 - D_1 = (f)$. Since $\deg(f) = 0$ for meromorphic functions f, deg is well defined on $\operatorname{Div}(M)$ and we define

$$\operatorname{Div}_k(M) = \deg^{-1}(k) \subset \operatorname{Div}(M)$$

for each $k \in \mathbb{Z}$.

In Definition 5.22 we introduced the point bundles at p of degree k on a compact Riemann surface M, called $L(kp)$. We will associate a unique (up to isomorphism) holomorphic line bundle to any divisor on M. Then we show that this map induces an isomorphism between the divisor group $\operatorname{Div}(M)$ and the Picard group, i.e.

Definition 8.16. The Picard *group* is defined by the isomorphism classes of holomorphic line bundles, i.e.

$$\operatorname{Pic}(M) = \{\text{holomorphic line bundles}\}/\{\text{holomorphic isomorphisms}\}.$$

First, we associate to every divisor a holomorphic line bundle by

Definition 8.17. Let M be a compact Riemann surface. Let $D = \sum_{j=1}^{k} n_j p_j$ be a divisor on M, then we define

$$L(D) := L(n_1 p_1) \otimes \ldots \otimes L(n_k p_k).$$

Remark 8.18. For any divisor D on M the equation $\deg(D) = \deg(L(D))$ holds.

Proposition 8.19. *The map*

$$L\colon \operatorname{Div}(M) \to \operatorname{Pic}(M)$$

is an isomorphism, if M is a compact Riemann surface. In particular, let ψ be a non–trivial meromorphic section and D its divisor. Then

$$L(D) \cong L \text{ as holomorphic line bundle.}$$

Proof. L is a group homomorphism between the group of divisors on M with respect to $+$ and the group of holomorphic line bundles with respect to $\otimes$. (Recall that the tensor product of holomorphic line bundles commutes.)

Let $D = \sum_{j=1}^{k} n_j p_j$ be a divisor on M, and let $s_{p_j}^{n_j} \in \Gamma(L(n_j p_j))$ be the "famous section", which is non zero and holomorphic on $M \setminus \{p_j\}$ and has a pole of order $-n_j$ if $n_j < 0$, or a zero of order n_j if $n_j > 0$ at p_j. Then

$$\psi = s_{p_1}^{n_1} \otimes \ldots \otimes s_{p_k}^{n_k}$$

is a meromorphic section of $L(D)$.

If $D = 0$ in $\operatorname{Div}(M)$ then $D = (f)$ for some meromorphic function. Then ψ/f is a non vanishing holomorphic section, thus $L(D) \cong M \times \mathbb{C}$. This means that L is a well defined map of $\operatorname{Div}(M)$ to $\operatorname{Pic}(M)$.

If $L(D) = 0$ in $\operatorname{Pic}(M)$ then $L(D) \cong M \times \mathbb{C}$ holomorphically, thus the section ψ is in fact a meromorphic function f on M, such that $D = (f)$. Hence L is one–to–one.

To show that L is onto, let $L \to M$ be a holomorphic line bundle. Then L has a non trivial meromorphic section ψ. This section defines a divisor $D = (\psi) = \sum_{j=0}^{k} \operatorname{ord}_{p_j} \psi$, where $\{p_1, \ldots, p_k\}$ is the set of all zeros and poles of ψ. We have to show that $L(D) \cong L$ holomorphically. But

$$L \otimes L(-D)$$

has a non vanishing holomorphic section, namely $\psi \otimes \psi_1 \otimes \ldots \otimes \psi_k$, where the $\psi_j = s_{p_j}^{-\operatorname{ord}_{p_j} \psi}$ are the famous holomorphic respectively meromorphic sections of $L((-\operatorname{ord}_{p_j} \psi) p_j)$. Thus $L \otimes L(-D) \cong M \times \mathbb{C}$, consequently

$$L \cong L(-D)^{-1} \cong L(D).$$

□

8.4 Kodaira embedding theorem

We first show the existence of a holomorphic map between any Riemann surface and $\mathbb{C}P^1$. The degree of this map is bounded by the genus of the Riemann surface.

Moreover, one may ask, if it is possible to embed each Riemann surface into a suitable projective space. The answer to this question is given by the KODAIRA embedding theorem.

Let $p \in M$ and $L = L((g+1)p)$. Then $\deg L = g+1$ and by the RIEMANN–ROCH theorem

$$h^0(L((g+1)p)) = h^0(L) \geq \deg L - g + 1 = 2.$$

Let ψ_1 be the famous section of $L((g+1)p)$, i.e. the holomorphic section with a zero of order $g+1$ at p and no other zero on $M \setminus \{p\}$. There is a linearly independent section to ψ_1, say ψ_2, because $h^0(L((g+1)p)) \geq 2$. The sections ψ_1 and ψ_2 define a meromorphic function $f\colon M \to \mathbb{C}$ by the equation $\psi_2 = \psi_1 f$, we use the short–hand notation $f = \psi_2/\psi_1$.

Assume that $\operatorname{ord}_p \psi_2 \geq g+1$. Then $\operatorname{ord}_q \psi_1 = \operatorname{ord}_q \psi_2$ for all $q \in M$, since $\sum_{q\in M} \operatorname{ord}_q \psi_2 = \deg(L) = g+1$. Hence f is holomorphic. This contradicts the linear independence of ψ_1 and ψ_2. Thus $\operatorname{ord}_p \psi_2 < g+1$.

The fact that $\operatorname{ord}_p f = \operatorname{ord}_p \psi_2 - \operatorname{ord}_p \psi_1$ implies

$$-g-1 \leq \operatorname{ord}_p f < g+1-g-1 = 0.$$

Hence f is holomorphic on $M \setminus \{p\}$ and has a pole of order at least 1 and at most $g+1$ at p. Thus we have shown

Proposition 8.20. *Let M be a compact Riemann surface of genus g. Then there exists a holomorphic map*

$$f\colon M \to \mathbb{C}P^1,$$

whose degree satisfies

$$1 \leq \deg f \leq g+1.$$

In the case of genus $g = 0$ this implies that M is biholomorphic to $\mathbb{C}P^1$.

Definition 8.21. Let L be a holomorphic line bundle over a compact Riemann surface with a non trivial holomorphic section, i.e. $h^0(L) > 0$. Let $H \subset H^0(L)$, $\dim H \geq 1$, then

$$\mathcal{B}_H = \{\, p \in M \mid \psi(p) = 0 \text{ for all } \psi \in H \,\}$$

is called the *base locus* of the *linear system* H, and $p \in \mathcal{B}_H$ a *base point.* For $H = H^0(L)$ we write $\mathcal{B}_L$ and call $p \in \mathcal{B}_L$ a base point of L.

Remark 8.22. $\mathcal{B}_L = \emptyset$ is equivalent to the exactness of the sequence

$$H^0(L) \longrightarrow L_p \longrightarrow 0$$

for all $p \in M$.

Suppose now that $\deg L > \deg K + 1$. Then $\deg KL^{-1} = \deg K - \deg L < -1$ and hence $h^0(KL^{-1}) = 0$. Thus the RIEMANN–ROCH theorem gives $h^0(L) = \deg L - g + 1 > 2g - 1 - g + 1 = g$. Thus L has a non trivial holomorphic section. Recall the long exact sequence on cohomology

$$0 \longrightarrow H^0(LL(-p)) \longrightarrow H^0(L) \longrightarrow L_p \longrightarrow H^1(LL(-p))$$

But $H^1(LL(-p)) \cong H^0(K(LL(-p))^{-1})$ by SERRE duality and $\deg K(LL(-p))^{-1} = \deg K - \deg L + 1 < 0$, thus $H^1(LL(-p)) = 0$ and $H^0(L) \longrightarrow L_p \longrightarrow 0$ is exact. Therefore, we have shown:

Proposition 8.23. *Let L be a holomorphic line bundle over a compact Riemann surface and $\deg L > \deg K + 1$, then $\mathcal{B}_L = \emptyset$.*

Lemma 8.24 (KODAIRA correspondence)**.** *Let $L \to M$ be a holomorphic line bundle over M and $H \subset H^0(L)$ be a linear system with $\dim H \geq 2$, then we can define a holomorpic map $f : M \backslash \mathcal{B}_L \to \mathbb{C}P^{n-1}$ via $f(p) = \mathrm{ev}_p^*(L_p^*)$*

Proof. Since $H \xrightarrow{\mathrm{ev}_p} L_p \longrightarrow 0$ is exact for $p \in M \backslash \mathcal{B}_L$ the map $\mathrm{ev}_p^* : L_p^* \to H^*$ is injective. Thus we have indeed a map $f : M \to P(H^*) \cong \mathbb{C}P^{n-1}$. □

Geometrically, any point, which is not a base point, defines the hyperplane H_p of all holomorphic sections vanishing at p. Let $\alpha \in P(H^*)$ be the unique dual form, such that $\psi \in H_p$ if and only $\alpha(\psi) = 0$. Then our map f is in fact the map $p \mapsto \alpha$.

In local coordinates the picture is as follows: choose a basis $s_1, \ldots, s_n$ of H and let $p \in M \backslash \mathcal{B}_L$. Let φ be a local nowhere vanishing holomorphic section in H around p. Then $f_i := \frac{s_i}{\varphi} : U \subset M \to \mathbb{C}$ are holomorphic maps, and since p is not a base point, $\begin{pmatrix} f_1(p) \\ \vdots \\ f_n(p) \end{pmatrix}$ defines a point in $\mathbb{C}P^{n-1}$. Note, that this point is independent of the chosen φ. By the identification

$$P(H^*) \cong \mathbb{C}P^{n-1}, \quad \alpha \longleftrightarrow \left[\begin{pmatrix} < \alpha, s_1 > \\ \vdots \\ < \alpha, s_n > \end{pmatrix}\right]$$

we see that $f(p) \in P(H^*)$ corresponds to $\left[\begin{pmatrix} f_1(p) \\ \vdots \\ f_n(p) \end{pmatrix}\right]$ since for $\alpha_p \in L_p^*$:

$$< \mathrm{ev}_p^*(\alpha_p), s_i >=< \alpha_p, \mathrm{ev}_p\, s_i >=< \alpha_p, s_i(p) >= f_i(p) < \alpha_p, \varphi(p) > .$$

★ die Abbildung kann auch in base points definiert werden.

Now, the natural question is whether this map defines an embedding. An answer to this question is given by

Theorem 8.25 (Kodaira embedding theorem)**.** *Let M be a compact Riemann surface and $L \to M$ a holomorphic line bundle over M such that* $\deg L > \deg K + 2$. *Then the map*

$$f : M \to P(H^0(L)^*) \cong \mathbb{C}P^{d-g},\ p \mapsto f(p) = L_p^*$$

is an embedding.

Remark 8.26. For any compact Riemann surface there exists a holomorphic line bundle of arbitrary degree (take a point bundle!). Thus, any compact Riemann surface can be embedded into a complex projective space.

Proof. Under the above assumptions we have $h^0(KL^{-1}) = 0$ and by the Riemann–Roch theorem hence $h^0(L) = d - g + 1 > 2g - 2 + 2 - g + 1 = g + 1 \geq 1$. Furthermore, since $\deg L > \deg K + 2 > \deg K + 1$ the bundle L is by Proposition 8.23 base point free. Thus the map f is well–defined by the above considerations.

Since M is compact it suffices to show that f is an injective immersion. Notice that $L_p^* = L_q^* \subset H^0(L)^*$ if and only if for $0 \neq \alpha \in L_p^*,\ 0 \neq \beta \in L_q^*$ there exists $\lambda \in \mathbb{C}\backslash\{0\}$ with $\mathrm{ev}_p^*(\alpha) = \lambda\, \mathrm{ev}_q^*(\beta)$. But this is equivalent to

$$< \alpha, \psi(p) >= \lambda < \beta, \psi(q) > \ \text{ for all } \psi \in H^0(L).$$

Hence f is injective if there exists for all $p, q \in M, p \neq q$, a holomorphic section ψ with $\psi(p) \neq 0$ but $\psi(q) = 0$.

Let $p, q \in M$ be arbitrary. Since $\deg LL(-q) = \deg L - 1 > \deg K + 1$ again $LL(-q)$ is base point free and there exists $\tilde{\psi} \in H^0(LL(-q))$ with $\tilde{\psi}(p) \neq 0$. Define $\psi = \tilde{\psi} s_q^1 \in H^0(L)$, where $s_q^1 \in H^0(L(q))$ is the famous holomorphic section which vanishes only at q and has order one at q. We see that $\psi(q) = 0$ but $\psi(q) \neq 0$ if $p \neq q$. Thus, f is indeed an injective map.

Now, if $p = q$, then ψ has a zero of first order at p. Since L is base point free we can choose a basis $s_1 = \psi, s_2, \ldots, s_n$ of holomorphic sections such that $s_2(p) \neq 0$. Define $f_i := \frac{s_i}{s_2}$ then

$$f = [f_1 : 1 : \ldots : f_{n+1}]$$

with $f_1 = zh$, where z is a local holomorphic coordinate around p, and h is a holomorphic map with $h(0) \neq 0$. Hence $f_1'(0) \neq 0$ and f is an immersion. □

Theorem 8.27 (Chows Theorem)**.** *Each compact complex submanifold of $\mathbb{C}P^n$ is the zero set of homogeneous polynomials, i.e. an* algebraic curve. ★ Quelle?

Corollary 8.28. *Each compact Riemann surface is algebraic.*

Proof. With Kodaira's theorem we can embed M into a projective space, hence with Chow's Theorem it is algebraic. □

8.5 The canonical curve

Theorem 8.29. *Assume that M is a compact Riemann surface of genus $g > 0$. Then the canonical bundle has no base points.*

★ ...

Proof. Consider the sequence

$$0 \longrightarrow H^0(K(-p)) \longrightarrow H^0(K) \longrightarrow K_p$$

Suppose p is a base point, then the map $H^0(K) \longrightarrow K_p$ vanishes. Thus

$$0 \longrightarrow H^0(K(-p)) \xrightarrow{\cong} H^0(K) \longrightarrow 0.$$

We use the the Riemann–Roch theorem to calculate

$$\begin{aligned} g &= h^0(K) = h^0(K(-p)) = h^0(L(p)) + \deg(K(-p)) - g + 1 \\ &= h^0(L(p)) + 2g - 3 - g + 1 = h^0(L(p)) + g - 2. \end{aligned}$$

Hence $h^0(L(-p)) = 2$ and there exists a meromoprhic function of degree 1, i.e. $M \cong \mathbb{C}P^1$, which contradicts $g > 0$. □

★ canonical curve

Remark 8.30. Thus use K as embedding ...

Chapter 9

Mittag–Leffler distributions and Weierstrass points

This chapter deals with some classical problems of Riemann surface theory. It gives a link between the language of line bundles on the one side and the classical theory based on the analysis of meromorphic functions in one complex variable ("Funktionentheorie") on the other side. Our treatment is quite short, because this chapter mainly serves as a preparation for the next chapter. For a classical treatment, see for example [Bo] or [FK]. ★ ???

9.1 Meromorphic functions versus line bundles

★ ich finde, das passt jetzt gar nicht mehr hier hin..

In this section we briefly explain how global statements formulated in the language of holomorphic line bundles and their sections can be stated using only meromorphic functions. This classical approach has many advantages when dealing with concrete calculations. An advantage of our approach is that it is more invariant.

We have seen that any holomorphic line bundle $L \to M$ over a Riemann surface admits a meromorphic section ψ (see Theorem 8.1) and that

$$L \cong L(D)$$

where $D = (\psi)$ is the divisor of ψ. Then

$$H^0(L) \cong \{f\colon M \to \mathbb{C} \cup \{\infty\} \text{ meromorphic with } (f) \geq -D\} =: \mathcal{M}_D$$

and the isomorphism is given by

$$\varphi \in H^0(L) \mapsto \varphi/\psi \in \mathcal{M}_D \qquad \text{and its inverse is} \qquad f \in \mathcal{M}_D \mapsto \psi \cdot f \in H^0(L).$$

("*Wir denken hier* $\varphi \in H^0(L)$, *Sascha denkt hier* $f \in \mathcal{M}_D$.")

Similarly we have

$$H^0(KL) \cong \{\omega \text{ meromorphic 1-form with } (\omega) \geq -D\} =: \Omega_D.$$

Here, the isomorphism is given by

$$\omega \in \Omega_D \mapsto \psi \cdot \omega.$$

The *divisor form of the Riemann-Roch theorem* reads as

$$\dim \mathcal{M}_D - \dim \Omega_{-D} = \deg D - g + 1.$$

9.2 Mittag–Leffler distributions

Let L be a holomorphic line bundle over a Riemann surface. We see a MITTAG–LEFFLER distribution of L as a collection of meromorphic sections $\psi_i\colon U_i \to L$ having the same principal parts on the overlaps of the open sets U_i of an open cover of M. A solution for a MITTAG–LEFFLER distribution is a global meromorphic section ψ having the same principal parts as the ψ_i's:

Definition 9.1. Let $E \to M$ be a holomorphic vector bundle over a Riemann surface. A MITTAG–LEFFLER *distribution* on E is given by

1. an open cover $\{U_i\}_{i \in I}$ of M and
2. local meromorphic sections $\psi_i \in \mathcal{M}(U_i, E)$ which satisfy

$$\psi_i - \psi_j\big|_{U_i \cap U_j} \in H^0(U_i \cap U_j, E).$$

A meromorphic section $\psi \in \mathcal{M}(E)$ is called a *solution of the* MITTAG–LEFFLER *distribution* (U_i, ψ_i) if

$$\psi\big|_{U_i} - \psi_i \in H^0(U_i, E).$$

We call $p \in M$ a *pole* of the MITTAG–LEFFLER distribution (U_i, ψ_i) if p is a pole of one of the ψ_i's.

Note that

Lemma 9.2. *On a compact Riemann surface there are only finitely many poles of a* MITTAG–LEFFLER *distribution.*

Proof. Let (ψ_i, U_i) be a MITTAG–LEFFLER distribution. By compactness of M we can choose a finite subcover $(V_j)_{j \in J}$ of $(U_i)_{i \in I}$. Choose for each $j \in J$ an U_{i_j} with $V_j \subset U_{i_j}$ and define $\varphi_j := \psi_{i_j}$. The MITTAG–LEFFLER distribution (φ_j, V_j) has clearly finitely many poles.

But each pole p of the MITTAG–LEFFLER distribution (ψ_i, U_i) is also pole of (φ_j, V_j):

If p is a pole of some ψ_i then $p \in V_j$ for some $j \in J$ and $\varphi_j - \psi_i = \psi_{i_j} - \psi_i \in H^0(U_{i_j} \cap U_i, E)$. But this means that p is a pole of φ_j, too. $\square$

Now, the natural question to ask is when a given MITTAG–LEFFLER distribution admits a solution, i.e. which further conditions we need to empose to the given principal parts to obtain a global meromorphic section. To state a condition we need the notion of residues of the poles of a MITTAG–LEFFLER distribution:

Definition 9.3. Let M be a compact Riemann surface. The *residue* of a MITTAG–LEFFLER distribution of meromorphic differentials (U_i, ω_i) is defined by

$$\operatorname{Res}(\omega_i) = \sum_{p \text{ pole of } (\omega_i)_i} \operatorname{Res}_p \omega_i.$$

Here, for each pole p of the MITTAG–LEFFLER distribution we choose one U_i with $p \in U_i$ and evaluate the residue of the corresponding meromorphic differential ω_i at the point p. Since the principal parts coincide on the overlaps of the U_i's this is independent of the choice we made. Furthermore, the sum is well defined, in fact, a finite sum, since on a compact surface there are only finitely many poles.

Theorem 9.4. *1. There exists a canonical map*

$$\{(U_i, \psi_i) \text{ MITTAG–LEFFLER } \textit{distribution}\} \xrightarrow{\delta} H^1(E).$$

2. *The* MITTAG–LEFFLER *distribution* (U_i, ψ_i) *has a solution if and only if* $\delta(U_i, \psi_i) = 0$.

3. *If M is a compact Riemann surface then the* MITTAG–LEFFLER *distribution* (U_i, ψ_i) *has a solution if and only if* $\operatorname{Res}(< \omega, \psi_i >)_i = 0$ *for all* $\omega \in H^0(KL^{-1})$.

Proof. 1. Let $\psi_{ij} = \psi_i - \psi_j \in \Gamma(U_i \cap U_j, E)$. Then the ψ_{ij}'s satisfy the cocycle condition (3.8). Thus there exist $\varphi_i \in \Gamma(U_i, E)$ such that

$$\psi_{ij} = \varphi_i - \varphi_j \text{ on } U_i \cap U_j,$$

compare Lemma 3.21. By the assumption that (U_i, ψ_i) is a MITTAG–LEFFLER distribution the ψ_{ij}'s are holomorphic sections and we have

$$\bar{\partial}\varphi_i - \bar{\partial}\varphi_j = \bar{\partial}\psi_{ij} = 0.$$

Since the $\bar{\partial}\varphi_i$'s coincide on the overlaps $U_i \cap U_j$ and $\bar{\partial}\varphi_i \in \Gamma(U_i, \bar{K}E)$ there exists a unique $\alpha \in \Gamma(\bar{K}E)$ with

$$\alpha\big|_{U_i} = \bar{\partial}\varphi_i.$$

We define

$$\delta(U_i, \psi_i) := \delta(\alpha) := \alpha + \bar{\partial}\Gamma(E) \in H^1(E).$$

This is well–defined, i.e. does not depend on the chosen φ_i's: Assume that $\psi_{ij} = \tilde{\varphi}_i - \tilde{\varphi}_j$ with $\tilde{\varphi}_i \in \Gamma(U_i, E)$ and let $\tilde{\alpha} \in \Gamma(\bar{K}E)$ with $\tilde{\alpha}\big|_{U_i} = \bar{\partial}\tilde{\varphi}_i$. But then

$$\tilde{\varphi}_i - \varphi_i = \tilde{\varphi}_j - \varphi_j \text{ on } U_i \cap U_j,$$

hence there exists $\varphi \in \Gamma(E)$ such that $\varphi\big|_{U_i} = \tilde{\varphi}_i - \varphi_i$. Thus

$$\tilde{\alpha}\big|_{U_i} = \bar{\partial}\tilde{\varphi}_i = \bar{\partial}\varphi\big|_{U_i} + \bar{\partial}\varphi_i = (\bar{\partial}\varphi + \alpha)\big|_{U_i}$$

and $\delta(\alpha) = \delta(\tilde{\alpha})$.

2. Let $\psi \in \mathcal{M}(E)$ be a solution of the Mittag–Leffler distribution (U_i, ψ_i), i.e. $\varphi_i := \psi\big|_{U_i} - \psi_i \in H^0(U_i, E)$. Since

$$\psi_{ij} = \psi_i - \psi_j = -(\psi\big|_{U_i} - \psi_i) + (\psi\big|_{U_j} - \psi_j) = \varphi_j - \varphi_i$$

we have $\delta(U_i, \psi_i) = \delta(\alpha) = 0$ for $\alpha\big|_{U_i} = \bar{\partial}(-\varphi_i) = 0$.

Conversely, let $\psi_{ij} = \varphi_i - \varphi_j$ and $\alpha \in \Gamma(\bar{K}E)$ such that $\alpha\big|_{U_i} = \bar{\partial}\varphi_i$ and $\delta(\alpha) = 0 \in H^1(E)$, i.e. $\alpha = \bar{\partial}\varphi$ for $\varphi \in \Gamma(E)$. Thus $\bar{\partial}\varphi_i = \alpha\big|_{U_i} = \bar{\partial}\varphi\big|_{U_i}$, i.e. $\tilde{\varphi}_i = \varphi\big|_{U_i} - \varphi_i$ is holomorphic and

$$\psi_i - \psi_j = \psi_{ij} = \varphi_i - \varphi_j = \varphi\big|_{U_i} - \tilde{\varphi}_i - (\varphi\big|_{U_j} - \tilde{\varphi}_j) = \tilde{\varphi}_j - \tilde{\varphi}_i.$$

In particular, the meromorphic sections $(\psi_i + \tilde{\varphi}_i)$ coincide on overlaps and there exists hence $\psi \in \mathcal{M}(E)$ with $\psi\big|_{U_i} = \psi_i + \tilde{\varphi}_i$. Thus ψ is a solution of the Mittag–Leffler distribution since $\psi\big|_{U_i} - \psi_i = \tilde{\varphi}_i \in H^0(U_i, E)$.

3. By the Serre duality theorem we know that $0 = \delta(\alpha) \in H^1(E)$ if and only if $\int_M < \omega \wedge \alpha >= 0$ for all $\omega \in H^0(KE^{-1})$. We calculate $< \omega \wedge \alpha >$:

Let $p_1, \ldots, p_n$ be the poles of the Mittag–Leffler distribution. Let $M_0 = M\backslash\{p_1, \ldots, p_n\}$, $\varphi_i \in \Gamma(U_i, E)$ as usual with $\psi_{ij} = \varphi_i - \varphi_j$ and $\alpha \in \Gamma(\bar{K}E)$ defined by $\alpha\big|_{U_i} = \bar{\partial}\varphi_i$. We have $\psi_i - \varphi_i \in \Gamma(U_i \cap M_0, E)$ and $\psi_i - \varphi_i = \psi_j - \varphi_j$ on $M_0 \cap U_i \cap U_j$. Hence there exists $\varphi \in \Gamma(M_0, E)$ which satisfies $\varphi\big|_{U_i \cap M_0} = \psi_i - \varphi_i$.

Now, $< \omega, \psi_i >$ is holomorphic for $\omega \in H^0(KE^{-1})$, i.e. $d < \omega, \psi_i >= 0$. Thus on $M_0 \cap U_i$:

$$\begin{aligned} < \omega \wedge \alpha > &= < \omega \wedge \bar{\partial}\varphi_i >= -d < \omega, \varphi_i >= -d < \omega, \psi_i - \varphi\big|_{U_i \cap M_0} > \\ &= d < \omega, \varphi\big|_{U_i \cap M_0} > . \end{aligned}$$

Now we take a holomorphic chart $z_k \colon V_k \to \mathbb{C}$ around every pole $p_1, \ldots, p_n$ such that $z_k(p_k) = 0$ and there is $i_k \in I$ with

$$B_\epsilon^k := \{p \in V_k \subset M : |z_k(p)| < \epsilon\} \subset U_{i_k} \text{ for all } 0 < \epsilon < 1.$$

$M_\epsilon = M \backslash (B_\epsilon^1 \cup \ldots \cup B_\epsilon^n)$. By STOKES theorem we have

$$\int_{M_\epsilon} < \omega \wedge \alpha > = - \int_{\partial M_\epsilon} < \omega, \varphi > = \sum_{k=1}^{n} \int_{\partial B_\epsilon^k} < \omega, \varphi >$$
$$= \sum_{k=1}^{n} \int_{\partial B_\epsilon^k} < \omega, \psi_{i_k} > - \sum_{k=1}^{n} \int_{\partial B_\epsilon^k} < \omega, \varphi_{i_k} > .$$

The last sum tends to zero for $\epsilon \to 0$, because $< \omega, \varphi_{i_k} >$ is smooth on U_{i_k}. The first sum is

$$\sum_{k=1}^{n} \int_{\partial B_\epsilon^k} < \omega, \varphi > = 2\pi i \sum_{k=1}^{n} \mathrm{Res}_{p_k} < \omega, \psi_{i_k} > .$$

Since $\int_{M_\epsilon} < \omega \wedge \alpha >$ tends to $\int_M < \omega \wedge \alpha >$ for $\epsilon \to 0$, the integral over M is zero if and only if $\mathrm{Res}(< \omega, \psi_i >)_i = 0$ for all $\omega \in H^0(KL^{-1})$.

□

Remark 9.5. For (1) we again follow indirectly a strategy of sheaf theory: we chase a diagram of cohomologies, cf. B.16.

Remark 9.6.

★ das hier ist noch ziemlich vergarbt

1. If $\deg E > \deg K$, every MITTAG LEFFLER distribution with values in $\mathcal{M}(E)$ admits a solution. The reason is that KE^{-1} has negative degree and by SERRE duality $0 = H^0(KE^{-1}) \cong H^1(E)$.

2. If $\deg E \leq \deg K$ and $0 \neq \xi \in H^1(E)$ then there are MITTAG LEFFLER distributions without solution: If $0 \neq \xi \in H^1(\mathcal{O}_L)$, then $H^1(E) = 0$ implies that there is a 0-chain $(\psi_i) \in C^0(\mathcal{U}, \mathcal{M}_L)$ such that $\xi = [(\psi_i - \psi_j)]$ in $H^1(\mathcal{O}_L)$. Now, (ψ_i) is a MITTAG LEFFLER distribution and the preceding Lemma implies that there is no solution for this distribution.

One important special case are MITTAG LEFFLER distributions for meromorphic functions on $\mathbb{C}P^1$. Here, the line bundle is the trivial bundle $L = \underline{\mathbb{C}}$. Because of $0 = \deg \underline{\mathbb{C}} > \deg K = -2$ we are in the case of the first remark and every such distribution has a solution. (This can easily be seen directly. A solution can be obtained by simply adding the prescribed principal parts. Compare the classical MITTAG LEFFLER theorem dealing with the non-compact Riemann surface $\mathbb{C}$, see for example [Ah], [FL] and [Ru].)

Corollary 9.7.

1. *A* MITTAG LEFFLER *distribution* (f_i) *of meromorphic functions admits a solution if and only if* $\mathrm{Res}(\omega f_i) = 0$ *for all* $\omega \in H^0(K)$.

2. *A* Mittag Leffler *distribution* (ω_i) *of meromorphic differentials admits a solution if and only if* $\mathrm{Res}(\omega_i) = 0$.

Proof. The first is a direct consequence of the preceding theorem. For the second part, the preceding theorem yields that (ω_i) has a solution if and only if $\mathrm{Res}(\alpha\omega_i) = 0$ for all $\alpha \in H^0(KK^{-1}) = H^0(\underline{\mathbb{C}}) = \mathbb{C}$. Since α is just a complex factor, this is equivalent to $\mathrm{Res}(\omega_i) = 0$. So, there isn't really much to show in this case either. $\square$

To see whether a Mittag Leffler distribution of meromorphic functions has a solution one has to check g linear equations, since $\dim H^0(K) = g$ where g is the genus of M. So, in the case $g = 0$, as we have seen earlier, there is really no condition to be satisfied: any distribution of meromorphic functions has a solution.

Example 9.8. *("...mag auch Sascha...")* In the case $g = 1$, we have $H^0(K) = \mathbb{C}dz$, i.e there is only one condition a distribution has to fulfill in order to have a solution. And this condition simply is that the sum of all residues has to be zero. A consequence of this is that on complex tori there are no meromorphic functions with only one pole!

We take n points $a_1, ..., a_n$ on a complex torus $\mathbb{C}/\Gamma$ and for every point we prescribe the principal part of a meromorphic function

$$f_i = \sum_{k=N_i}^{-1} C_k^j \frac{1}{(z - a_i)^k}.$$

Since $H^0(K_{\mathbb{C}/\Gamma}) = \mathbb{C}dz$, this Mittag Leffler distribution has a solution if and only if $\mathrm{Res}(f_i dz) = \sum_{i=1}^n C_{-1}^i = 0$.

9.3 Abelian differentials

Abelian differentials constitute a classical part of Riemann surface theory. We will only need them later for the study of the Jacobi variety. But the existence of so called elementary Abelian differentials is a nice application of the results obtained in the preceding section, that's why they fit in here quit well.

Definition 9.9. Let M be a compact Riemann surface.

1. An *Abelian differential of the first kind* is an element of $H^0(K)$.

2. An *Abelian differential of the second kind* is a meromorphic section of K without residues.

3. An *Abelian differential of the third kind* is a meromorphic section of K with residues.

An *elementary differential of the second kind* is a meromorphic 1-Form $\omega_p^{(N)}$ having a pole of order N with $\operatorname{Res}_p \omega_p^{(N)} = 0$ at $p \in M$ and being holomorphic elsewhere. To show the existence for a fixed $p \in M$ we take a holomorphic chart $z\colon U \to \mathbb{C}$ with $z(p) = 0$. A Mittag Leffler distribution of meromorphic differentials for the open cover $U_1 = U$ and $U_2 = M \backslash \{p\}$ of M is defined by

$$\omega_1 = \sum_{k=2}^{N} \frac{a_k}{z^k} \qquad \text{and} \qquad \omega_2 = 0.$$

Since there is no residue, the corollary of Theorem 3 implies that there is a solution. This solution has the stated properties, i.e. a pole of order N at p and is holomorphic elsewhere. Of course, such elementary differentials of the second kind are by no means uniquely determined.

An *elementary differential of the third kind* is a meromorphic 1-Form ω_{pq} having poles of order 1 at $p, q \in M$ with $\operatorname{Res}_p \omega_{pq} = 1$ and $\operatorname{Res}_q \omega_{pq} = -1$ and being holomorphic elsewhere. Again, we use the corollary of Theorem 3 to prove the existence of such differentials for two fixed points $p \neq q$. We take two charts $z_1\colon U_1 \to \mathbb{C}$ and $z_2\colon U_2 \to \mathbb{C}$ with $p \in U_1$ and $q \in U_2$ such that $U_1 \cap U_2 = \emptyset$. An open cover of M is defined by U_1, U_2 and $U_3 = M \backslash \{p, q\}$ and a Mittag Leffler distribution is defined by

$$\omega_1 = \frac{1}{z_1} \qquad \text{and} \qquad \omega_2 = -\frac{1}{z_2} \qquad \text{and} \qquad \omega_3 = 0.$$

Since the sum of the residues is zero, the distribution has a solution.

9.4 Weierstrass points of a holomorphic line bundle L

In this section we define Weierstrass points of arbitrary holomorphic line bundles.

Definition 9.10. Let $L \to M$ be a holomorphic line bundle over a compact Riemann surface. The Wronskian *determinant* $W(\psi_1, \ldots, \psi_n) \in \Gamma(K^{\frac{n(n-1)}{2}} L^n)$ of $\psi_1, \ldots, \psi_n \in \Gamma(L)$ is defined by

$$W(\psi_1, \ldots, \psi_n) := \det(\partial^{(k)} \psi_i)_{\substack{i=1,\ldots,n \\ k=0,\ldots,n-1}} = \begin{vmatrix} \psi_1 & \ldots & \psi_n \\ \partial\psi_1 & \ldots & \partial\psi_n \\ \ldots & \ldots & \ldots \\ \partial^{(n-1)}\psi_1 & \ldots & \partial^{(n-1)}\psi_n \end{vmatrix}$$

where ∂ is induced by arbitrary ∂-operators on K and L.

Remark 9.11. The WRONSKIAN determinant does not depend on the choice of ∂, since any other ∂-operator $\tilde{\partial}$ is of the form $\tilde{\partial} = \partial + \alpha$ with $\alpha \in \Gamma(K)$ (use induction to calculate the WRONSKIAN for $\partial + \alpha$!).

Moreover, for any $A \in \mathrm{GL}(n, \mathbb{C})$ and $(\tilde{\psi_1}, \ldots, \tilde{\psi_n}) = (\psi_1, \ldots, \psi_n) \cdot A$ we have

$$W(\tilde{\psi_1}, \ldots, \tilde{\psi_n}) = \det(A) W(\psi_1, \ldots, \psi_n).$$

Thus, up to a non-zero multiplicative constant it depends only on the subspace spanned by the sections $\psi_1, \ldots, \psi_n \in \Gamma(L)$.

Lemma 9.12. *The* WRONSKIAN *determinant of holomorphic sections is again holomorphic, i.e. for $\psi_1, \ldots, \psi_n \in H^0(L)$ we have $W(\psi_1, \ldots, \psi_n) \in H^0(K^{\frac{n(n-1)}{2}} L^n)$.*

Proof. Let $\nabla = \bar{\partial} + \partial$ and $\omega \in \Omega^2(M)$ the curvature form of ∇, i.e. $\mathcal{R}^\nabla \psi = \omega\psi$. We calculate for $\psi \in H^0(L)$ and a local holomorphic vector field $X \in H^0(TM)$

$$\begin{aligned} \mathcal{R}^\nabla_{X,JX} \psi &= \nabla_X(\bar{\partial}_{JX}\psi) + \nabla_X(\partial_{JX}\psi) - \nabla_{JX}(\bar{\partial}_X\psi) - \nabla_{JX}(\partial_X\psi) - \nabla_{[X,JX]}\psi \\ &= J\bar{\partial}_X(\partial_X\psi) + J\partial_X(\partial_X\psi) - \bar{\partial}_{JX}(\partial_X\psi) - \partial_{JX}(\partial_X\psi) \\ &= 2J\bar{\partial}_X(\partial_X\psi) \end{aligned}$$

Thus $\omega_{X,JX}\psi = 2J\bar{\partial}_X\partial_X\psi$. Now, we see

$$\begin{aligned} \bar{\partial}_X(W(\psi_1, \psi_2)_{JX}) &= \bar{\partial}_X \begin{vmatrix} \psi_1 & \psi_2 \\ \partial_{JX}\psi_1 & \partial_{JX}\psi_2 \end{vmatrix} \\ &= \begin{vmatrix} \bar{\partial}_X\psi_1 & \psi_2 \\ \bar{\partial}_X\partial_{JX}\psi_1 & \partial_{JX}\psi_2 \end{vmatrix} + \begin{vmatrix} \psi_1 & \bar{\partial}_X\psi_2 \\ \partial_{JX}\psi_1 & \bar{\partial}_X\partial_{JX}\psi_2 \end{vmatrix} \\ &= \begin{vmatrix} 0 & \psi_2 \\ J\omega_{X,JX}\psi_1 & \partial_{JX}\psi_2 \end{vmatrix} + \begin{vmatrix} \psi_1 & 0 \\ \partial_{JX}\psi_1 & J\omega_{X,JX}\psi_2 \end{vmatrix} \\ &= -\begin{vmatrix} J\omega_{X,JX}\psi_1 & \partial_{JX}\psi_2 \\ 0 & \psi_2 \end{vmatrix} + \begin{vmatrix} \psi_1 & 0 \\ \partial_{JX}\psi_1 & J\omega_{X,JX}\psi_2 \end{vmatrix} = 0 \end{aligned}$$

Use the same arguments for the general case. ∎

Lemma 9.13. *Let $\psi_1, \ldots, \psi_n \in H^0(L)$ be holomorphic sections of a holomorphic line bundle $L \to M$. The sections are linearly independent if and only if the* WRONSKIAN *determinant $W(\psi_1, \ldots, \psi_n)$ does not vanish identically.*

Proof. If the sections are linearly dependent, one of the columns in the determinant is linearly dependent of the others and the determinant vanishes identically.

The other direction is proved by induction. For $n = 1$ nothing is to prove. Suppose $\psi_1, \ldots, \psi_n$ are linearly independent. We consider the equation

$$\begin{vmatrix} \psi_1 & \ldots & \psi_{n-1} & \varphi \\ \partial\psi_1 & \ldots & \partial\psi_{n-1} & \partial\varphi \\ \ldots & \ldots & \ldots & \ldots \\ \partial^{(n-1)}\psi_1 & \ldots & \partial^{(n-1)}\psi_{n-1} & \partial^{(n-1)}\varphi \end{vmatrix} = 0.$$

Developing this determinant with respect to the last column yields

$$a_{n-1}\partial^{(n-1)}\varphi + a_{n-2}\partial^{(n-2)}\varphi + \ldots + a_1\partial\varphi + a_0\varphi = 0$$

with coefficients a_i depending on the derivatives of the ψ_j's. This equation is a $(n-1)^{th}$-order ordinary differential equation with the principal coefficient $a_{n-1} = W(\psi_1, \ldots, \psi_{n-1})$. By induction, a_{n-1} has only isolated zeros. The space of solutions is $(n-1)$-dimensional and $\psi_1, \ldots, \psi_{n-1}$ are solutions. Thus, being linearly independent of the other sections, ψ_n can not be a solution, i.e. $W(\psi_1, \ldots, \psi_n) \neq 0$. □

Let $L \to M$ be holomorphic line bundle over a compact Riemann surface of genus g and let $H \subset H^0(L)$ be a linear system of dimension n, i.e. a subspace of the space of holomorphic sections of L. We take a basis $\psi_1, \ldots, \psi_n$ of H. The WRONSKIAN determinant $W(\psi_1, \ldots, \psi_n)$ is then a non-vanishing holomorphic section of $K^{\frac{n(n-1)}{2}}L^n$. Up to a constant factor it is independent of the choice of basis.

Definition 9.14. Let L be a holomorphic line bundle over a compact Riemann surface. Let $H \subset H^0(L)$ be a n-dimensional subspace. The WEIERSTRASS *divisor of* H is the divisor $W(H) = \sum_{p\in M} \eta_p p$ of the WRONSKIAN determinant $W(\psi_1, \ldots, \psi_n)$ for a basis $\psi_1, \ldots, \psi_n$ of H. A WEIERSTRASS *point of* H is a point $p \in M$ with $\eta_p \neq 0$. η_p is called its *weight*.

Remark 9.15. Counted with their weights, the number of WEIERSTRASS points of a subspace $H \subset H^0(L)$ is

$$\deg W(H) = \deg K^{\frac{n(n-1)}{2}}L^n = n(n-1)(g-1) + n\deg L.$$

Lemma 9.16. *Let $L \to M$ be holomorphic line bundle over a compact Riemann surface and let $H \subset H^0(L)$ be a n-dimensional subspace of the space of holomorphic sections of L. For a fixed $p \in M$, there are numbers*

$$0 \leq \nu_1 < \ldots < \nu_n$$

such that for all $\psi \in H$ we have $\operatorname{ord}_p \psi = \nu_i$ for one $i \in \{1, \ldots, n\}$. Furthermore, we can choose a basis $\psi_1, \ldots, \psi_n$ of H satisfying $\operatorname{ord}_p \psi_i = \nu_i$ for all i.

Proof. Set $H_1 := H$. We take $\nu_1 := \min_{\psi\in H_1} \operatorname{ord}_p \psi$. The set of sections vanishing to higher order than ν_1 is a hyperplane $H_2 \subset H_1$. We define $\nu_2 := \min_{\psi\in H_2} \operatorname{ord}_p \psi$ and proceed inductively. Since H has dimension n, we obtain exactly n numbers. □

Definition 9.17. The sequence $\nu_1 < \ldots < \nu_n$ is called the WEIERSTRASS *gap sequence* of H at $p \in M$. If $\nu_{i+1} - \nu_i > 1$ we say that there is a gap in the sequence.

Lemma 9.18. *Let $L \to M$ be a holomorphic line bundle over a compact Riemann surface and let $H \subset H^0(L)$ be a n-dimensional subspace of the holomorphic sections of L. Then*

$$ord_p W(\psi_1, \ldots, \psi_n) = \sum_{i=1}^{n} (\nu_i - i + 1) = \sum_{i=1}^{n} \nu_i - \frac{n(n-1)}{2}$$

where $\psi_1, \ldots, \psi_n$ is a basis of H and $\nu_1 < \ldots < \nu_n$ is the Weierstrass *gap sequence at p.*

Proof. Without loss of generality we can suppose that the basis $\psi_1, \ldots, \psi_n$ satisfies $\operatorname{ord}_p \psi_i = \nu_i$ for all i. We can chose a local chart $z\colon U \to \mathbb{C}$ with $z(p) = 0$ and a local holomorphic section $\varphi \in \Gamma(U, \mathcal{O}_L)$ with no zeros. There are holomorphic functions $f_i\colon U \to \mathbb{C}$ with

$$\psi_i = \varphi \cdot f_i$$

for all i. Normalizing the ψ_i's we have the local representation

$$f_i(z) = z^{\nu_i} + o(|z|^{\nu_i})$$

for all i. With these choices, the Wronskian determinant becomes

$$W(\psi_1, \ldots, \psi_n) = \underbrace{\begin{pmatrix} f_1 & \cdots & f_n \\ \frac{\partial f_1}{\partial z} & \cdots & \frac{\partial f_n}{\partial z} \\ \cdots & \cdots & \cdots \\ \frac{\partial^{(n-1)} f_1}{\partial z^{(n-1)}} & \cdots & \frac{\partial^{(n-1)} f_n}{\partial z^{(n-1)}} \end{pmatrix}}_{B:=} dz^{\frac{n(n-1)}{2}} \varphi^n.$$

A short calculation shows that

$$B = \begin{pmatrix} 1 & & & & \\ & z^{-1} & & & \\ & & z^{-2} & & \\ & & & \cdots & \\ & & & & z^{-n+1} \end{pmatrix} \cdot \tilde{B} \cdot \begin{pmatrix} z^{\nu_1} & & & & \\ & z^{\nu_2} & & & \\ & & z^{\nu_3} & & \\ & & & \cdots & \\ & & & & z^{\nu_n} \end{pmatrix}$$

where

$$\tilde{B} = \begin{pmatrix} 1 & \cdots & 1 \\ \nu_1 & \cdots & \nu_n \\ \nu_1(\nu_1 - 1) & \cdots & \nu_n(\nu_n - 1) \\ \cdots & \cdots & \cdots \\ \nu_1 \cdots (\nu_1 - n + 2) & \cdots & \nu_n \cdots (\nu_n - n + 2) \end{pmatrix} + o(1).$$

The determinant of $\tilde{B}$ is non–zero, because it is the Wronskian determinant of the linearly independent functions $e^{\nu_1 x}$,...,$e^{\nu_n x}$ evaluated at $x = 0$. Thus the determinant of B has the given order. $\square$

Corollary 9.19. *Generically (besides finitely many points) the* WEIERSTRASS *gap sequence of a n-dimensional subspace $H \subset H^0(L)$ is $0, 1, \dots, n-1$.* WEIERSTRASS *points of H are exactly the points with gaps in the* WEIERSTRASS *gap sequence.*

9.5 Weierstrass points of Riemann surfaces

("Weierstraßpunkte kommen mit der Fläche. Gott hat die da draufgemalt.")

There are two bundles coming for free with each Riemann surface M: The canonical bundle K and the trivial bundle $\underline{\mathbb{C}}$. We now apply the general theory developed above to $L = K$. WEIERSTRASS points of this bundle are special points of a Riemann surface. They can be characterized by a singular behavior of the vanishing orders of holomorphic differentials. Generically, the vanishing orders of holomorphic differentials at a point $p \in M$ are $0, \dots, g-1$ where g is the genus of M. A WEIERSTRASS point is a point where this so called WEIERSTRASS gap sequence has gaps, i.e. where there are holomorphic differentials with other vanishing orders than the generic ones.

Another possible characterization of WEIERSTRASS points of Riemann surfaces of genus $g \geq 1$ is via meromorphic functions. A point $p \in M$ is a WEIERSTRASS point if there is a meromorphic functions having a pole of order $\leq g$ at p and being holomorphic elsewhere. On other points, the possible orders of meromorphic functions with a single pole at the point are $g+1$, $g+2$, $g+3, \dots$.

We take a basis $\omega_1, \dots, \omega_g$ of $H^0(K)$. The WRONSKIAN determinant

$$W(\omega_1, \dots, \omega_g) \in H^0(K^{\frac{n(n+1)}{2}})$$

is not identically zero and up to a constant factor it is independent of the choice of basis.

Definition 9.20. The WEIERSTRASS *divisor* $W = \sum_{p \in M} \eta_p p$ of a compact Riemann surface is the divisor of $W(\omega_1, \dots, \omega_g)$ for a basis $\omega_1, \dots, \omega_g$ of $H^0(K)$. A WEIERSTRASS *point* $p \in M$ is a point with $\eta_p \neq 0$. η_p is called its *weight*.

Remark 9.21.

1. Counted with their weights, there are

   $$\deg K^{\frac{n(n+1)}{2}} = (g-1)g(g+1)$$

   WEIERSTRASS points. In particular, if M is the sphere or a torus, there are no WEIERSTRASS points at all.

2. Above we proved in a more general setting that generically the Weierstrass gap sequence of $H^0(K)$ is

$$0, 1, \dots, g-1$$

and that Weierstrass points of M are characterized by the property that there are true gaps in the gap sequence.

Another characterization of Weierstrass points in the case of a Riemann surface of genus $g \geq 1$ is in terms of meromorphic functions with a single pole of a certain order at the point.

Let $p \in M$ be an arbitrary point on a compact Riemann surface. The considerations of the first section applied to the point bundle $L(np)$ over M yield

$$H^0(L(np)) \cong \mathcal{M}_{np} = \{f\colon M \to \mathbb{C} \cup \{\infty\} : \text{ meromorphic with } (f) \geq -np\}.$$

The isomorphism is given by

$$\varphi \in H^0(L(np)) \mapsto \varphi/\psi \qquad \text{and} \qquad f \mapsto \psi \cdot f$$

where $\psi \in H^0(L(np))$ is the unique (up to constant factor) holomorphic section with $\operatorname{ord}_p \psi = n$, which is non–zero on $M \backslash \{p\}$.

For $L((g+1)p)$, the Riemann-Roch theorem implies that

$$\dim H^0(L((g+1)p)) \geq \deg L((g+1)p) - g + 1 \geq 2.$$

Thus, there is beside the famous holomorphic section another holomorphic section. These two section define always a meromorphic function f on M with a pole of order $\leq g+1$ which is holomorphic on $M \backslash \{p\}$. The question is: for a fixed $p \in M$, is there a meromorphic function f with a pole of order $\leq g$ and holomorphic in $M \backslash \{p\}$?

On a torus, the answer is no, since the sum of the residues has to be zero. In the case of genus $g \geq 2$, such meromorphic functions exist if and only if the point p is a Weierstrass point. (Remember that an the sphere, there is a meromorphic function for each Mittag Leffler distribution. In particular, for any fixed point p there are meromorphic functions with one single pole of arbitrary order at p. Nevertheless, there are no Weierstrass points.)

Theorem 9.22. *Let M be a Riemann surface of genus $g \geq 1$. A point p in M is not a* Weierstrass *point if and only if for all numbers $g+1, g+2, g+3, \dots$ there is a meromorphic function $f : M \to \mathbb{C} \cup \{\infty\}$ with one single pole at p and holomorphic elsewehre. Then point $p \in M$ is a* Weierstrass *point if and only if there is a meromorphic function $f\colon M \to \mathbb{C} \cup \{\infty\}$ with one single pole of order $\leq g$ and holomorphic elsewhere.*

Proof. The proof is a straightforward application of the corollaries of Theorem 3 and Lemma 9.18. Let $p \in M$ be a fixed point. Take a holomorphic chart $z\colon U \to \mathbb{C}$ with $z(p) = 0$ and the open cover $U_1 = U$ and $U_2 = M \backslash \{p\}$. A Mittag Leffler distribution of meromorphic functions is defined by

$$f_1 = \sum_{k=1}^{N} \frac{a_k}{z^k} \qquad \text{and} \qquad f_2 = 0.$$

Any solution is a meromophic function with one single pole at p and holomorphic elsewhere. Using the corollary of Theorem 3 we know that such a distribution admits a solution if and only it satisfies the linear equations

$$\operatorname{Res}_p \omega_1 f_1 = 0 \qquad \ldots \qquad \operatorname{Res}_p \omega_g f_1 = 0$$

where $\omega_1, \ldots, \omega_g$ is a basis of the holomorphic differentials $H^0(K)$. Above, we have seen that we can chose such a basis with the property $\operatorname{ord}_p \omega_i = \nu_i$ for all i where $\nu_1 < \ldots < \nu_g$ is the Weierstrass gap sequence of $H^0(K)$ at p. Thus, the linear equations are equivalent to

$$a_{\nu_1+1} = 0 \qquad \ldots \qquad a_{\nu_g+1} = 0$$

and the distribution (f_i) of meromorphic functions admits a solution if and only if these coefficient vanish!

The corollary of Lemma 9.18 implies that p is not a Weierstrass point if and only if $\nu_i = i - 1$ for all i. Then the distribution (f_i) has a solution if and only if $a_1 = a_2 = \ldots = a_g = 0$.

Conversely, if p is a Weierstrass point, then the coefficients of the distribution (f_i) can be chosen such that it has a solution with a single pole of order $\leq g$ in p. □

9.6 Hyperelliptic surfaces

An *elliptic surface* is a Riemann surface of genus $g = 1$, i.e. a complex torus.

Definition 9.23. A compact Riemann surface M is called *hyperelliptic* if there is a holomorphic map $f\colon M \to \mathbb{C}P^1$ of degree 2.

Corollary 9.24. *Each surface of genus 2 is hyperelliptic.*

Proof. By Theorem 9.22 there is a meromorphic function with pole of order ≤ 2 at a Weierstraß point. Since it can not have a pole of order 1 we are done. □

Example 9.25. Consider

$$P(z) = y^2 = \Pi_{i=1}^{2d}(z - a_i), \;\; a_i \neq a_j, a_i \in \mathbb{C}. \tag{9.1}$$

We like to show that this defines a compact hyperelliptic submanifold.

Consider the open subsets $U_0 = \mathbb{C}P^1 \backslash \{\infty\}$, $U_\infty := \mathbb{C}P^1 \backslash \{0\}$ in $\mathbb{C}P^1$, compare Remark 5.19). On

$$M_0 = \{(\begin{bmatrix} z \\ 1 \end{bmatrix}, y) \in U_0 \times \mathbb{C} \mid y^2 = P(z)\}$$

we see

$$2ydy - P'(z)dz \neq 0$$

(since $P'(z) \neq 0$ for $y = 0$, otherwise clear) hence M_0 is a submanifold of $U_0 \times \mathbb{C}$. Now we treat the point ∞. Write $w = \frac{1}{z}$ then

$$y^2 = P(z) \iff w^{2d} y^2 = Q(w) = \Pi_{i=1}^{2d}(1 - wa_i).$$

Again,

$$M_\infty = \{(\begin{bmatrix} 1 \\ w \end{bmatrix}, \tilde{y}) \in U_\infty \times \mathbb{C} \mid \tilde{y}^2 = Q(w)\}$$

is a submanifold of $U_\infty \times \mathbb{C}$ and M_0 and M_∞ are glued together as a complex manifold M via the isomorphism

$$M_0 \ni (\begin{bmatrix} z \\ 1 \end{bmatrix}, y) \mapsto s_\infty(\begin{bmatrix} z \\ 1 \end{bmatrix}) = (\begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}, z^{-d}y) \in M_\infty, \ \begin{bmatrix} z \\ 1 \end{bmatrix} \in U_0 \cap U_\infty$$

The manifold obtained this way is a complex submanifold of $\mathcal{O}(d)$ where $\mathcal{O}(d)$ is the holomorphic line bundle of degree d over $\mathbb{C}P^1$. Since $\mathcal{O}(d)$ is given uniquely up to isomorphism, compare Remark 5.19, we can see $\mathcal{O}(d) = L(d\infty)$ with transition function

$$g_{0\infty}(\begin{bmatrix} z \\ 1 \end{bmatrix}) = z^d, \ (\begin{bmatrix} z \\ 1 \end{bmatrix}) \in U_0 \cap U_\infty.$$

Define

$$s_0(\begin{bmatrix} z \\ 1 \end{bmatrix}) = (\begin{bmatrix} z \\ 1 \end{bmatrix}, y) \in M_0, \ s_\infty(\begin{bmatrix} z \\ 1 \end{bmatrix}) = (\begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}, z^{-d}y) \in M_\infty, \ \begin{bmatrix} z \\ 1 \end{bmatrix} \in U_0 \cap U_\infty$$

then we see that this defines a section of $\mathcal{O}(d)$ since the transition function is given by $s_0 = z^d s_\infty$.

Consider now the projection $\pi : \mathcal{O}(d) \to \mathbb{C}P^1$, $(\begin{bmatrix} z \\ 1 \end{bmatrix}, y) \mapsto \begin{bmatrix} z \\ 1 \end{bmatrix}$. Then $\pi\big|_M$ is a 2–fold covering of $\mathbb{C}P^1$. The branch points are given by $a_1, \ldots, a_{2d}$ since $d\pi = dz = 0$ if and only if $\frac{2ydy}{P'(z)} = 0$ i.e. $y = 0$ resp. $z = a_i$ for some i.

We obtain the genus of M via the Riemann–Hurwitz formula: The degree of $\pi\big|_M$ is 2 and the ramification order $b(\pi) = \sum_{p \in M} \operatorname{ord}_p d\pi = 2d$ hence

$$g_M = \frac{1}{2} b(\pi) + \deg \pi (g_{\mathbb{C}P^1} - 1) = d - 1.$$

A basis for the Abelian differentials of the first kind is given by

$$\omega_k = \frac{z^{k-1}dz}{y}, \; k = 1, \dots, d-1, (\text{on } M_0).$$

This shows that ω is holomorphic on $M \backslash \{\pm\infty\}$, but the transformation

$$\omega_k = \frac{w^{d-k+1}}{\tilde{y}}(-\frac{dw}{w^2}) = -\frac{w^{d-k-1}dw}{\tilde{y}}$$

shows that they can be extended to M_∞ and are holomorphic at $\pm\infty$ as well.

In fact, any hyperelliptic surface can be written in the form (9.1); one only has to replace $2d$ by $2d+1$ to allow ∞ as branch point.

Chapter 10

Jacobi variety and the Abel-Jacobi map

In this section our aim is to understand the connection between a compact oriented Riemann surface of genus $g > 1$ and it's attached JACOBI variety, which is often abbreviated as the JACOBIAN. The JACOBI variety could be understood as a g-dimensional complex tori. We will see that the JACOBI variety inherits many of the properties of the Riemann surface and is a gratifying tool in the study of the Riemann surfaces together with an attached line bundle. As we know from section 8.3 every holomorphic line bundle has a associated divisor and vice versa. In the following section we will use this fact without mentioning it anymore. ★

Hier muss man noch mal sehen, ob die Einleitung dann zum tatsaechlichen Kapitel passen wird

To be more explicit we will see that the space of all holomorphic line bundle with fixed degree, over a compact Riemann surface M of genus $g > 1$ is isomorhic to the Jacobian $\mathbb{C}^g/\Lambda$, where Λ is a $2g$ $\mathbb{R}$-lattice. Further we will proof the ABEL Theorem, which says that the degree zero PICARD group can be embedded into the JACOBI variety. Translating this into the divisor language, it says something about the determination of necessary and sufficient conditions for a divisor of degree zero to be principal. At the end of this section we will proof the JACOBI Inversion Theorem which says that the JACOBI variety as a group is isomorphic to the group of divisor of degree zero modulo it's subgroup of principal divisors. Again translated into the divisor language it says that every point in the Jacobi variety is the image of a integral divisor. ★

find a non divisor description.

10.1 Differential topological background

From differential topology we know that every two dimensional orientable compact manifold M has only one topological invariant, the genus g. Hence every such manifold could be topologically identified with a sphere joined with g handles.

$$H_1(M, \mathbb{Z}) \cong \mathbb{Z}^{2g} \cong \frac{\pi_1(M)}{[\pi_1(M), \pi_1(M)]}$$

We also know that for every compact topological space M of genus $g \geq 1$ there exist a basis $\{a_1, \cdots, a_g, b_1, \cdots, b_g\}$ for $H_1(M, \mathbb{Z})$ where the a_i and b_i can be identified with closed $\mathbb{R}$-curve in M where only a_i and b_i are intersecting once[1]. This basis is then called the *canonical basis*. Even further there is a differential choice. See picture **??**. Another important alternative for these basis curves we get if we chose one point in the topological space M and require that all basis curve will meet there and only there. If we cut the surface along this basis curves we get the *fundamental domain* Δ.
From the Uniformisation Theorem we know that for any compact two dimensional manifold M of genus $g \geq 2$ there is a Riemannian metric of negative constant curvature, without loss of generality we can assume $K = -1$, i.e. The hyperbolic space is the universal covering of all these surfaces, for a proof see [Sp, page 430ff]. Each of these surfaces give us a tiling of the hyperbolic space by equilateral $4g$-gons, The fundamental domain can be understand as one of these $4g$-gon where each interior angle equals $2\pi/4g$. Even further if M is a compact Riemann surface it is biholomorphic to the $\mathcal{H}^2/G$, where G is the group generated by biholomorphic maps from hyperbolic space to itself which fixes the vertexes of the tiling.[2]
From algebraic geometry we know that on a compact Riemann surface $H^1_{dR}(\mathbb{C}) = H^0(K) \oplus H^0(\overline{K})$, where $H^0(K)$ are the holomorphic one forms and $H^0(\overline{K})$ the antiholomorpic one forms [3]. Take the basis $\{\omega_1, \cdots, \omega_g\}$ for the holomorphic one forms as the dual to the canonical homological basis, i.e $\int_{a_i} \omega_j = \delta_{i,j}$. We define the so called *periods* as $\Pi_{i,j} := \int_{b_i} \omega_j$. As we will see later these periods carry very important data from the Riemann surface.

★ nasty grammar

10.2 Riemann bilinear relations

★ introduction?!

Theorem 10.1 (RIEMANN bilinear relations). *Let $\alpha, \beta \in \Omega^1(M, \mathcal{A})$ be closed 1–forms with values in an $\mathbb{R}$ algebra $\mathcal{A}$. Further let $S\colon M \to \mathcal{A}$ be a complex structure, i.e $S^2 = -id$. Furthermore, let $a_1, \ldots, a_g, b_1, \ldots, b_g \in H_1(M, \mathbb{Z})$ a basis. Then*

★ Voraussetzung ueber die Schnittzahl? Sowohl die Bibel, als auch Farkas/Kra setzen die as und bs als die Randzykel vom Fundamental domain voraus

$$\int_M \alpha \wedge \beta = \sum_{i=1}^{g} \left(\int_{a_i} \alpha \int_{b_i} \beta - \int_{a_i} \beta \int_{b_i} \alpha \right)$$

Proof. On the fundamental domain Δ, which is simply connected $\alpha|_\Delta = dA$

[1] to be precise it is the identification with a representant in the associated element of the fundamental group $\pi_1(M)$. In the following text we will identify them all and refer to them as the basis curves, basis cycles or just by a_i and b_i.

[2] Again following Spivak [Sp, page 430ff] the maps are reflections on the polygon sides. It could be that these group are the so-called Fuchsian group.

[3] $w \in H^0(K)$ is exact precisely when there is an holomorhic function g and $dg = w$. Since every holomorhic function on a compact M is constant dg vansishes.

where $A\colon \Delta \to \mathcal{A}$ is smooth. Thus

$$\begin{aligned}\int_M \alpha \wedge \beta &= \int_\Delta dA \wedge \beta = \int_\Delta d(A\beta) = \int_{\partial\Delta} A\beta \\ &= \sum_{i=1}^{g}\Big(\int_{a_i} A\beta + \int_{b_i} A\beta + \int_{{a_i}^{-1}} A\beta + \int_{{b_i}^{-1}} A\beta\Big)\end{aligned} \tag{10.1}$$

We calculate $\int AB$ along the a–cycles by parametrizing... ★ picture!?

$$\begin{aligned}\int_{a_i} A\beta + \int_{{a_i}^{-1}} A\beta &= \int_0^1 A(t,0)\beta(t,0)dt - \int_0^1 A(t,1)\beta(t,1)dt \\ &= \int_0^1 \underbrace{(A(t,0) - A(t,1))}_{=-\int_{b_i}\alpha}\beta(t,1)dt \\ &= -\int_{b_i}\alpha \int_{a_i}\beta\end{aligned} \tag{10.2}$$

Together with the same calculation for the b-cycle integrals we get the claim. □

We obtain as simple conclusions:

Corollary 10.2. *1. If $\alpha, \beta \in H^0(K)$ then*

$$\sum_{i=1}^{g}\Big(\int_{a_i}\alpha \int_{b_i}\beta - \int_{b_i}\alpha\int_{a_i}\beta\Big) = 0.$$

2. If $\alpha \in \Omega^1(M,\mathbb{C})$ and $d\alpha = 0$ then

$$\frac{1}{2i}\int_M \alpha \wedge \overline{\alpha} = \operatorname{Im}\left(\sum_{i=0}^{g}\int_{a_i}\alpha\int_{b_i}\overline{\alpha}\right).$$

In particular, if $\alpha \in H^0(K)$, $\alpha \neq 0$, then

$$\frac{1}{2i}\int_M \alpha \wedge \overline{\alpha} = \operatorname{Im}\left(\sum_{i=0}^{g}\int_{a_i}\alpha\int_{b_i}\overline{\alpha}\right) < 0.$$

3. If $\alpha \in H^0(K)$ and $\int_{a_i}\alpha = 0$ for all $i = 1, \cdots, g$ then $\alpha = 0$.

Proof. 1. By type $\alpha \wedge \beta = 0$ thus $\int_M \alpha \wedge \beta = 0$.

2. Part one of the second claim follows from the following computations.

$$\begin{aligned}\int_M \alpha\wedge\overline{\alpha} &= \sum_{i=1}^{g}\int_{a_i}\alpha\int_{b_i}\overline{\alpha} - \int_{a_i}\overline{\alpha}\int_{b_i}\alpha = \sum_{i=1}^{g}\int_{a_i}\alpha\int_{b_i}\overline{\alpha} - \overline{\int_{a_i}\alpha\int_{b_i}\overline{\alpha}} \\ &= 2i\,\mathrm{Im}\left(\sum_{i=0}^{g}\int_{a_i}\alpha\int_{b_i}\overline{\alpha}\right)\end{aligned} \tag{10.3}$$

Part two follows from the local fact: let $\alpha = f_\alpha dz$ then

$$i\,\alpha\wedge\overline{\alpha} = i\, f_\alpha\overline{f_\alpha}dz\wedge d\overline{z} = 2\,|f_\alpha|^2\,dx\wedge dy \;>\; 0. \tag{10.4}$$

3. Clear.

□

10.3 The period matrix and the reciprocity formula

A further important applications of the Riemann bilinear relations is that it allows us to distinguish a certain basis of the holomorphic 1–forms adapted to the basis of the homology.

Proposition 10.3. *1. Let $\underline{\omega} := (\omega_1,\cdots,\omega_g)$ be a basis for $H^0(K)$ and $A_k := \int_{a_k}\underline{\omega}\in\mathbb{C}^g$ then $A := (A_1,\cdots,A_g)\in GL(g,\mathbb{C})$.*

2. There is a basis $\underline{\omega} := (\omega_1,\cdots,\omega_g)$ for $H^0(K)$ such that $\int_{a_i}\omega_k = \delta_{ik}$. This basis is called canonically normalized *or* adapted.

3. Let a_i, b_i be the basis cycles, $\underline{\omega}$ the adapted basis of $H^0(K)$ and let $B_k := \int_{b_k}\underline{\omega}\in\mathbb{C}^g$ then $B := (B_1,\cdots,B_g)\in GL(g,\mathbb{C})$ is symmetric and $\mathrm{Im}\ B > 0$.

Proof. 1. Let be $\sum c_k A_k = 0$ then for $\alpha = \sum c_k\omega_k$ the integral $\int_{a_i}\alpha$ vanishes. So by Corollary 10.2, (3), α vanishes, and since $\underline{\omega}$ is a basis all c_k vanish.

2. Take any basis $\underline{\omega} = (\omega_1,\ldots,\omega_g)$ and define $A = (\int_{a_j}\omega_i)\in$ GL$(g,\mathbb{C})$. Then $\tilde{\underline{\omega}} := \underline{\omega}A^{-1}$ gives the sought basis.

3. B is symmetric by Corollary 10.2, (1), and Im B is positive definite with Corollary 10.2, (2).

★ check!

$$\begin{aligned}0 = \int_M \omega_i\wedge\omega_j &= \sum_{k=1}^{g}\left(\int_{a_k}\omega_i\int_{b_k}\omega_j - \int_{a_k}\omega_j\int_{b_k}\omega_i\right) \\ &= \int_{b_i}\omega_j - \int_{b_j}\omega_i = B_{i,j} - B_{j,i}.\end{aligned} \tag{10.5}$$

Let ω be the sum $\sum_{j=1}^{g} c_j\omega_j$ where all $c_j \in \mathbb{R}$ then $0 > \text{Im } \sum_{k,j=1}^{g} c_k c_j \overline{B_{k,j}} = -\sum_{k,j=1}^{g} c_k c_j \text{Im } B_{k,j}$. Hence Im $B > 0$

□

Definition 10.4. Let a_i, b_i be the basis cycles, $\underline{\omega}$ the adapted basis of $H^0(K)$ and $B := \left(\int_{b_k} \underline{\omega}\right)_{k=1,\cdots,g} \in GL(g, \mathbb{C})$. The matrix $(\text{id}_g, B) \in Mat(g \times 2g, \mathbb{C})$ is called the from M.

Lemma 10.5. *Given $p, q \in M$ there is an unique elementary meromorphic differential of 3^{rd} kind, i.e. a meromorphic differential with only 2 poles of order 1 and residues $+1$ and -1, and*

$$\int_{a_i} \omega_{pq} = 0 \tag{10.6}$$

Proof. The existence already was proven in Section 9.3 uniqueness follows from the condition (10.6).

□ ★

Theorem 10.6 (Reciprocity formula)**.** *Let $a_1, \ldots, a_g, b_1, \ldots, b_g$ the basic cycles and let ω_{pq} the unique adapted elementary meromorphic differential of the 3^{rd} kind. Then*

$$\int_{b_k} \omega_{pq} = 2\pi i \int_p^q \omega_k.$$

Proof. Recall the crucial step in the proof of the Riemann bilinear relations: Write $\omega_k = dA$ on the simply connected fundamental domain Δ where A is holomorphic. Then

$$\int_{\partial\Delta} A\omega_k = \sum_{i=1}^{g} \int_{a_i} \omega_k \int_{b_i} \omega_{pq} - \int_{b_i} \omega_k \int_{a_i} \omega_{pq} = \int_{b_k} \omega_{pq}$$

On the other hand

$$\int_{\partial\Delta} A\omega_k = 2\pi i(\text{Res}_p(A\omega_{pq}) + \text{Res}_q(A\omega_{pq})) = 2\pi i(A(p) - A(q)) = 2\pi i \int_p^q \omega_k.$$

□

Lemma 10.7. *The columns in* (id_g, B) *are* $\mathbb{R}$ *linearly independent.*

Proof. $0 = \sum c_k e_k + \sum d_k B_k$. Since Im $B > 0$ all d_k vanish, hence all c_k too. □

10.4 Jacobi variety

Recall the definition of the Picard group, 8.16:

$$\text{Pic}(M) = \{L \to M \text{ holomorphic line bundle}\}/\text{holomorphic equivalence}.$$

This group is isomorphic to the divisor group and is abelian (use tensor product as group action) hence a $\mathbb{Z}$–modul. The degree map $\deg : \mathrm{Pic}(M) \to \mathbb{Z}$ is a surjective group homomorphism and we can define $\mathrm{Pic}_k(M) = \deg^{-1}(k)$. $\mathrm{Pic}_k(M)$ is an affine $\mathbb{Z}$–subspace with underlying subgroup $\mathrm{Pic}_0(M)$.

Our aim is to find an analytic description of $\mathrm{Pic}_0(M)$.

Recall the basic definitions of homology theory: the closed cycles are given by $Z_1(M, \mathbb{Z}) :=$ free $\mathbb{Z}$–modul generated by all loops in M. Define the pairing

$$H^1_{dR}(M, \mathbb{C}) \times Z_1(M, \mathbb{Z}) \to \mathbb{C}, \ ([\omega], \gamma) \mapsto \int_\gamma \omega.$$

Definition 10.8. $\gamma \in Z_1(M, \mathbb{Z})$ is called *homologeous* to zero, $\gamma \sim 0$, if and only if

$$\int_\gamma \omega = 0, \quad \text{for all } [\omega] \in H^1_{dR}(M).$$

The *first homology group* is defined by

$$H_1(M, \mathbb{Z}) = Z_1(M, \mathbb{Z}) / \sim .$$

We cite the well known

Lemma 10.9. *Let M be a compact, oriented 2–dimensional manifold of degree g. Then*

$$H_1(M, \mathbb{Z}) \cong \mathbb{Z}^{2g}.$$

Theorem 10.10. *The map $j : H_1(M, \mathbb{Z}) \to H^0(K)^{-1} \cong (\mathbb{C}^g)$, $[\gamma] \mapsto j([\gamma])$ where*

$$j([\gamma])(\omega) := \int_\gamma \omega$$

maps $H_1(M, \mathbb{Z})$ to a lattice in $\mathbb{C}^g$.

Proof. Let $\underline{\omega} = (\omega_1, \ldots, \omega_g)$ the basis of $H^0(K)$ adapted to $a_1, \ldots, a_g, b_1, \ldots, b_g$. Then $H^0(K)^{-1} \cong \mathbb{C}^g$ via $\underline{\omega}$ and $j([a_i], [b_i]) = (I_g, B)$. By Proposition 10.3, (3), the image of the a and b cycles under j are $2g$ $\mathbb{R}$ linearly independent vectors since $\mathrm{Im}\ (j[a_i], [b_i]) = (0, \mathrm{Im}\ B)$.

□

So we can define

Definition 10.11. The Jacobi *variety* of M is the torus defined by

$$\mathrm{Jac}(M) := \mathbb{C}^g / j(H_1(M, Z)).$$

The lattice $j(H_1(M, Z))$ is called the *period lattice*.

10.5 Abel map

We define now the ABEL map which maps a zero divisor into a point of the JACOBI variety. Since the group of zero divisors is isomorphic to the PICARD group, cf. **??**, we obtain an injective group homomorphismus of the PICARD GROUP to the JACOBI variety. We will see that the Riemann surface can be embedded holomorphically into the JACOBI variety via the ABEL map.

Theorem 10.12 (ABEL's Theorem)**.** *The* ABEL *map*

$$A : \mathrm{Div}_0(M) \ \rightarrow \ \mathrm{Jac}(M), \ D \mapsto A([D])$$

where

$$A([\sum_{k=1}^{n} n_k p_k]) := \left[\sum_{k=1}^{n} n_k \int_{p_0}^{p_k}\right], \ p_0 \in M,$$

is well–defined and an injective group homomorphism.

Proof. First, A is independent of p_0 since $\int_{p_0}^{q_0} \in j(H_1(M,\mathbb{Z}))$. We need to show that $D = (f)$ implies $A([D]) \in j(H_1(M,\mathbb{Z}))$. Since D is a zero divisor we can assume without loss of generality that $D = (f) = \sum_{i=1}^{n}(p_i - q_i)$ where $p_i \neq q_i$, $p_i, q_i \in M$. We consider the logarithmic derivative of f, i.e $\frac{df}{f}$ which is a meromorphic 1–form with poles of order 1 and residue at p given by the order of the pole/zero of f at p. Thus, using the unique meromorphic differentials of the 3^{rd} kind we can write $\frac{df}{f}$ as

$$\frac{df}{f} = \sum_{i=1}^{n} \omega_{p_i q_i} + \sum_k c_k \omega_k .$$

Since we integrate a logarithmic derivative we know that $\int_{a_j} \frac{df}{f}, \int_{b_j} \frac{df}{f} \in 2\pi i\mathbb{Z}$. Thus there exists integers $n_j, m_j \in \mathbb{Z}$ with

$$2\pi i \ n_j = \int_{a_j} \frac{df}{f} = \sum_{i=1}^{n} \int_{a_j} \omega_{p_i q_i} + \sum_k c_k \int_{a_j} \omega_k = c_j,$$

and

$$2\pi i \ m_j = \int_{b_j} \frac{df}{f} = 2\pi \sum_k \int_{b_j} \omega_{p_k q_k} + 2\pi i \sum_k n_k \int_{b_j} \omega_k .$$

Using the reciprocity formula we get with $m = (m_1, \ldots, m_g), n = (n_1, \ldots, n_g) \in \mathbb{Z}^g$:

$$\sum_k \int_{q_k}^{p_k} \underline{\omega} = \mathrm{Id}\, m - Bn \in j(H_1(M,\mathbb{Z})).$$

But the left hand side is exactly the matrix of $A([D])$ using the chosen basis of $H^0(K)$ to identify $H^0(K)^{-1} \cong \mathbb{C}^g$.

Remains to prove the injectivity of the ABEL map, so assume that $A([D]) = 0$ for the divisor $D = \sum(p_i - q_i)$. We need to show that there exists a meromorphic function f such that $(f) = D$. Using the above considerations we easily see how to define f: Define first the logarithmic derivative of f as the meromorphic 1–form $\eta := \sum_{i=1}^n \omega_{p_i q_i} - 2\pi i \sum_{J=1}^g n_j \omega_j$ and integrate

$$f(p) := \exp(\int_{p_0}^p \eta).$$

By construction, f has divisor $(f) = D$.

□

Remark 10.13. Short–version of the second part of the proof, compare [GH]

$$\mathbb{C}P^1 \to (\lambda_1 f - \lambda_0) \to A(\lambda_1 f - \lambda_0)$$

★

hier fehlt ja wohl noch was

Our next project is to show that the ABEL map gives a holomorphic embedding of the Riemann surface M into the JACOBI variety of M:

Theorem 10.14. *Let M be a compact Riemann surface of genus $g > 0$. Then the* ABEL *map*

$$A : M \to \mathrm{Jac}(M),\ p \mapsto \int_{p_0}^p \bmod H_1(M, \mathbb{Z})$$

is a holomorphic embedding.

Proof. Clearly, the ABEL map is a holomorphic map (it is defined as an integration). We show that A is an injective immersion then by compactness it is an embedding. So assume $p_1 \neq p_2$ and $A(p_1) = A(p_2)$, i.e. $A((p_1 - p_2)) = 0$. By ABELs Theorem there exists a meromorphic function f with the same divisor, i.e $(p_1 - p_2) = (f)$. Thus $f : M \to \mathbb{C}P^1$ is a meromorphic function of degree 1, i.e. $M \cong \mathbb{C}P^1$ which contradicts the assumption $g > 0$.

Next, $d_p A : T_p M \to T_{A(p)} \mathrm{Jac}(M) \cong H^0(K)^{-1}$ and for $\omega \in H^0(K)$ we have

$$< d_p A, \omega >= d_p < A, \omega >= d_p \int \omega = \omega_p.$$

Since K has no base points, cf. Theorem 8.29, A is an immersion. □

10.6 The Picard group and the Jacobi variety

Now, our aim is to show that the above defined injective group homomorphism between the PICARD group and the JACOBI variety is in fact an isomorphism.

We consider

$$
\begin{aligned}
&\mathrm{Pic}_0(M)\\
&\quad = \{L \to M \text{ holomorphic line bundle of degree } 0\}/\text{holomorphic equivalence}\\
&\quad = \{\text{all holomorphic structures on } M \times \mathbb{C}\}/\text{holomorphic equivalence}\\
&\quad = \{[\bar\partial_0 + \alpha] \mid \alpha \in \Gamma(\bar K), \bar\partial_0 \text{ trivial holomorphic structure on } M \times \mathbb{C}\}
\end{aligned}
$$

We investigate when two given holomorphic structures are equivalent: Let $[\bar\partial_0 + \alpha] = [\bar\partial_0]$ where $\alpha \in \Gamma(\bar K)$. In fact, this means that there exists a function $f \in C^\infty(M, \mathbb{C}_*)$ such that

$$\bar\partial_0 + \frac{\bar\partial f}{f} = \bar\partial_0 + \alpha.$$

Clearly we can factor out the functions with zero winding number, i.e. $f = e^g$ where $g : M \to C$. For these we obtain $\frac{\bar\partial f}{f} = \bar\partial g$.

Lemma 10.15. *Let $\alpha \in \Gamma(\bar K)$. Then there exists a unique holomorphic $\omega \in H^0(K)$ such that*

$$\alpha + \mathrm{Im}\ \bar\partial = \bar\omega + \mathrm{Im}\ \bar\partial.$$

I.e.

$$H^0(K)^* \cong \overline{H^0(K)}.$$

Proof. □

★ go on! Leichter gesagt, als getan: ab hier wurden die gesammelten Mitschriften etwas wirr; ich habe momentan keine Lust, und steige bei Theta wieder ein!

Chapter 11

Theta functions

★ bisher schreibe ich nur wörtlich die Aufzeichnungen ab: man sollte das alles hier nicht so ernst nehmen

We start with a general definition of theta functions on certain complex tori and will later apply it to the special case where the complex torus is the Jacobi variety of a Riemann surface. Theta functions have applications by solving Heat equations and KdV. We will use them to construct meromorphic functions of tori into complex projective spaces. Using the Abel map we obtain thus embeddings of Riemann surfaces into complex projective spaces. We follow essentially [?].

11.1 Definition

Definition 11.1. Let $B \in \mathrm{GL}(g,\mathbb{C})$, $B^T = B$, $\mathrm{Im}\, B > 0$. Consider the g dimensional torus $\mathbb{C}^g/\Lambda$ where the lattice is defined by

$$\Lambda = \mathrm{span}_{\mathbb{Z}}\{e_1, \dots, e_g, B_{-1}, \dots, B_{-g}\} \cong \mathbb{Z}^g$$

where $\{e_1, \dots, e_g\}$ denotes the standard basis of $\mathbb{C}^g$ and $B_{-k} := Be_k$ is the k–th column of B. The *theta function* of $\mathbb{C}^g/\Lambda$ is the map $\theta : \mathbb{C}^g \to \mathbb{C}$ defined by

$$\theta(z) := \sum_{n\in\mathbb{Z}^g} e^{\pi i(<n,Bn>+2<n,z>)},\ z \in \mathbb{C}^g.$$

We state first some simple properties

Lemma 11.2. *1. $\theta : \mathbb{C}^g \to \mathbb{C}$ is holomorphic and $\theta \not\equiv 0$.*

2. $\theta(z) = \theta(-z)$ for $z \in \mathbb{C}^g$.

3. $\theta(z) = \theta(z + e_k)$ for $z \in \mathbb{C}^g$.

4. $\theta(z) = \exp -\pi i(B_{kk} + 2z_k))\theta(z + B_{-k})$ for $z \in \mathbb{C}^g$

Proof. 1. θ is absolut convergent on compact subsets K from $\mathbb{C}^g$ since for $z = x + iy \in K \subset \mathbb{C}^g$

$$|e^{\pi i<n,Bn>}| = e^{-\pi<n,\mathrm{Im}\ Bn>} \leq e^{-\pi c|n|^2}$$

and

$$|e^{2\pi i<n,z>}| = e^{-2\pi<n,x>} \leq e^{-2\pi\tilde{c}|n|}$$

for some constants $c, \tilde{c}$ depending only on K. Thus the theta function is majorized by a convergent series and hence a holomorphic map. Since θ is a Fourier series with coefficients $\neq 0$ the series $\theta \not\equiv 0$.

2./3. Simple calculation.

4. Calculating first the exponent of the addend of the theta function at $z+B_{-k}$ we obtain

$$\begin{aligned} < n, Bn > +2 < n, z + Be_k > &= < n + e_k, B(n + e_k) > +2 < n + e_k, z > \\ &\quad -2z_k - B_{kk} \end{aligned}$$

thus

$$\begin{aligned} \theta(z + B_{-k}) &= \sum_{n\in\mathbb{Z}^g} e^{\pi i(<n,Bn>+2<n,z+Be_k>)} \\ &= \sum_{\tilde{n}=n+e_k\in\mathbb{Z}^g} e^{\pi i(<\tilde{n},B\tilde{n}>+2<\tilde{n},z>)} e^{\pi i(-2z_k-B_{kk})} \\ &= \theta(z)e^{-\pi i(2z_k+B_{kk})} \end{aligned}$$

□

11.2 General nonsense: multipliers

Let $\pi : \tilde{M} \to M$ the universal covering map and $\Gamma = \pi_1(M)$. Denote by $\Gamma \times \tilde{M} \to \tilde{M}$, $(\gamma, x) \mapsto \gamma x$ the *deck transformation*. Let $V \to M$ be a vector bundle over M and $\tilde{V} = \pi^* V$ the pull back of V to the universal covering $\tilde{M}$. Then $\tilde{V}_{\gamma x} = V_{\pi\gamma x} = V_{\pi x} = \tilde{V}_x$.

Definition 11.3. Assume that $\tilde{V}$ is trivializable over $\tilde{M}$, i.e. there exists a global trivialization map $\Phi : \tilde{V} \to \tilde{M} \times \mathbb{C}^r$. There is a commutative diagram

$$\begin{tikzcd} & \mathbb{C}^r \arrow[dd, "\Phi_{\gamma x}\Phi_x^{-1} \in \mathrm{GL}(\mathbb{C}^r)"] \\ \tilde{V}_x = \tilde{V}_{\gamma x} \arrow[ur, "\Phi_x"] \arrow[dr, "\Phi_{\gamma x}"'] & \\ & \mathbb{C}^r \end{tikzcd}$$

The *multiplier* of V with respect to Φ is given by

$$\varphi : \Gamma \times \tilde{M} \to \mathrm{GL}(\mathbb{C}^r),\ (\gamma, x) \mapsto \Phi_{\gamma x}\Phi_x^{-1}.$$

Notice that $\varphi(\gamma_1\gamma_2, x) = \varphi(\gamma_1, \gamma_2 x)\varphi(\gamma_2, x)$, in particular $\varphi(1, x) = 1$.

If $V_1 \sim V_2$ are isomorphic vector bundles and assume $\tilde{V}_1, \tilde{V}_2$ are trivializable with trivalization maps $\Phi_k : \tilde{V}_k \to \tilde{M} \times \mathbb{C}^r$. The so defined mulitpliers φ_1, φ_2 satisfy

$$\varphi_2(\gamma, x) = f(\gamma x)\varphi_1(\gamma, x)f(x)^{-1} \text{ for some map } f : \tilde{M} \to \mathrm{GL}(\mathbb{C}^r).$$

Thus we define an equivalence relation on multipliers *varphi* $\sim \tilde{\varphi}$ if and only if $\tilde{\varphi}(\gamma, x) = f(\gamma x)\varphi(\gamma, x)f(x)^{-1}$ for some $f : \tilde{M} \to \mathrm{GL}(\mathbb{C}^r)$. In particular we have a well–defined map

$$\begin{aligned} \{V \to M \text{trivializable}\}/\text{Isom.} &\to \{\text{multipliers}\}/\sim, \\ V &\mapsto \varphi. \end{aligned}$$

Conversely, let $\varphi : \Gamma \times \tilde{M} \to \mathrm{GL}(\mathbb{C}^r)$ be given. Then there exists a up to isomorphism unique vector bundle $V \to M$ which has φ as multiplier: it is given by $V := \tilde{M} \times \mathbb{C}^r / \sim$ where $(x, \xi) \sim (y, \eta)$ if and only if $y = \gamma x$ and $\eta = \varphi(\gamma, x)\xi$.

★ holomorph?

We calculate the sections of this bundle:

Let $\psi \in \Gamma(V)$ then $\pi^*(\psi) = \psi \circ \pi \in \Gamma(\tilde{V})$. Define $f := \Phi \circ (\pi^*\psi) : \tilde{M} \to \mathbb{C}^r$. Then

$$f(\gamma x) = \Phi_{\gamma x}\psi_x = \Phi_{\gamma x}\Phi_x^{-1}(\Phi_x\psi_x) = \varphi(\gamma, x)f(x).$$

Conversely, any f with the above transformation behavior gives rise to a section in V, hence

$$\Gamma(V) \cong \{f : \tilde{M} \to \mathbb{C}^r \mid f(\gamma x) = \varphi(\gamma, x)f(x)\}.$$

11.3 Lefschetz embedding

We use the above general nonsense to see

Lemma 11.4. *Let* $M = \mathbb{C}^g/\Lambda$, $\Lambda = \mathrm{span}_{\mathbb{Z}}\{e_1, \dots, e_g, B_{-1}, \dots, B_{-g}\}$, $B \in GL(g, \mathbb{C})$, $B^T = B$, $\mathrm{Im}\ B > 0$. *Then there exists a unique holomorphic line bundle* L_B*, the* θ line bundle *over* $\mathbb{C}^g/\Lambda$*, such that* $0 \not\equiv \theta \in H^0(L_B)$.

Proof. Define the multiplier $\varphi : \Lambda \times \mathbb{C}^g \to \mathrm{GL}(\mathbb{C}) = \mathbb{C}_*$ by $\varphi(e_k, z) = 1, \varphi(B_{-k}, z) = e^{-\pi i(B_{kk}+2z_k)}$. Since $\theta(e_k+z) = \theta(z) = \varphi(e_k, z)\theta(z)$ and $\theta(B_{-k}+z) = \varphi(B_{-k}, z)\theta(Z)$ the theta functions is a holomorphic section of L_B. ☐ ★

und wieder: holomorph?

Lemma 11.5. *Let $L_B \to \mathbb{C}^g/\Lambda$ the θ–line bundle. Then*

$$\dim H^0(L_B) = 1.$$

Proof. Let $\tilde{\theta} \in H^0(L_B)$, $\tilde{\theta} : \mathbb{C}^g \to \mathbb{C}$ and $\tilde{\theta}(z + e_k) = \tilde{\theta}(z)$. The Fourier series of $\tilde{\theta}$ is hence

$$\tilde{\theta}(z) = \sum a_n e^{2\pi i <z,n>}.$$

Since $\tilde{\theta}(z_k + B_{-k}) \overset{!}{=} e^{-\pi i (B_{kk} + 2z_k)} \tilde{\theta}(z)$ one gets comparing the coefficients of

$$\sum a_n e^{2\pi i <z + Be_k, n>} = \sum a_n e^{-\pi i (B_{kk} + 2z_k - 2<z,n>)}$$

that

$$a_{n+e_k} e^{-\pi i B_{kk}} = a_n e^{2\pi i <Be_k, n>},$$

thus the coefficients a_n are determined by a_0. This applies also to the Fourier series $\tilde{\theta} - a_0 \theta$ thus the coefficients all vanish. $\square$

Lemma 11.6. $\dim H^0(L_B^r) = r^g$.

Proof. The multipliers of the tensor product are given by

$$\varphi^{L^r}(e_k, z) = 1,\ \varphi^{L^r}(B_{-k}, z) = e^{-i\pi r (B_{kk} + 2z_k)}.$$

The *theta function of order r* which are given by

$$\theta_{r,s} = \sum e^{\pi i (<B(n + \frac{s}{r}), s + rn> + 2<z, s + rn>)}$$

where $s = (s_1, \ldots, s_g),\ 0 \le s_i \le r - 1$ are a basis of $H^0(L^r)$. $\square$

Theorem 11.7 (Lefschetz embedding)**.** *Let $\mathbb{C}^g/\Lambda$ the usual torus. Then there exists an embedding*

$$F : \mathbb{C}^g/\Lambda \hookrightarrow \mathbb{C}P^{3^g - 1},\ F = [F_1, \ldots, F_N]$$

where $\{F_i\}$ is a basis of $H^0(L^3)$.

As applications of the Lefschetz embedding theorem we mention

Corollary 11.8. 1. *The* Lefschetz *embedding $F : \mathbb{C}/\Lambda \to \mathbb{C}P^2$ is given by the* Weierstrass *$\wp$ function $F = [1 : \wp : \wp']$.*

2. *Let M be a compact Riemann surface of genus g. Then we have the following diagram*

$$\begin{tikzcd}
M \arrow[r, hook, "A", "Abel"'] \arrow[drr] & \mathrm{Jac}(M) \arrow[r, hook, "F", "Lefschetz"'] & \mathbb{C}P^{3^g-1} \arrow[d] & \\
 & & \mathbb{C}P^3 \arrow[r, "Twistor"] \arrow[d, "immersion"] & \mathbb{H}P^1 \\
 & & \mathbb{C}P^2 &
\end{tikzcd}$$

For the proof of the LEFSCHETZ embedding theorem we need the following

Lemma 11.9. *1. Given $z_1, z_2 \in \mathbb{C}^g$ the map $a \mapsto \frac{\theta(z_1+a)}{\theta(z_2+a)}$ is nowhere vanishing holomorphic*

2. If $z \mapsto \frac{\theta(z+w)}{\theta(z)}$ is a nowhere vanishing holomorphic map then $w \in \Lambda$.

Proof. 1. Let $a_0 \in \mathbb{C}^g$ then there eixsts an open neighborhood U of $a_0, b \in \mathbb{C}^g$, wiht $\theta(z_k + b) \neq 0,\ \theta(z_k - a - b) \neq 0$ for all $a \in U$.

2. $\frac{\theta(z+w)}{\theta(z)} = e^{h(z)}$ for some holomorphic function h on $\mathbb{C}^g$ with $h(z+e_k) - h(z) = 2\pi i n_k$. Moreover, $h(z + B_{\_n}) - h(z) + 2\pi i \omega_k = 2\pi i m_k$. Since $\partial_{z_i} h : \mathbb{C}^g/\Lambda \to \mathbb{C}$ is holomorphic we have $\partial h \equiv const$ thus $h = h_0 + \sum_{i=1}^k h_i z_i$ and $h(z+e_k) - h(z) = h_k = 2\pi i n_k$. For the second kind of periods we see $h(z + B_{\_k}) - h(z) = \sum h_i B_{ik} = 2\pi i m_i - 2\pi i m_k$. We obtain $\omega_k = -\sum n_i B_{ik} + m_k$ and $\omega = Bn + Idm$, which means $\omega \in \Lambda$.

$\square$

Proof of the LEFSCHETZ *embedding theorem.* 1. Since $F_l : \mathbb{C}^g \to \mathbb{C},\ F_l(z + \gamma) = a(\gamma, z) F_l(z)$ the map $[F_0, \ldots, F_n]$ is well–defined if L^3 has no base points.

2. L^3 has no base points: Let $a, b \in \mathbb{C}^g$ and define

$$\theta_{a,b}(z) := \theta(z+a)\theta(z+b)\theta(z-(a+b)).$$

First, $\theta_{a,b} \in \Gamma(L^3)$ since

$$\begin{aligned} \theta_{a,b}(z + e_k) &= \theta_{a,b}(z) \\ \theta_{a,b}(z + B_{-k}) &= e^{-i\pi(2z_k + 2a_k + B_{kk} + 2z_k + 2b_k + B_{kk} + 2z_k - 2(a_k+b_k) + B_{kk})} \theta_{a,b}(z) \\ &= e^{-i\pi 3(2z_k + B_{kk})} \theta_{a,b}(z) \end{aligned}$$

Let $z_0 \in \mathbb{C}^g$. We need to show that there exists a $f \in H^0(L^3)$ such that $f(z_0) \neq 0$. If $\theta(z_0) \neq 0$ then $\theta_{a,b} \neq 0$ in a neighborhood of z_0, in particular there exist $a, b \in \mathbb{C}^g$ with $\theta_{a,b}(z_0) \neq 0$.

Now let $\theta(z_0) = 0$. One knows that $\theta(z) = 0$ defines a hyperplane and again there exist $a, b \in \mathbb{C}^g$ such that $\theta_{a,b}(z_0) \neq 0$. ★ who knows?

3. F is injective: We need to show that $F(z_1) = F(z_2)$ implies $z_1 - z_2 \in \Lambda$. But $F(z_1) = F(z_2)$ implies $F_l(z_1) = \rho F_l(z_2)$ for $\rho \neq 0$. Since $\{F_l\}$ is a basis we have $\theta_{a,b}(z_1) = \rho \theta_{a,b}(z_2)$ for all $a, b \in \mathbb{C}^g$ which implies

$$\frac{\theta(z_1 + a)}{\theta(z_2 + a)} = \rho \frac{\theta(z_2 + b)\theta(z_2 - (a+b))}{\theta(z_1 + b)\theta(z_1 - (a+b))}$$

Use Lemma 11.9.

4. F is an immersion: For $z_0 \in \mathbb{C}^g$ we have $d_{z_0}F(\mathbb{C}^g) \subset \mathrm{Hom}(\Sigma_{z_0}, \mathbb{C}^{N+1}/\Sigma_{z_0})$ if and only if $\delta_{z_0}\begin{pmatrix} F_0 \\ \vdots \\ F_N \end{pmatrix}(\mathbb{C}^g) \subset \mathbb{C}^{N+1}/\Sigma_{z_0}$ if and only if

$$\operatorname{rank}\begin{pmatrix} F_0 & \partial_{z_1}F_0 & \dots & \partial_{z_g}F_0 \\ \vdots & & \dots & \vdots \\ F_N & \partial_{z_1}F_N & \dots & \partial_{z_g}F_N \end{pmatrix} = g+1.$$

Suppose there exists c_k not all zero such that $c_0F(z_0) = \sum_{k=1}^{g} c_k\partial_{z_k}F(z_0)$. Then $c_0\theta_{a,b}(z_0) = \sum c_k\partial_{z_k}\theta_{a,b}(z_0)$ for all $a, b \in \mathbb{C}^g$. Thus

$$c_0\theta(z_0+a)\theta(z_0+b)\theta(z_0-(a+b)) = \sum c_k(\partial_{z_k}\theta(z_0+a))\theta(z_0+b)\theta(z_0-(a+b)),$$

so finally $c_0 = \sum c_k(\partial_{z_k}(\ln\theta(z_0+a)) + \partial_{z_k}(\ln\theta(z_0+b)) + \partial_{z_k}(\ln\theta(z_0+(a-b)))$. Let $\varphi := \sum c_k\partial_{z_k}\theta$ i.e. $c_0 = \varphi(z_0+a) + \varphi(z_0+b) + \varphi(z_0-a-b)$.

★ hier sind die gesammelten Aufzeichnungen äußerst kryptisch, z.B.: Bildchen von a_0 und Θ...?!?!?!?!?!

The map $a \mapsto \varphi(z_0+a)$ is holomorphic. Lemma 11.9 thus implies that $\varphi(z+e_k) = \varphi(z)$ and $\varphi(z+B_{-k}) = \varphi(z) - 2\pi i c_k$. The same trick applies to $\partial\varphi$ which is holomorphic on $\mathbb{C}^g/\Lambda$, i.e. $\partial\varphi$ is constant. Hence, writting $\varphi = \varphi_0 + \sum e_i z_i$, we have $\varphi = \varphi_0$, thus $c_k = 0$ and $\varphi = 0, c_0 = 0$.

□

11.4 Theta Divisor

Since the theta function θ is a holomorphic section in the theta bundle L_B, i.e. $\theta \in H^0(L_B)$, it is natural to investigate its divisor Θ. Before we do so, we have to note that the difference between divisors on Riemann surfaces and divisors on general manifolds is that instead of points the formal sum goes over irreduzible analytic codimension 1 subsets of the manifold.
Throughout this section get to know following objects with there properties and there relationships:

1. The translation of the theta divisor in the Jacobian, $\Theta_\zeta := \Theta + \zeta$ where $\zeta \in \mathrm{Jac}(M)$, Θ_ζ is the divisor of a holomorphic section in the translated line bundle $T_\zeta^* L_B$, which will be abbreviated by L_ζ, i.e. $\Theta_\zeta = (\theta(z - \zeta))$ and $\theta(z - \zeta) \in H^0(L_\zeta)$.

2. An other object is the map $\Psi_\zeta : p \mapsto \theta(A_{P_0}(p) - \zeta)$ which is a section in pullback $A_{P_0}^*(L_\zeta)$, which we abbreviated often by Ψ and its divisor D_ζ, which is then a divisor on a Riemann surface.

3. Last but not least $W_k := \{A_{P_0}(D) \mid D \text{ effective divisor of degree } k\}$ an analytic subset of the Jacobian for each k. A effective divisor, also integral divisor, is a divisor $D = \sum n_j p_j$ where all $n_j \geq 0$.

Before we go into details we want note some technical facts, which are easy to proof, see Appendix G, to make the rest of this section more instructive.

Lemma 11.10 (Technical). *Let a_j, b_j be the basis curves of M. Let $P_j \in a_j$ and P'_j the corrosponding point in a_j^{-1} and Q_j, Q'_j analog in b_j respective b_j^{-1}. Further R_1, R_2, R_3 the vertexes of the fundamental domain Δ where a_j lies between R_1, R_2 and b_j lies between R_2, R_3. Note that everything which works for a Q – couples works for R_1, R_2 and analogous with the P – couples and R_2, R_3. See picture* **??**.

1. $A_k(P'_j) = A_k(P_j) + B_{j,k}$

2. $A_k(Q'_j) = A_k(Q_j) + \delta_{j,k}$

3. $d \log \Psi(Q'_j) = d \log \Psi(Q_j)$

4. $d \log \Psi(P'_j) = d \log \Psi(P_j) - 2\pi i \, \omega_j(P_j)$

5. $\Psi(P'_j) = \Psi(P_j) \exp(-2\pi i \, (A_j(P_j) - \zeta_j + \frac{1}{2} B_{j,j}))$

6. $\int_{a_j} d\log\Psi(P) = 0$

7. $\int_{b_j} d\log\Psi(Q) = 2\pi i\,(\zeta_j - A_j(R_2) - \frac{1}{2}B_{j,j}))$

8. $\int_{a_j+a_j^{-1}}(d\log\Psi(P)A_k(P)) = 2\pi i\,\int_{a_j}(\omega_j(P)A_k(P)) + 2\pi i\,B_{j,k}$

9. $\int_{b_j+b_j^{-1}}(d\log\Psi(Q)A_k(Q)) = \delta_{j,k}\,2\pi i\,(\zeta_j - A_j(R_2) - \frac{1}{2}B_{j,j}))$

10. $\int_{a_j}(\omega_k(P)A_k(P)) = \delta_{j,k}(A_k(R_2) - \frac{1}{2})$

Lemma 11.11. *The function Ψ either vanishes identically on M or has exactly g zeros counting the multiplicity, i.e.* $\deg L_\zeta = g$.

To proof the lemma we need following fact from complex analysis: Let h be a meromorphic function with $M < \infty$ poles and $N < \infty$ zeros, both counted with multiplicity, then $N - M = \frac{1}{2\pi i}\int_\gamma d\log h(z)dz$, where γ is a simply connected curve around the zeros and poles.

Proof. Suppose $\Psi \not\equiv 0$. Remember the notation and the situation sketched in the Riemann bilinear relation section, see e.g. the picture **??**, i.e the number of zeros in the fundamental domain Δ are

$$\deg(\Psi) = \frac{1}{2\pi i}\int_{\partial\Delta} d\log\Psi = \frac{1}{2\pi i}\sum_{i=1}^{g}\int_{a_i+a_i^{-1}+b_i+b_i^{-1}} d\log\Psi,$$

where we can assume that no zeros occure on $\partial\Delta$.
Using fact 3 and 4 from the technical Lemma 11.10 we get

$$\int_{b_i+b_i^{-1}} d\log\Psi = 0 \quad \text{and} \quad \int_{a_i+a_i^{-1}} d\log\Psi = 2\pi i\int_{a_i}\omega_i = 2\pi i.$$

Recapitulating we have

$$\deg(\Psi) = \frac{1}{2\pi i}\sum_{i=1}^{g} 2\pi i = g,$$

which is the claim.

∎

Theorem 11.12. *If M is a compact Riemann surface of genus g and $A(M) \not\subset \Theta_\zeta$ then $A(M)\cap\Theta_\zeta$ consists of g points, counted with multiplicities. Further, we have* $\deg D_\zeta = g$ *and* $A(D_\zeta) = \zeta - \kappa$, *where* $\kappa \in \mathrm{Jac}(M)$ *is the vector of Riemann constance.*

★correct?

Remark 11.13. The vector of Riemann constance κ only depends on the base point chosen for the ABEL-JACOBI map and the basic curves. The components are[1]

$$\kappa_k = \frac{1}{2}(B_{k,k}+1) - \sum_{j\neq k}\int_{a_j}(\omega_j(P)A_k(P)).$$

Remark 11.14. Before we start to prove Theorem 11.12 we want to remember some facts from complex analysis:

1. The Residium Theorem: Let h be a meromorphic function then $\operatorname{Res}_p h = \frac{1}{2\pi i}\int_{\gamma_p} h$, where γ_p is a simply closed curve with goes around p but not around any other pole.

2. Let h be a meromorphic function with a pole at p of residium 1 and f a holomorphic function with no zero at p, then hf has a pole at p of residium $f(p)$.

3. Let Δ a n-gon, further there are $m \leq n$ interior points p_j then for all interior point p_0 different from the p_js there exists m pathes from p_0 to the vertex point of the n-gon such that the Δ decomposes into m parts T_j where in every part one and only one point p_j lies in the interior, see picture **??**.
   As an application we have for a meromorphic function h with $m \leq g$ poles p_i
   $$\sum_{j=1}^{g}\int_{a_i+b_i+a_i^{-1}+b_i^{-1}} h = \int_{\partial\Delta} h = \sum_{j=1}^{m}\int_{T_j} h = \sum_{p\in\text{poles}}\int_{\gamma_p} h.$$

Proof. Since $A(M) \not\subset \Theta_\zeta$ its divisor is $D_\zeta = \sum_{j=1}^{g} P_j$. From the Remark 11.14 if take h to be $d\log\Psi$ and f to be A_k we have

$$\begin{aligned} A_k(D_\zeta) = \sum_{j=1}^{g} A_k(P_j) &= \frac{1}{2\pi i}\int_{\partial\Delta}(d\log\Psi(Z)A_k(P)) \\ &= \frac{1}{2\pi i}\sum_{j=1}^{g}\int_{a_j+b_j+a_j^{-1}+b_j^{-1}}(d\log\Psi(Z)A_k(P)) \end{aligned}$$

Using fact 8 and 9 from the technical Lemma 11.10 we get

$$A_k(D_\zeta) = \frac{1}{2\pi i}\sum_{j=1}^{g}(\int_{a_j}(2\pi i\ \omega_j(P)A_k(P)) + 2\pi i\ B_{j,k} + \delta_{j,k}\ 2\pi i\ (\zeta_j - A_j(R_2) - \frac{1}{2}B_{j,j}))$$

[1] For BOBENKO $\kappa_k = \pi i + \frac{1}{2}B_{kk} - \frac{1}{2\pi i}\sum_{j\neq k}\int_{a_j}\omega_j\int_{P_0}^{P}\omega_k$. But, we have different definitions for θ, ω_k, ... – they differ by the factor $\frac{1}{2\pi i}$ –.

Since its enough to compute the $A_k(D_\zeta)$ modulo periods we could cancel the $B_{j,k}$ term, i.e.

$$A_k(D_\zeta) = \sum_{j=1}^{g} \int_{a_j} (\omega_j(P)A_k(P)) + (\zeta_k - A_k(R_2) - \frac{1}{2}B_{k,k}).$$

Using again the technical Lemma 11.10, fact 10, we get

$$\begin{aligned} A_k(D_\zeta) &= \sum_{j\neq k} \int_{a_j} (\omega_j(P)A_k(P)) + (A_k(R_2) - \frac{1}{2}) + (\zeta_k - A_k(R_2) - \frac{1}{2}B_{k,k})) \\ &= \sum_{j\neq k} \int_{a_j} (\omega_j(P)A_k(P)) - \frac{1}{2} - \frac{1}{2}B_{k,k} + \zeta_k \end{aligned}$$

Hence we have

$$A(D_\zeta) = \zeta - \kappa \text{ where } \kappa_k = \frac{1}{2}(B_{k,k}+1) - \sum_{j\neq k} \int_{a_j} (\omega_j(P)A_k(P))$$

□

Theorem 11.15. *$\Theta = W_{g-1} + \kappa$, where κ is the vector of Riemann constance.*

★ dense?

Proof. First we show that $W_{g-1} + \kappa \subset \Theta$. Let $D = \sum^g P_i$ such that $A(M) \not\subset \Theta_\zeta$ for $\zeta := A(D) + \kappa$. Note these divisors are a dense subset, hence its enough to show it for them. Since $A(D) = \zeta - \kappa$ and the Theorem 11.12, $A(D_\zeta) = \zeta - \kappa$, we know $D = D_\zeta$. Hence

$$0 = \theta(A(P_g) - \zeta) = \theta(-A(D - P_g) - \kappa) = \theta(A(D - P_g) + \kappa).$$

Since all divisor of W_{g-1} are of the form $D - P_g$ we have $W_{g-1} + \kappa \subset \Theta$.
Conversly we show that $\Theta \subset W_{g-1} + \kappa$. Let $\zeta \in \Theta$ a zero.
Suppose there exist a point $P \in M$ such that $\theta(A(z) - A(P) - \zeta) \not\equiv 0$ in z. Define $D = D_{A(P)+\zeta} = (\theta(A(z) - A(P) - \zeta))$ its divisor. Since P is a zero, $D' := D - P$ has degree $g - 1$. Again from Theorem 11.12 we know $A(D) = (A(P) + \zeta) - \kappa$, i.e. $A(D') = A(D - P) = \zeta - \kappa$. Hence $\zeta = A(D') + \kappa$ which is the claim for this case.
Remains the case that for all points $P \in M$, $\theta(A(z) - A(P) - \zeta) \equiv 0$. Let k be the least integer such that $\theta(A(D_0) - A(D_1) - \zeta) \equiv 0$ for all effective divisor D_0, D_1 of degree k. Since the Abel-Jacobi map is onto for $Div^g(M)$, i.e. $\theta(A(D_0) - A(D_1) - \zeta) \not\equiv 0$, we know that $k < g$. Now let D_0, D_1 be effective divisors of degree $k+1$ with $\theta(A(D_0) - A(D_1) - \zeta) \neq 0$. Suppose $D_0 + D_1$ consists of $2k+2$ distinct points. For $D_0 =: P_0 + D'_0$ the map $F : z \mapsto \theta(A(z) + A(D'_0) - A(D_1) - \zeta) \not\equiv 0$, e.g. $F(P_0) \neq 0$.

★ abel section: A, S^g, dA

Let $D := D_{-A(D'_0)+A(D_1)+\zeta} = (F)$ the divisor of F. Then from Theorem 11.12

we know $\deg D = g$ and since $\zeta \in \Theta$ that $A(D) = A(D_1) - A(D_0') + \zeta - \kappa$. Further $F(P_1) = 0$ for all $P_1 \in D_1$ since $\deg(D_1 - P_1) = k$, i.e. $F(P_1) = \theta(A(D_0') - A(D_1 - P_1) - \zeta)$. Hence there exists a divisor D_2 of degree $g - k - 1$ such that $D = D_1 + D_2$.
Now we know $A(D) = A(D_1) - A(D_0') + \zeta - \kappa$ and $A(D) = A(D_1) + A(D_2)$, hence $\zeta = A(D_2 - D_0') + \kappa$ and the degree is $\deg(D_2 - D_0') = g - k - 1 + k = g - 1$. Thus we have $\Theta \subset W_{g-1} + \kappa$. □

Remark 11.16. The above Theorem relates the Theta divisor Θ, a purley algebraically object, with another divisor W_{g-1} on the JACOBIAN which involves the geometry of the Riemann surface M and the geometry of the ABEL - JACOBI-map A.

Using the relationship between the algebraic Θ divisor and the geometric W_{g-1} divisor we get following interesting

Theorem 11.17. *$A(K) = -2\kappa$, where K is the canonical divisor associated with the canonial bundle K and κ is again the vector of Riemann constance.*

Proof. Let $D \geq 0$ be a divisor of degree $g-1$. Since $D \geq 0$ we know $h^0(L(D)) \geq 1$. Using RIEMANN-ROCH Theorem we have $h^0(KL(-D)) \geq 1$, i.e. $K - D$ is effective. Further we know that $\deg K = 2g-2$, hence the degree of $K - D$ is $g-1$, i.e. $A(K-D) = A(K) - A(D) \in W_{g-1}$. Now then we have $A(K) - W_{g-1} \subset W_{g-1}$. On the other side for $D' \sim K - D$ we have $A(D) = A(K) - A(D') \in A(K) - W_{g-1}$, hence $W_{g-1} \subset A(K) - W_{g-1}$. Summa summarum we have $A(K) - W_{g-1} = W_{g-1}$. From this together with Theorem 11.15 and the fact that θ is a even function we get

$$\Theta = W_{g-1} + \kappa = -\Theta = -W_{g-1} - \kappa = W_{g-1} - A(K) - \kappa = \Theta - (A(K) - 2\kappa)$$

Since Θ is not left invariant on the JACOBIAN, see 11.9, the claim is proven. □

Theorem 11.18. *$A(M) \subset \Theta_\zeta$ if and only if there exist a special divisor $D \geq 0$ of degree g such that $A(D) = \zeta - \kappa$.*

Remark 11.19. A divisor D is called special if $h^0(KL(-D)) > 0$. The number $h^0(KL(-D))$ is also called the index of speciality and noted by $i(D)$. For a special divisor D of degree $> g - 1$ we have by RIEMANN-ROCH $h^0(L(D)) > 0$ and for degree $\geq g$ the existence of a non-vanishing holomorphic section.

★ rest!?

Proof. $A(M) \subset \Theta_\zeta$ if and only if for all $P \in M$ we have $A(P) - \zeta$, i.e. $\zeta - A(P) \in \Theta = W_{g-1} + \kappa$. Since there exists a degree $g-1$ divisor D with $A(D) + \kappa = \zeta - A(P)$ the condition above is equivalent to the following. For all P there exists a degree g divisor D' such that $P \in D'$ and $A(D) = \zeta - \kappa$. □

Corollary 11.20 (JACOBI Inversion Problem). *If $\zeta \in \mathrm{Jac}(M)$ and $A(M) \not\subset \Theta_\zeta$, then there exists a unique effective divisor D of degree g such that $A(D)+\kappa = \zeta$. Further D is given by the zero divisor of $\theta(A(z)-\zeta)$.*

Theorem 11.21 (Riemann's Factorisation Theorem). *Let f be a non constant meromorphic function on M, with divisor $D = \sum_{k=1}^r P_k - \sum_{k=1}^r Q_k$. Then there exists a one form $\omega \in H^0(K)$ such that for most $\zeta \in \Theta$ we have*

$$f(z) = c\exp(\int_{P_0}^z \omega)\prod_{k=1}^r \frac{\theta(A(z)-A(P_k)-\zeta)}{\theta(A(z)-A(Q_k)-\zeta)},$$

where P_0 is the base point chosen for the definition of the ABEL- JACOBI *map.*

★

abel section: define Y'!

Proof. Let f be a non constant meromorphic function on M and $D := \sum_{k=1}^r P_k - \sum_{k=1}^r Q_k$ its divisor. Let further $\zeta \in \Theta$ with

$$\zeta \notin \bigcup_{k=1}^r (Y' + \kappa - A(P_k)) \cup \bigcup_{k=1}^r (Y' + \kappa - A(Q_k)).$$

We know from Theorem 11.15 there exists a divisor $D_0 := \sum_{j=1}^{g-1} Q_{0,j}$ such that $\zeta = A(D_0) + \kappa$.
Define a function on the fundamental domain Δ by

$$F(z) = \prod_{k=1}^r \frac{\theta(A(z)-A(P_k)-\zeta)}{\theta(A(z)-A(Q_k)-\zeta)}$$

All $Q_{0,j} \in D_0$ are zeros of $\theta(A(z)-A(P_k)-\zeta)$ and of $\theta(A(z)-A(Q_k)-\zeta)$, since $\theta(A(Q_{0,j})-A(P_k)-\zeta) = \theta(A(Q_{0,j})-A(P_k)-A(D_0)-\kappa) = \theta(A(D_0+P_k-Q_{0,j})+\kappa)$ this is zero because its argument is of the form $W_{g-1}+\kappa$ and this by Theorem 11.15 the theta divisor Θ. Hence the divisor of F is

$$(F) = (\sum_{k=1}^r P_k + rD_0) - (\sum_{k=1}^r Q_k + rD_0) = \sum_{k=1}^r P_k - \sum_{k=1}^r Q_k = (f) = D$$

Since F is defined on Δ we have to modify the map so, that it fits together on the boundary. There is nothing to do on the b-sides, because for the corresponding point R_j and R'_j we know $A(R'_j) = A(R_j) + e_j$ and $\theta(z) = \theta(z+e_j)$.
For the a-sides we have $A(R'_j) = A(R_j) + B_j$ and the periodicity for θ, Lemma 11.10 fact 1 and 5, i.e. we get

$$\begin{aligned} F(R'_j) &= \frac{\prod_{k=1}^r \exp(-2\pi i(A_j(R_j) - A_j(P_k) - \zeta_j))}{\prod_{k=1}^r \exp(-2\pi i(A_j(R_j) - A_j(Q_k) - \zeta_j))}\, F(R_j) \\ &= \exp(-2\pi i \sum_{k=1}^r (A_j(P_k) - A_j(Q_k)))\, F(R_j) \end{aligned}$$

By the Abel Theorem 10.12, since $\deg D = 0$ and $D = (f)$, we know $A(D) \in \Lambda$, i.e. $A(D) = A(\sum_{k=1}^r P_k - \sum_{k=1}^r Q_k) = \sum_{j=1}^g n_j e_j + \sum_{j=1}^g m_j B_j$. Hence

$$\begin{aligned} F(R_j') &= \exp(-2\pi i(n_j + \sum_{i=1}^g m_i B_{i,j}))\ F(R_j) \\ &= \exp(-2\pi i \sum_{i=1}^g m_i B_{i,j})\ F(R_j) \end{aligned}$$

Now define a holomorphic one form by $\omega = \sum_{j=1}^g m_j \omega_j$, where the ω_j are the ususal dual to the a_j. Further define $\varphi(z) = \int_{P_0}^z \omega$, where P_0 is the base point of the Abel map.
It remains to show that $\exp(2\pi i\ \varphi(z))$ is almost the missing multiplier for F to be f. Let see what happends on the a-sides using fact 2:

$$\begin{aligned} \exp(2\pi i\ \varphi(R_k')) &= \exp(2\pi i\ \sum_{j=1}^g (m_j \int_{P_0}^{R_k'} \omega_j)) \\ &= \exp(2\pi i\ \sum_{j=1}^g (m_j A_j(R_k'))) \\ &= \exp(2\pi i\ \sum_{j=1}^g (m_j (A_j(R_k) + \delta_{k,j}))) \\ &= \exp(2\pi i\ \sum_{j=1}^g (m_j A_j(R_k)))\exp(2\pi i\ m_k) \\ &= \exp(2\pi i\ \varphi(R_k)) \end{aligned}$$

Hence $\exp(2\pi i\ \varphi(z))F(z)$ does it for the a-sides. But the crucial point are the b-sides, using fact 1 we get

$$\begin{aligned} \exp(2\pi i\ \varphi(R_k')) &= \exp(2\pi i\ \sum_{j=1}^g (m_j A_j(R_k'))) \\ &= \exp(2\pi i\ \sum_{j=1}^g (m_j (A_j(R_k) + B_{j,k}))) \\ &= \exp(2\pi i\ \sum_{j=1}^g (m_j A_j(R_k)))\exp(2\pi i\ \sum_{j=1}^g (m_j B_{j,k})) \\ &= \exp(2\pi i\ \varphi(R_k))\exp(2\pi i\ \sum_{j=1}^g (m_j B_{j,k})) \end{aligned}$$

Since $G(z) := \exp(2\pi i\ \varphi(z))F(z)$ and $f(z)$ have the same divisor, $G(z)/f(z)$ is a non–vanishing holomorphic function, hence constant. Id est there exists a complex number c such that $c\,G(z)$ does it.

□

Appendix A

Sheaves and presheaves

A short trip to history of mathematics: Sheave theory was invented by JEAN LERAY. Being prisoned in 1940 by the Germans he changed his mathematical field of interest from fluid dynamics to algebraic topology to make absolutely sure that Nazis could not benefit of his research for purpose of war. He then introduced sheaves and spectral sequences.

A good intuition of thinking of sheaves is as a way of organizing functions and forms which satisfy local properties. Often sheaves are introduced in this intuitive sense via presheaves, cf. for example [Mi], but one ends up with a interminable definition of sheaves by adding to the presheaf definition a not very illuminating sheaf condition. Following for example [Wa], [H] or [Gu] we prefer the more elegant way of defining sheaves as in Definition A.1 but have to work a little to relate this to the intuitive ideas.

A.1 Sheaves

In this section, if not stated otherwise M can be an arbitrary topological space.

Definition A.1. • A *sheaf of R–modules* (over a fixed ring R) on M is given by a topological space $\mathcal{G}$ and a continuous map $\pi\colon \mathcal{G} \to M$ such that

1. π is surjective and a local homeomorphism,
2. the *stalk* $\mathcal{G}_p = \pi^{-1}(p)$ over $p \in M$ is a R–module,
3. the maps

$$\dot{\bigcup_{p\in M}} \mathcal{G}_p = \mathcal{G} \to \mathcal{G}, \alpha \mapsto \lambda\alpha, \text{ for } \lambda \in R,$$

and

$$\mathcal{G} \times \mathcal{G} \supset \dot{\bigcup_{p\in M}} \mathcal{G}_p \times \mathcal{G}_p \to \mathcal{G}, \qquad (f, g) \mapsto (f - g)$$

are continuous.

- A continuous map $s\colon U \to \mathcal{G}$ satisfying $\pi \circ s = \mathrm{id}_U$ is called a *section* of $\mathcal{G}$ over the open set $U \subset M$. We denote by $\Gamma(U, \mathcal{G})$ the set of sections of $\mathcal{G}$ over U and by $\Gamma(\mathcal{G})$ the set of global sections.

- A *subsheaf* of a sheaf $\mathcal{G}$ is an open subset $\mathcal{F} \subset \mathcal{G}$ such that $\mathcal{F}_p := \mathcal{F} \cap \mathcal{G}_p$ is for all $p \in M$ a submodule of G_p.

Notice, that the topology on $\mathcal{G}$ induces the discrete topology on every stalk.

★ change it!

Remark A.2. A sheaf of abelian groups is clearly given by $R = \mathbb{Z}$. Nevertheless, one could define a sheaf of abelian groups or R–algebras by using only a modified version (to the corresponding algebraic structures) of the continuity conditions 3) .

We denote by 0 the *zero–section* $p \to 0 \in \mathcal{G}_p$. Since π is a local homeomorphism, π is an open map and so is any local section of a sheaf $\mathcal{G}$. If a local section of $\mathcal{G}$ vanishes in one point p, it has to vanish completely in a neighborhood of p. This follows from the fact that if two sections coincide in a point, they have to coincide in a whole neighborhood of that point.

Example A.3. 1. Let G be a R–module (abelian group, R–algebra) carrying the discrete topology. Then $\mathcal{G} := M \times G$ with the product topology is the so called *constant sheaf*. One can interpret a section $s \in \Gamma(U, \mathcal{G})$ as a map from $U \to G$ by $p \mapsto \mathrm{pr}_2(s(p))$.

2. Let G be an abelian group (R–module, R–algebra). Define

$$\mathcal{G} = \dot{\bigcup_{p\in M}} G_p \text{ where } \mathcal{G}_p := \begin{cases} 0 & p \neq p_0 \\ G & p = p_0. \end{cases}$$

Equipping $\mathcal{G}$ with the unique topology making the projection a local homeomorphism we obtain a sheaf, the so called *skyscraper sheaf* (“Out of the drab skyline the group G rises majestically at the chosen point p_0”, [Mi]).

3. Recall the definition of a *germ of a C^k function in a point* $p \in M$ where M is a manifold: two local C^k functions f and g define the same germ at p if they coincide in some neighborhood of p. We denote by $[f]_p$ the germ of f at p, i.e. the equivalence class of all local functions with the same germ at p as f, and write

$$\mathcal{C}^k_p := \{\text{germs of } C^k \text{ functions at } p\}.$$

For $U \subset M$ open, $f \in C^k(U, \mathbb{R})$, i.e. $f : U \to \mathbb{R}$ is a C^k function, we define

$$U_f := \{[f]_p \mid p \in U\}.$$

It is an easy exercise to show that this defines a basis of a topology on

$$\mathcal{C}^k := \dot{\bigcup_{p \in M}} \mathcal{C}^k_p$$

such that $\mathcal{C}^k$ with the projection $\pi : \mathcal{C}^k \to M, [f]_p \mapsto p$ is a sheaf, the *sheaf (of germs) of C^k functions.*

Furthermore, the set of local sections of the sheaf of C^k functions can be identified with local C^k functions: $C^k(U, \mathbb{R}) \cong \Gamma(U, \mathcal{C}^k),\ f \in C^k(U, \mathbb{R}) \mapsto (s : U \to \mathcal{C}^k, p \mapsto [f]_p \in \mathcal{C}^k_p)$.

As in the real case we define the sheaf of $\mathbb{C}$–valued C^k functions; similarly one defines the sheaf of germs of holomorphic functions and of sections of vector bundles. We will do this later as part of a general construction, cf. Section A.3.

4. Let $\mathcal{F}$ be a subsheaf of $\mathcal{G}$. Clearly, $\mathcal{F}$ is itself a sheaf over M with projection map $\pi_{\mathcal{F}} = \pi_{\mathcal{G}}|_{\mathcal{F}}$.

   In particular, the zero–sheaf, defined up to isomorphism by $M \times 0$, where 0 is the module consisting only of the identity element, is a subsheaf of every sheaf.

5. Let $\mathcal{F} \subset \mathcal{G}$ be a subsheaf. The *quotient sheaf* is defined by

$$(\mathcal{G}/\mathcal{F})_p = \mathcal{G}_p/\mathcal{F}_p \quad \text{(use the quotient topology on } \mathcal{G}/\mathcal{F}).$$

Definition A.4. A *sheaf homomorphism* of sheaves of R–modules $\mathcal{G}_1$ to $\mathcal{G}_2$ on M is a continuous map

$$\alpha : \mathcal{G}_1 \to \mathcal{G}_2$$

such that

$$\pi_2 \circ \alpha = \pi_1 \tag{A.1}$$

and the maps

$$\alpha_p = \alpha|_{\mathcal{G}_{1p}} : \mathcal{G}_{1p} \to \mathcal{G}_{2p}$$

are R–homomorphism.

Remark A.5. Notice that (A.1) implies that α is a local homeomorphism from $\mathcal{G}_1$ to $\mathcal{G}_2$. Furthermore, for any sheaf homomorphism $\alpha : \mathcal{G}_1 \to \mathcal{G}_2$ both $\ker \alpha$ and Im α define subsheaves (of $\mathcal{G}_1$ resp. $\mathcal{G}_2$) where the stalks over p are exactly the kernel resp. image of the R–homomorphism α_p.

Definition A.6. A sequence of sheaves

$$\ldots \to \mathcal{G}_i \overset{\alpha_i}{\to} \mathcal{G}_{i+1} \overset{\alpha_{i+1}}{\to} \mathcal{G}_{i+2} \to \ldots$$

(where α_i are sheaf homomorphisms) is called exact if $\ker \alpha_{i+1} = \text{Im } \alpha_i$.

A *short exact sequence of sheaves* is an exact sequence of sheaves of the form

$$0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 0.$$

Observe that a sequence of sheaves

$$\ldots \to \mathcal{G}_i \xrightarrow{\alpha_i} \mathcal{G}_{i+1} \xrightarrow{\alpha_{i+1}} \mathcal{G}_{i+2} \to \ldots$$

is (by definition) exact if and only if at each point $p \in M$ the sequence

$$\ldots \to \mathcal{G}_{ip} \xrightarrow{\alpha_{ip}} \mathcal{G}_{i+1p} \xrightarrow{\alpha_{i+1p}} \mathcal{G}_{i+2p} \to \ldots$$

on the stalks is exact.

To remind the ideas of exact sequences we explain in more detail:

Example A.7. Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be sheaves. We investigate under which conditions the sequence

$$0 \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \xrightarrow{\gamma} \mathcal{H} \xrightarrow{\delta} 0.$$

is a short exact sequence:

Exactness at $\mathcal{F}$ requires that $0 = \text{Im } \alpha_p = \ker \beta_p$, i.e. the map β has to be injective. Hence, we can interpret $\mathcal{F}$ as subsheaf of $\mathcal{G}$ by considering $\beta = \text{incl}$ as the inclusion map of $\mathcal{F}$ into $\mathcal{G}$.

At $\mathcal{G}$ we need that $\mathcal{F}_p = \text{Im incl}_p = \text{Im } \beta_p = \ker \gamma_p$. Hence γ can be understood as a projection map from $\mathcal{G}$ to $\text{Im } \gamma = \mathcal{G}/\mathcal{F} \subset \mathcal{H}$. By exactness at $\mathcal{H}$ the map γ is surjective since $\text{Im } \gamma_p = \ker \delta_p = (\mathcal{H})_p$. Hence, any short exact sequence can be seen as a sequence

$$0 \xrightarrow{\alpha} \mathcal{F} \overset{\text{incl}}{\hookrightarrow} \mathcal{G} \xrightarrow{\text{proj}} \mathcal{G}/\mathcal{F} \to 0.$$

A.2 Presheaves

Now we want to explain how sheaves can be used as a mechanism to pass local information to global information. Given for example a vector bundle $E \to M$ over a manifold M we have the local sections $\mathcal{C}^k_E(U) := \Gamma(E_U)$ as local data. In particular, for any $s \in \mathcal{C}^k_E(U)$ and for any open subset $V \subset U$ we have a map $\tilde{s} \in \mathcal{C}^k_E(V)$, given by the restriction $\tilde{s} = s|_V$. We first formalize this to:

★ ???

Definition A.8. Let M be a topological space. A *presheaf of R–modules* on M consists of a collection of R–modules $\mathcal{P}(U)$ (for all open sets $U \subset M$) and a homomorphisms $\rho^V_U \colon \mathcal{P}(V) \to \mathcal{P}(U)$ (for all inclusions $U \subset V \subset M$ of open sets) such that

1. $\mathcal{P}(\emptyset) = 0$
2. $\rho_U^U = \mathrm{id}_{\mathcal{P}(U)}$ for any open set $U \subset M$ and
3. $\rho_U^W = \rho_U^V \circ \rho_V^W$ for all inclusions $U \subset V \subset W \subset M$ of open sets.

The maps ρ_U^V are called the *restriction maps*. We use the short–hand notation $f|_U = \rho_U^V(f)$ for $f \in \mathcal{P}(V)$. A *homomorphism* α *of presheaves* $\mathcal{P}_1$ *to* $\mathcal{P}_2$ is a collection of homomorphisms $\alpha_U \colon \mathcal{P}_1(U) \to \mathcal{P}_2(U)$ (for $U \subset M$ open) such that for all inclusions $V \subset W \subset M$ of open sets the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{P}_1(W) & \xrightarrow{\rho_V^W} & \mathcal{P}_1(V) \\ {\scriptstyle \alpha_W}\downarrow & & {\scriptstyle \alpha_V}\downarrow \\ \mathcal{P}_2(W) & \xrightarrow{\rho_V^W} & \mathcal{P}_2(V) \end{array}$$

Remark A.9. Again we can define presheaves of R–algebras.

Example A.10. 1. Let $\mathcal{G}$ be a sheaf. The sections of $\mathcal{G}$ define a presheaf, the *canonical presheaf* ˜

G$ofGby\tilde{y}$

G(U) := $\Gamma(U, \mathcal{G})$ for open subsets $U \subset M$(and the restriction map ρ_V^U is just the usual restriction of a map $U \to \mathcal{G}$ to V).

Let $E \to M$ be a (differentiable) vector bundle over $\mathbb{K}$ on a differentiable manifold. The presheaf of C^k-sections of E denoted by $\mathcal{C}_E^k$ assigns to an open subset $U \subset M$ the $\mathbb{K}$-module

$$\mathcal{C}_E^k(U) := \Gamma^k(U, E)$$

(this was just our motivation to define presheaves!).

An important special case of this is the presheaf

$$\Omega_E^p = \mathcal{C}_{\tilde{E}}^{\infty}$$

where $\tilde{E} = \Lambda^p TM^* \otimes E$.

Let $E \to M$ be a holomorphic vector bundle over a complex manifold. The presheaf $\mathcal{O}_E$ of holomorphic sections of E is defined by

$$\mathcal{O}_E(U) := H^0(E_U) = \Gamma_{hol}(U, E)$$

for all open subsets $U \subset M$. In the case of the trivial bundle $E = \underline{\mathbb{C}}$, the presheaf $\mathcal{O}_E =: \mathcal{O}$ is called the *presheaf of holomorphic functions*.

More generally, one can define the *structure presheaf* of a manifold M as the presheaf of certain functions on M, depending on the structure (differentiable, holomorphic, analytic) of the manifold.

Let M be complex manifold We define the presheaf $\mathcal{O}^*$ by

$$\mathcal{O}^*(U) = \{f \in \mathcal{O}(U) \mid f(p) \neq 0 \text{ for all } p \in U\}.$$

Notice that $\mathcal{O}^*(U)$ is a presheaf of abelian groups, each $\mathcal{O}^*(U)$ being an abelian group by pointwise multiplication.

Given an R–module V we define the skyscraper presheaf by

$$\mathcal{P}(U) := \begin{cases} V, & p_0 \in U \\ 0, & p_0 \notin U \end{cases}$$

with restriction maps

$$\rho^U_{\tilde{U}}(v) = \begin{cases} v, & p_0 \in \tilde{U} \\ 0, & p_0 \notin \tilde{U} \end{cases}, \text{ where } v \in \mathcal{P}(U).$$

The *constant presheaf* of a commutative ring R is given by $R(U) = R$ where the restriction maps are the identity.

We like to give also examples for homomorphism of presheaves:

1. Given a vector bundle homomorphism $\alpha : E_1 \to E_2$ we obtain an induced homomorphism of presheaves $\alpha : C^\infty_{E_1} \to C^\infty_{E_2}$ by

$$\alpha_U : \Gamma(E_{1U}) \to \Gamma(E_{2U}),\ s \mapsto \alpha \circ s|_U.$$

2. More generally, we obtain: if $\alpha : \mathcal{G}_1 \to \mathcal{G}_2$ is a sheaf homomorphism then α induces a homomorphism of presheaves on the canonical presheaves $\tilde{\alpha} : \tilde{\mathcal{G}}_1 \to \tilde{\mathcal{G}}_2$.

3. Another homomorphism of presheaves is for example: $\mathcal{O} \overset{\exp}{\to} \mathcal{O}^*$ where $\exp(f)(z) := \exp(2\pi i f(z))$.

A.3 From sheaves to presheaves and back?

We already noticed that any sheaf $\mathcal{G}$ induces a presheaf $\tilde{\mathcal{G}}$, the canonical presheaf, by $\tilde{\mathcal{G}}(U) = \Gamma(U, \mathcal{G})$. Now we ask if we can reverse this construction.

First, let $\mathcal{P}$ be a presheaf and let

$$\hat{\mathcal{P}}_p = \{\text{germs of } \mathcal{P} \text{ at } p \in M\} = \bigcup_{p \in U} \mathcal{P}(U)/\sim,$$

where the equivalence relation $\sim$ for $f_1 \in \mathcal{P}(U_1), f_2 \in \mathcal{P}(U_2)$ is defined by

$$\begin{aligned} f_1 \sim f_2 \quad &\Longleftrightarrow \quad \text{there exists an open subset } W \subset U_1 \cap U_2 \text{ such that } p \in W \text{ and} \\ &\qquad \rho_W^{V_1}(f_1) = \rho_W^{V_2}(f_2) \\ &\Longleftrightarrow \quad f_1|_W = f_2|_W. \end{aligned}$$

As in the case of germs of C^k–functions we define the topology on $\hat{\mathcal{P}} = \bigcup_{p \in M} \hat{\mathcal{P}}_p \to M$ by the basis given by

$$U_f = \{[f]_p \mid p \in U\}$$

where $U \subset M$ open, $f \in \mathcal{P}(U)$ and $[f]_p$ is the germ of f at p. There is an R–module structure on $\hat{\mathcal{P}}_p$ induced by addition and scalar multiplication on representatives. So, $\hat{\mathcal{P}}$ is a sheaf.

Definition A.11. The sheaf $\hat{\mathcal{P}}$ is called the *sheaf of (germs of)* $\mathcal{P}$.

Definition A.12. By this construction the presheaves in Example A.10, (2), (3) induce the *sheaf* $\mathcal{C}_E^k$ *of (germs of)* C^k *sections resp. the sheaf* $\mathcal{O}_E$ *of (germs of) holomorphic sections.*

Lemma A.13. *If α is a homomorphism of presheaves then α induces a sheaf homomorphism on the induced sheaves via $\hat{\alpha}([f]_p) = [\alpha_U(f)]_p$.*

Now, given a sheaf $\mathcal{G}$ we have the construction

$$\mathcal{G} \rightsquigarrow \tilde{\mathcal{G}} \rightsquigarrow \hat{\tilde{\mathcal{G}}}$$

In fact, $\hat{\tilde{\mathcal{G}}} \cong \mathcal{G}$: Let $f \in \tilde{\mathcal{G}}(U) = \Gamma(U, \mathcal{G})$ be a representative of $[f]_p \in \hat{\tilde{\mathcal{G}}}$ then the isomorphism is given by $[f]_p \mapsto f(p) \in \mathcal{G}$.

On the other hand, given a presheaf $\mathcal{P}$ consider

$$\mathcal{P} \rightsquigarrow \hat{\mathcal{P}} \rightsquigarrow \tilde{\hat{\mathcal{P}}}.$$

Is $\tilde{\hat{\mathcal{P}}} \cong \mathcal{P}$? Let $\mathcal{P}(U) = R$ be a presheaf where R is a commutative ring with restriction maps $\rho_V^U = 0$ if $V \subsetneq U$. Then $\tilde{\hat{\mathcal{P}}} = 0$ hence in general

$$\tilde{\hat{\mathcal{P}}} \ncong \mathcal{P}.$$

Clearly, we have the following necessary conditions

Definition A.14. Let $\mathcal{P}$ be a presheaf. It is called *complete* if the following *sheaf conditions* are satisfied:

(S1) Let $U = \bigcup_{i\in I} U_i$, $f \in \mathcal{P}(U)$, where U_i are open sets.

$$\text{If } f|_{U_i} = \rho^U_{U_i}(f) = 0 \text{ for all } i \in I \text{ then } f = 0.$$

(S2) Let $U = \bigcup_{i\in I} U_i$, $f_i \in \mathcal{P}(U_i)$ where U_i are open sets.

$$\text{If } f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j} \text{ then there exists a } f \in \mathcal{P}(U) \text{ with } f|_{U_i} = f_i.$$

These sheaf conditions are in fact sufficient:

Proposition A.15. *If $\mathcal{P}$ is a complete presheaf then $\mathcal{P} \cong \tilde{\hat{\mathcal{P}}}$ via $f \in \mathcal{P}(U) \mapsto (p \mapsto [f]_p) \in \Gamma(U, \hat{\mathcal{P}})$.*

Proof. The map $\mu : f \in \mathcal{P}(U) \mapsto (p \mapsto [f]_p) \in \Gamma(U, \hat{\mathcal{P}})$ is injective:

Let $s = \mu(f) \in \Gamma(U, \hat{\mathcal{P}})$ for some $f \in \mathcal{P}(U)$ with $s = 0$. For $p \in U$ so $s(p) = [f]_p = 0$ i.e. there exists an open subset $U_p \subset U$ with $f\big|_{U_p} = 0$. But $U = \cup_{p\in U} U_p$, hence (S1) gives $f = 0$.

μ is surjective:

Let $s \in \Gamma(U, \hat{\mathcal{P}})$, then $s(p) \in \hat{\mathcal{P}}_p$, i.e. there exists an open set U_p and $f_p \in \mathcal{P}(U_p)$ with $s(q) = [f_p]_q$ for all $q \in U_p$. Let $p, q \in U$ and $\tilde{q} \in U_p \cap U_q$. Then

$$[f_p]_{\tilde{q}} = s(\tilde{q}) = [f_q]_{\tilde{q}}.$$

Hence for each $\tilde{q}$ there exists an open set $U_{\tilde{q}}$ such that $f_p\big|_{U_{\tilde{q}}} = f_q\big|_{U_{\tilde{q}}}$ hence by (S1)

$$f_p\big|_{U_p\cap U_q} = f_q\big|_{U_p\cap U_q}.$$

With (S2) there exists $f \in \mathcal{P}(U)$ with $\rho^U_{U_p}(f) = f_p$ for all $p \in U$ hence: $[f]_p = [f_p]_p = s(p)$ i.e. $\mu(f) = s$.

□

Remark A.16. As already remarked in the beginning one can define a sheaf by this construction: $\mathcal{G}$ is a sheaf if $\mathcal{G} = \hat{\mathcal{P}}$ for some complete presheaf.

Example A.17. a) The skyscraper presheaf (for a R–module V) is complete:

$$\mathcal{P}(U) := \begin{cases} V, & p_0 \in U \\ 0, & p_0 \notin U \end{cases}$$

with restriction maps

$$\rho^U_{\tilde{U}}(v) = \begin{cases} v, & p_0 \in \tilde{U} \\ 0, & p_0 \notin \tilde{U} \end{cases}, \text{ where } v \in \mathcal{P}(U).$$

(S1) Let $U = \cup U_i$ and $v \in \mathcal{P}(U)$. If $p_0 \notin U$ then $v = 0$. Assume that $v\big|_{U_i} = 0$ for all i and let $p_0 \in U_{i_0} \subset U$ for some i_0. Then $v\big|_{U_{i_0}} = v$. So in any case $v = 0$.

(S2) Let $U = \cup U_i$ and $v_i \in \mathcal{P}(U_i)$ with $v_i\big|_{U_i \cap U_j} = v_j\big|_{U_i \cap U_j}$. If $p_0 \notin U$ then define $v = 0$ hence $v\big|_{U_i} = v_i\big|_{U_i}$. If $p_0 \in U$ then there exists i_0 with $p_0 \in U_{i_0}$. Define $v \in \mathcal{P}(U)$ by $v = v_{i_0}$.

This is also clear if one realizes that the skyscraper presheaf is the canonical presheaf of the skyscraper sheaf.

b) Let $\mathcal{B}$ the presheaf given by

$$\mathcal{B}(U) = \{f : U \to \mathbb{C} \mid f \text{ is bounded and holomorphic.}\}$$

with canonical restriction maps. Let $U_i = \{z \mid |z| < i\}, i \in \mathbb{N}$, and define $f_i(z) = z,\ f_i \in \mathcal{B}(U_i)$. But $\mathbb{C} = \cup U_i$ and there exists no $f \in \mathcal{B}(\mathbb{C})$ with $f\big|_{U_i} = f_i$.

Hence $\mathcal{B}$ violates (S2). This comes from the fact that $\mathcal{B}$ consists of functions defined by a global property.

A.4 Sheaf sequences

Lemma A.18. *Let $\mathcal{P}_1 \overset{\alpha}{\to} \mathcal{P}_2 \overset{\beta}{\to} \mathcal{P}_3$ where $\mathcal{P}_k$ are presheaves satisfying* (S1). *Then we have a sequence of the associated sheaves*

$$\hat{\mathcal{P}}_1 \overset{\hat{\alpha}}{\to} \hat{\mathcal{P}}_2 \overset{\hat{\beta}}{\to} \hat{\mathcal{P}}_3 \tag{A.2}$$

The sequence (A.2) *is exact if and only if for all $U \subset M$ open:*

1. $\beta_U \circ \alpha_U = 0$
2. *if $\beta_U(f) = 0$ for some $f \in \mathcal{P}_2(U)$ then there exists an open covering $U = \cup U_i$ such that $f\big|_{U_i} \in \operatorname{Im} \alpha_{U_i}$.*

Proof. The sequence (A.2) is exact if and only if $\ker \hat{\beta}_p = \operatorname{Im} \hat{\alpha}_p$ for all $p \in M$. So we only need to show that exactness of the sequence implies (1) and (2).

Let $f \in \mathcal{P}_1(U)$ then

$$[\beta_U(\alpha_U(f))]_p = \hat{\beta}\hat{\alpha}[f]_p = 0, \quad \text{for all } p \in U.$$

But then $\beta_U \alpha_U(f) = 0$ at all $p \in U$. (S1) gives $f = 0$.

Since $\beta_U(f) = 0$ we see $\hat{\beta}([f]_p) = 0$. Since the sequence is exact we have $[f]_p = \hat{\alpha}_p[g]_p = [\alpha g]_p$. Hence there exists an open set U_p with $f\big|_{U_p} = \alpha g\big|_{U_p}$. □

Example A.19. 1. *(DE RHAM sequence)* Let M be a differentiable manifold of dimension m. The sequence

★
sheaf
$\Omega^{0,1}$def.

$$0 \to \mathbb{R} \overset{\text{incl}}{\hookrightarrow} \mathcal{C}^\infty_E \xrightarrow{d} \Omega^1_E \xrightarrow{d} \dots \xrightarrow{d} \Omega^m_E \to 0$$

is exact. This is a consequence of the Poincaré Lemma.

2. *(DOLBEAULT sequence)* Let M be a Riemann surface and $E \to M$ a holomorphic vector bundle on M. Consider the sequence

$$0 \to \mathcal{O}_E \hookrightarrow \Gamma(E) \xrightarrow{\bar\partial} \Omega^{(0,1)}_E \to 0.$$

At $\Gamma(E)$ the sequence is exact since $\varphi \in \mathcal{O}_E$ if and only if $\bar\partial\varphi = 0$. At $\Lambda^{(0,1)}_E = \mathcal{C}^\infty_{\bar{K}E}$ the exactness means the surjectivity of $\bar\partial$: For given $\alpha \in \mathcal{C}^\infty_{\bar{K}E}(U) = \Omega^{(0,1)}(U,E)$ and $p \in U$ we can solve locally $\bar\partial\psi = \alpha\big|_{\tilde U}$ with $\psi \in \Gamma(\tilde U, E)$, $p \in \tilde U \subset U$, see Chapter 4. Hence Lemma A.18 gives the result.

3. *(Exponential sequence)* Consider the sequence

$$0 \to \underline{\mathbb{Z}} \overset{\text{incl}}{\hookrightarrow} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

on a complex manifold M where $\exp(f)(z) = \exp(2\pi i f(z))$ and $\underline{\mathbb{Z}}$ is the constant sheaf. This sequence is exact:

Let $g_p \in \mathcal{O}^*_p$. Then there exists $g \in \mathcal{O}^*(U)$ with $[g]_p = g_p$ on some simply connected U. Choose a branch of logarithm and define $f = \frac{1}{2\pi i}\log g \in \mathcal{O}(U)$, i.e. $\exp(f)_p = g_p$. On the other hand if $\exp_p f_p = 0$ ("0" is the identity element in the abelian group $\mathcal{O}^*(U)$!) then $\exp 2\pi f(z) = 1$ for all $z \in U$ and all representatives f on U. Then f is constant and, in fact, an integer.

The presheaf Im exp satisfies not (S2): Let $M = \mathbb{C}\backslash\{0\} = U_1 \cup U_2$ where $U_1 = \mathbb{C}\backslash\{z \in \mathbb{R} \mid z \le 0\}$, $U_2 = \mathbb{C}\backslash\{z \in \mathbb{R} \mid z \ge 0\}$. Then $U_1 \cap U_2 = \mathbb{C}\backslash\mathbb{R}$. Define $f(z) = z$, then $f|_{U_k} = \exp g_k, k = 1,2$ with $g \in \mathcal{O}(U_k)$, i.e. $f \in$ Im exp and $\exp g_1\big|_{U_1\cap U_2} = \exp g_2\big|_{U_1\cap U_2}$ but there exists no $g \in \mathcal{O}(\mathbb{C}\backslash\{0\})$ with $\exp g = f$.

★
Diskusionsbedarf!

4. *(Skyscraper Sequence)* Let M be a Riemann surface, $E \to M$ a holomorphic vector bundle and $L(-p)$ the holomorphic line bundle defined in Remark 5.19, (**??**). Let again s^{-1} denote the meromorphic section with pole of order -1 at p and which is holomorphic on $M\backslash\{p\}$ and let $\psi \in \mathcal{O}_L$. Then

$$\psi \otimes s^{-1} \in \mathcal{O}_{EL(-p)} \text{ if and only if } \psi \text{ has a zero at } p, \text{ cf. Example 2.} \quad \text{(A.3)}$$

Thus s^1, the holomorphic section which vanishes only at p and with order 1, induces a map $\mathcal{O}_{EL(-p)} \to \mathcal{O}_L$ by $(\psi \otimes s^{-1}) \mapsto \psi \otimes s^{-1} \otimes s^1 \cong \psi$.

Now, the skyscraper sequence is the exact sheaf sequence

$$0 \to \mathcal{O}_{EL(-p)} \stackrel{\otimes s^1}{\to} \mathcal{O}_E \stackrel{ev_p}{\to} E_p \to 0$$

where E_p denotes the skyscraper sheaf with vector space E_p at p and ev_p is the evaluation map, i.e. $ev_p(\psi) = \psi(p)$ for $\psi \in \mathcal{O}_E(U)$ and $p \in U$.

In fact, for exactness we only need to show the injectivity of $\otimes s^1$ since the sequence is exact at $\mathcal{O}_E$ with (A.3).

Let $(\psi \otimes \varphi), (\tilde{\psi} \otimes \tilde{\varphi}) \in \mathcal{O}_{EL(-p)}$ and $\varphi = s^{-1} f,\ \tilde{\varphi} = s^{-1} \tilde{f}$. Then

$$(\psi \otimes \varphi) \otimes s^1 = (\tilde{\psi} \otimes \tilde{\varphi}) \otimes s^1 \text{ if and only if } \psi f = \tilde{\psi} \tilde{f}.$$

But then $\tilde{\psi} \otimes \tilde{\varphi} = \psi f \tilde{f}^{-1} \otimes s^{-1} \tilde{f} = \psi f \otimes s^{-1} = \psi \otimes \varphi$.

Appendix B

Čech sheaf cohomology

We consider sheaves as a tool to formulate local statements easily – however in geometry one is usually interested in global information. We attach cohomology modules to sheaves which measure more global information.

Assume that we can solve a problem locally, by finding a section in some sheaf. But we are searching for a global solution, i.e. a global section of the sheaf. The sheaf conditions (S1) and (S2) guarantee that a global section exists if the local sections coincide on the overlap domains. We turn this into an algebraic construction by given a homomorphism whose kernel is exactly the set of those sections which agree on overlap domains.

B.1 Sheaf cohomology over open covers

Let M be a topological space, $\mathcal{U} = (U_i)_{i \in I}$ an open cover. Let $\mathcal{G}$ be a sheaf over M.

Definition B.1. The set of *k–cochains* (for $\mathcal{G}$ over the open cover $\mathcal{U}$) is given by

$$C^k(\mathcal{U}, \mathcal{G}) = \prod_{i=(i_0,\ldots,i_k) \in I^{k+1}} \mathcal{G}(U_{i_0} \cap U_{i_1} \ldots \cap U_{i_k}).$$

Remark B.2. A cochain $f = (f_i)_{i \in I^{k+1}} \in C^k(\mathcal{U}, \mathcal{G})$ consists of

$$f_{i_0 \ldots i_k} \in \mathcal{G}(U_{i_0} \cap \ldots \cap U_{i_k}) = \Gamma(U_{i_0} \cap \ldots \cap U_{i_k}, \mathcal{G}).$$

The $C^k(\mathcal{U}, \mathcal{G})$ are R–modules for $k \geq 0$. For $k < 0$ we define $C^k(\mathcal{U}, \mathcal{G})$ to be zero.

Definition B.3. The *coboundary operator* d^k on k–cochains is defined by

$$d^k : C^k(\mathcal{U}, \mathcal{G}) \to C^{k+1}(\mathcal{U}, \mathcal{G}),\ (d^k f)_{i_0 \ldots i_{k+1}} = \sum_{j=0}^{k+1} (-1)^j f_{i_o,\ldots,\hat{i_j},\ldots,i_{k+1}}\big|_{U_{i_0} \cap \ldots \cap U_{i_{k+1}}}.$$

Lemma B.4. *The d^k's are coboundary operators, i.e*

$$d^{k+1}d^k = 0,$$

in particular $\text{Im } d^k \subset \ker d^{k+1}$.

Proof.

$$\begin{aligned}
d^{k+1}(d^k f)_{i_0\dots i_{k+2}} &= \sum_{j=0}^{k+2}(-1)^j (d^k f)_{i_0\dots\hat{i_j}\dots i_{k+2}}\big|_{U_{i_0}\cap\dots\cap U_{i_{k+2}}} \\
&= \sum_{j=0}^{k+2}(-1)^j \left(\sum_{n<j}(-1)^n f_{i_0\dots\hat{i_n}\dots\hat{i_j}\dots i_{k+2}}\big|_{U_{i_0}\cap\dots\cap U_{i_{k+2}}}\right. \\
&\qquad \left. + \sum_{n>j}(-1)^{n-1} f_{i_0\dots\hat{i_j}\dots\hat{i_n}\dots i_{k+2}}\big|_{U_{i_0}\cap\dots\cap U_{i_{k+2}}}\right) \\
&= 0
\end{aligned}$$

□

Definition B.5. The k^{th} *coboundaries* are defined by

$$B^k(\mathcal{U},\mathcal{G}) = \text{Im } d^{k-1}.$$

The k^{th} *cocycles* are

$$Z^k(\mathcal{U},\mathcal{G}) = \ker d^k.$$

The k^{th} *cohomology group* is defined by

$$H^k(\mathcal{U},\mathcal{G}) = Z^k(\mathcal{U},\mathcal{G})/B^k(\mathcal{U},\mathcal{G}).$$

Remark B.6. The k^{th} cohomology group is in fact a module.

We consider some special cases just to get familiar with the notations:

Example B.7.

k=0: $0 \to C^0(\mathcal{U},\mathcal{G}) \xrightarrow{d} C^1(\mathcal{U},\mathcal{G})$,

$$(df)_{ij} = (f_j - f_i)\big|_{U_i\cap U_j} \in \mathcal{G}(U_i\cap U_j) \text{ for } f = (f_i)_{i\in I} \in C^0(U,\mathcal{G}),\ f_i \in \mathcal{G}(U_i),$$

i.e. $df = 0$ if and only if $f_i = f_j$ on $U_i \cap U_j$.
Hence, if $df = 0$ then by (S2) there exists $\tilde{f} \in \tilde{\mathcal{G}}(M) = \Gamma(M,\mathcal{G})$ ($M = \bigcup U_i$) with $\tilde{f}\big|_{U_i} = f_i$. Since $B^0(\mathcal{U},\mathcal{G}) = 0$ we can identify

$$H^0(\mathcal{U},\mathcal{G}) = Z^0(\mathcal{U},\mathcal{G}) = \Gamma(M,\mathcal{G}).$$

<u>k=1:</u> $C^1(\mathcal{U},\mathcal{G}) \xrightarrow{d} C^2(\mathcal{U},\mathcal{G})$,

$$(df)_{ijk} = (f_{jk} - f_{ik} + f_{ij})\big|_{U_i \cap U_j \cap U_k} \in \mathcal{G}(U_i \cap U_j \cap U_k),$$

and $df = 0$ if and only if $f_{ik} = f_{ij} + f_{jk}$ on $U_i \cap U_j \cap U_k$.

We consider now the special case where M is a manifold, $\mathcal{U}$ an open cover of M and $\mathcal{G} = (C^\infty)^*$, i.e. $\mathcal{G}(U) = \{f : U \to \mathbb{C}\backslash\{0\} \mid f \in C^\infty\}$.

Then we have a map

$$H^1(\mathcal{U},(C^\infty)^*) \to \{C^\infty \text{ isomorphism classes of } \mathbb{C}\text{–line bundles}\}$$

defined by

$$[(g_{ij})] \mapsto L, \text{ where } L \text{ is given by the transition functions } g_{ij}.$$

This is in fact well defined: Since each representative (g_{ij}) of $[(g_{ij})]$ satisfies the cocycle condition (3.1), i.e.

$$g_{ij} \cdot g_{jk} = g_{ik} \text{ on } U_i \cap U_j \cap U_k,$$

each (g_{ij}) defines a line bundle by Theorem 3.8. But if $(\tilde{g}_{ij}) \in C^1(\mathcal{U},\mathcal{G})$, $(g_{ij}) \sim (\tilde{g}_{ij})$ then there exists $(f_i)_{i\in I} \in C^0(\mathcal{U},\mathcal{G})$ such that

$$\tilde{g}_{ij} = g_{ij}(df)_{ij} = f_i^{-1} g_{ij} f_j \tag{B.1}$$

(we have a multiplicative structure on $\mathcal{G}$!). Theorem 3.7 shows that the defined line bundle is independent, up to isomorphism, of the choice of the representative.

Notice that up to now the map depends on the cover we have chosen. A cover independent definition of the sheaf cohomology group will later give an isomorphism between the first sheaf cohomology group and the isomorphism classes of $\mathbb{C}$ line bundles over M. ★

A similar construction can be done for holomorphic line bundles: There is a map from the first cohomology group of the sheaf of non–vanishing holomorphic functions to the set of isomorphism classes of holomorphic line bundles. neu!

However, notice that for rank r vector bundles the situtation is more complicated: one would have to replace $\mathbb{C}^*$ by $\mathrm{GL}(r)$ and hence the argument in (B.1) does not work (non–commutativity).

Lemma B.8. *Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves over M. Any sheaf homomorphism $\alpha : \mathcal{G} \to \mathcal{F}$ induces a map on cohomology $\alpha : H^k(\mathcal{U},\mathcal{G}) \to H^k(\mathcal{U},\mathcal{F})$.*

Proof. Define $(\alpha(f))_{i_0,\ldots,i_k} := \alpha(f_{i_0,\ldots,i_k})$. □

B.2 Refinements

So far, our definition of cohomology depends on the chosen open cover. To find an independent definition we consider how the cohomology module changes under refinement of the cover. Hereby, we define a refinement as:

Definition B.9. Let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover. An open cover $\mathcal{V} = (V_j)_{j\in J}$ together with a map $t : J \to I$ is called a *refinement* of $\mathcal{U}$, short–hand $\mathcal{V} \prec_t \mathcal{U}$, if $V_j \subset U_{t(j)}$ for all $j \in J$.

Remark B.10. Any subcover of an open cover is a refinement. In particular, a refinement can arise (counterintuitive!) by omitting some of the original open sets and need not to come from a diminishment of the originally sets.

Lemma B.11. *Given a refinement $\mathcal{V} \prec_t \mathcal{U}$ define $t : C^k(\mathcal{U}, \mathcal{G}) \to C^k(\mathcal{V}, \mathcal{G})$ by*

$$t(f)_{j_0,\ldots,j_k} := f_{t(j_0),\ldots,t(j_k)}\big|_{V_{j_0}\cap\ldots V_{j_k}}$$

Then $d \circ t = t \circ d$ and there is an induced map

$$t : H^k(\mathcal{U}, \mathcal{G}) \to H^k(\mathcal{V}, \mathcal{G}).$$

Proof. Simple calculation. □

Lemma B.12 (Homotopy Lemma)**.** *Let s and t be* homotopic *i.e. there exists a* homotopy h_k *such that $s - t = d^{k-1} \circ h_k + h_{k+1} \circ d^k$, i.e.*

$$\begin{tikzcd} C^k(\mathcal{U},\mathcal{G}) \arrow[r, "d^k"] \arrow[d, "h_k"'] \arrow[dr, "s", "t"'] & C^{k+1}(\mathcal{U},\mathcal{G}) \arrow[d, "h_{k+1}"] \\ C^{k-1}(\mathcal{V},\mathcal{G}) \arrow[r, "d^{k-1}"'] & C^k(\mathcal{V},\mathcal{G}) \end{tikzcd}$$

Then the induced maps on the cohomology group coincide.

Proof. Let $f \in H^k(\mathcal{U}, \mathcal{G})$. Then $(s - t)f \in H^k(\mathcal{V}, \mathcal{G})$ and for $f \in Z^k(\mathcal{U}, \mathcal{G})$, i.e. $d^k f = 0$, one has

$$(s - t)(f) = d^{k-1}h_k f + h_{k+1}d^k f = d^{k-1}h_k f \in \text{Im } d^{k-1} = B^k(\mathcal{U}, \mathcal{G}).$$

□

Corollary B.13. *Let $V \prec U$ with two refinement maps $s, t : J \to I$. Then the induced maps on the cohomology group coincide.*

Proof. We only need to show that s and t are homotopic. At the beginning some abbreviations and definitions:

$$(h_{k+1}f)_{j_0,\dots,j_k} := \sum_{l=0}^{k}(-1)^l f_{s(j_0),\dots,s(j_l),t(j_l),\dots,t(j_k)} \tag{B.2}$$

$$(d^k f)_{i_0,\cdots,i_{k+1}} := \sum_{j=0}^{k+1}(-1)^j \; f_{i_0,\cdots,\widehat{i_j},\cdots,i_{k+1}} \tag{B.3}$$

$$f^{(l\leq n)} := f_{s_{j_0},\cdots,s_{j_l},t_{j_l},\cdots,\widehat{t_{j_n}},\cdots,t_{j_k}} \tag{B.4}$$

$$f_{(l\geq n)} := f_{s_{j_0},\cdots,\widehat{s_{j_n}},\cdots,s_{j_l},t_{j_l},\cdots,t_{j_k}} \tag{B.5}$$

We take three steps to show that h_k is a homotopy. First calculate $h \circ d$, second $d \circ h$ and third add them.

$$\begin{aligned}(h^{k+1}(d^k f))_{j_0,\cdots,j_k} &\overset{(B.2)}{=} \sum_{l=0}^{k}(-1)^l \; (d^k f)_{s_{j_0},\cdots,s_{j_l},t_{j_l},\cdots,t_{j_k}} \\ &\overset{(B.3)}{=} \sum_{l=0}^{k}(-1)^l \left(\sum_{n=0}^{l}(-1)^n f_{(l\geq n)} + \sum_{n=l+1}^{k+1}(-1)^n f_{s_{j_0},\cdots,s_{j_l},t_{j_l},\cdots,\widehat{t_{j_{n-1}}},\cdots,t_{j_k}}\right) \\ &= \sum_{l=0}^{k}\sum_{n=0}^{l}(-1)^{l+n} f_{(l\geq n)} + \sum_{l=0}^{k}\sum_{n=l}^{k}(-1)^{l+n+1} f^{(l\leq n)}\end{aligned} \tag{B.6}$$

$$\begin{aligned}(d^{k-1}(h^k f))_{j_0,\cdots,j_k} &\overset{(B.3)}{=} \sum_{n=0}^{k}(-1)^n (h^k f)_{j_0,\cdots,\widehat{j_n}\cdots,j_k} \\ &\overset{(B.2)}{=} \sum_{n=0}^{k}(-1)^n \left(\sum_{l=0}^{n-1} f^{(l\leq n)} + \sum_{l=n}^{k-1}(-1)^l f_{s_{j_0},\cdots,\widehat{s_{j_n}},\cdots,s_{j_{l+1}},t_{j_{l+1}},\cdots,t_{j_k}}\right) \\ &= \sum_{n=0}^{k}\sum_{l=0}^{n-1}(-1)^{l+n} f^{(l\leq n)} + \sum_{n=0}^{k}\sum_{l=n+1}^{k}(-1)^{l+n+1} f_{(l\geq n)}\end{aligned} \tag{B.7}$$

$$
\begin{aligned}
(h^{k+1}(d^k f) + d^{k-1}(h^k f))_{j_0,\cdots,j_k} &\overset{(B.6),(B.7)}{=} \sum_{n=0}^{k} (-1)^{2n} f_{(n\geq n)} + \sum_{n=0}^{k} (-1)^{2n-1} f^{(n\leq n)} \\
&= \sum_{n=0}^{k} \left(f_{(n\geq n)} - f^{(n\leq n)} \right) \\
&= \sum_{n=0}^{k} \left(f_{s(j_0),\dots,s(j_{n-1}),t(j_n),\dots,t(j_k)} - f_{s(j_0),\dots,s(j_n),t(j_{n+1}),\dots,t(j_k)} \right) \\
&= f_{s(j_0),\dots,s(j_k)} - f_{t(j_0),\dots,t(j_k)}
\end{aligned}
\tag{B.8}
$$

□

B.3 Cohomology groups of a sheaf

Definition B.14. The k^{th} *cohomology group* of a sheaf $\mathcal{G}$ is defined by

$$H^k(\mathcal{G}) := \dot{\bigcup_{\mathcal{U} \text{ open cover}}} H^k(\mathcal{U}, \mathcal{G}) / \sim$$

where the equivalence relation is defined for $[f] \in H^k(\mathcal{U}, \mathcal{G})$, $[g] \in H^k(\mathcal{V}, \mathcal{G})$ by $[f] \sim [g]$ if and only if there exists a common refinement $\mathcal{W} \prec_s \mathcal{U}$ and $\mathcal{W} \prec_t \mathcal{V}$ such that

$$s[f] = t[g] \text{ in } H^k(\mathcal{W}, \mathcal{G}).$$

Lemma B.15. *Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of M and $\mathcal{V} = \{V_j\}_{j\in J}$ a refinement of $\mathcal{U}$, with refinement map $t\colon J \to I$. Then the induced maps*

$$t\colon H^1(\mathcal{U}, \mathcal{F}) \longrightarrow H^1(\mathcal{V}, \mathcal{F})$$

is one–to–one and consequently

$$t\colon H^1(\mathcal{U}, \mathcal{F}) \longrightarrow H^1(\mathcal{F})$$

is one–to–one.

Proof. Let $f := (f_{i_0 i_1})_{(i_0,i_1)\in I^2} \in Z^1(\mathcal{U}, \mathcal{F})$ such that $t(f) \in B^1(\mathcal{V}, \mathcal{F})$, i.e. there exists $a = (a_j)_{j\in J} \in C^0(\mathcal{V}, \mathcal{F})$ such that

$$f_{t(j_0),t(j_1)}\Big|_{V_{j_0}\cap V_{j_1}} = t(f)_{j_0 j_1} = (a_{j_1} - a_{j_0})\Big|_{V_{j_0}\cap V_{j_1}}$$

for j_1 and $j_2 \in J$. From $(df)_{t(j_0),i,t(j_1)} = 0$ follows

$$a_{j_1} - a_{j_0} = f_{t(j_0),t(j_1)} = f_{t(j_0),i} + f_{i,t(j_1)}$$

on $V_{j_0}\cap V_{j_1}\cap U_i$ for all $i\in I$. That means $a_{j_1}+f_{t(j_1),i}=a_{j_0}+f_{t(j_0),i}$ on $V_{j_0}\cap V_{j_1}\cap U_i$. Therefore there exists $h=(h_i)\in C^0(\mathcal{U},\mathcal{F})$, such that

$$h_i\Big|_{V_j\cap U_i}=a_j+f_{t(j),i}\Big|_{U_i}.$$

But on $U_{i_0}\cap U_{i_1}\cap V_j$ the 0–cocycle h satisfies

$$h_{i_1}-h_{i_0}=a_j+f_{t(j),i_1}-a_j-f_{t(j),i_0}=f_{t(j),i_1}+f_{i_0,t(j)}=f_{i_0i_1}.$$

Hence $f=dh$ and $f=0$ in $H^1(\mathcal{U},\mathcal{F})$. □

Theorem B.16 (Snake Lemma)**.** *Let M be paracompact. Let*

$$0\longrightarrow\mathcal{E}\xrightarrow{\alpha}\mathcal{F}\xrightarrow{\beta}\mathcal{G}\longrightarrow 0,$$

be a short exact sequence of sheaves. Then there is a homomorphism $\delta: H^k(\mathcal{G})\to H^{k+1}(\mathcal{E})$, the so called connecting homomorphism*, such that there is a long exact sequence on cohomology*

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{E}) \arrow[r, "\alpha"] & H^0(\mathcal{F}) \arrow[r, "\beta"] & H^0(\mathcal{G}) \arrow[dll, "\delta"'] \\
& H^1(\mathcal{E}) \arrow[r, "\alpha"] & H^1(\mathcal{F}) \arrow[r, "\beta"] & H^1(\mathcal{G}) \arrow[dll, "\delta"'] \\
& H^1(\mathcal{E}) \arrow[r, "\alpha"] & H^1(\mathcal{F}) \arrow[r, "\beta"] & \cdots
\end{tikzcd}$$

Proof. 1. Let $U\subset M$ be open. We prove that

$$0\longrightarrow\Gamma(U,\mathcal{E})\xrightarrow{\alpha}\Gamma(U,\mathcal{F})\xrightarrow{\beta}\Gamma(U,\mathcal{G})$$

is exact. Let $\beta f=0$ then $\beta_p f_p=0$ for all $p\in U$. With Lemma A.18 there exists an open neighborhood $U_p\subset U$ of p and $e_p\in\Gamma(U_p,\mathcal{E})$ with $\alpha e_p=f\big|_{U_p}$. For $p,q\in U$ we have $\alpha(e_p-e_q)\big|_{U_p\cap U_q}=(f\big|_{U_p}-f\big|_{U_q})\big|_{U_p\cap U_q}=0$. We already know that α is one to one hence $e_p=e_q$ on $U_p\cap U_q$. With (S2) there exists $e\in\Gamma(U,\mathcal{E})$ with $\alpha e=f$ on $U_p\cap U_q$.

★ Na schön, ich glaube es, es muß wohl bewiesen werden...

2. Let $\mathcal{U}$ be an open cover of M. Define $\bar{C}^k(\mathcal{U},\mathcal{G}):=\operatorname{Im}\beta\subset C^k(\mathcal{U},\mathcal{G})$. Then the sequence

$$0\longrightarrow C^k(\mathcal{U},\mathcal{E})\xrightarrow{\alpha}C^k(\mathcal{U},\mathcal{F})\xrightarrow{\beta}\bar{C}^k(\mathcal{U},\mathcal{G})\longrightarrow 0$$

is exact. In fact, we will show that *any* short exact sequence as above induces a connecting homomorphism and a long exact sequence on cohomology.

Now, d maps $\bar{C}^k(\mathcal{U},\mathcal{G})$ to $\bar{C}^{k+1}(\mathcal{U},\mathcal{G})$ since $d\beta = \beta d$ and we have

$$\begin{tikzcd}
 & C^k(\mathcal{U},\mathcal{E}) \arrow[r, "\alpha"] \arrow[d, "d"] & C^k(\mathcal{U},\mathcal{F}) \arrow[r, "\beta"] \arrow[d, "d"] & \bar{C}^k(\mathcal{U},\mathcal{G}) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & C^{k+1}(\mathcal{U},\mathcal{E}) \arrow[r, "\alpha"] & C^{k+1}(\mathcal{U},\mathcal{F}) \arrow[r, "\beta"] & \bar{C}^{k+1}(\mathcal{U},\mathcal{G}) &
\end{tikzcd}$$

Let $g \in \bar{Z}^k(\mathcal{U},\mathcal{G})$, and $f \in C^k(\mathcal{U},\mathcal{F})$ with $\beta f = g$. Then, since $dg = 0$ also $\beta df = 0$, hence there exists $e \in C^{k+1}(\mathcal{U},\mathcal{E})$ with $df = \alpha e$ ($df \in C^{k+1}(\mathcal{U},\mathcal{F})$ and exactness of the sequence). Since $0 = d(df) = d\alpha e = \alpha de$ we have $de \in \ker\alpha = 0$, i.e. $e \in Z^{k+1}(\mathcal{U},\mathcal{E})$. We define $\delta : \bar{H}^k(\mathcal{U},\mathcal{G}) \to H^{k+1}(\mathcal{U},\mathcal{E})$ by $\delta(g) = e$ and have to show that this is a well–defined homomorphism i.e. does not depend on the choice of f. Let hence $f, \tilde{f} \in C^k(\mathcal{U},\mathcal{F})$ with $\beta f = \beta\tilde{f} = g$ and $e, \tilde{e} \in C^{k+1}(\mathcal{U},\mathcal{E})$ with $\alpha(e) = df, \alpha(\tilde{e}) = d\tilde{f}$. Then $\tilde{f} - f \in \ker\beta$ and it exists a $\hat{e} \in C^k(\mathcal{U},\mathcal{E})$ with $\alpha(\hat{e}) = \tilde{f} - f$. Since $\alpha(\tilde{e}) - \alpha(e) = d\tilde{f} - df = d(\tilde{f} - f) = d\alpha(\hat{e}) = \alpha(d\hat{e})$ and α is one to one we see $\tilde{e} - e = d\hat{e}$ hence a coboundary. This shows that $\delta : \bar{C}^k(\mathcal{U},\mathcal{G}) \to H^{k+1}(\mathcal{U},\mathcal{E})$ is well defined. Similarly one obtains $\bar{B}^k(\mathcal{U},\mathcal{G}) \subset \ker\delta$ by diagram chasing.

3. Next, we need to show, that the diagram

$$\begin{tikzcd}
\bar{H}^k(\mathcal{U},\mathcal{G}) \arrow[r, "\delta"] \arrow[d, "t"] & H^{k+1}(\mathcal{U},\mathcal{E}) \arrow[d, "t"] \\
\bar{H}^k(\mathcal{V},\mathcal{G}) \arrow[r, "\delta"] & H^{k+1}(\mathcal{V},\mathcal{E})
\end{tikzcd}$$

commutes for every refinement $\mathcal{V} \prec_t \mathcal{U}$. If so then there exists an induced map δ with

$$0 \longrightarrow H^0(\mathcal{E}) \xrightarrow{\alpha} H^0(\mathcal{F}) \xrightarrow{\beta} \bar{H}^0(\mathcal{G}) \xrightarrow{\delta} H^1(\mathcal{E}) \longrightarrow \cdots$$

Consider the following diagram

$$\begin{tikzcd}
 & C^k(\mathcal{U},\mathcal{E}) \arrow[rr, "\alpha"] \arrow[dl, "t"] \arrow[dd, "d"] & & C^k(\mathcal{U},\mathcal{F}) \arrow[rr, "\beta"] \arrow[dl, "t"] \arrow[dd, "d"] & & \bar{C}^k(\mathcal{U},\mathcal{G}) \arrow[dl, "t"] \arrow[dd, "d"] \\
C^k(\mathcal{V},\mathcal{E}) \arrow[rr, "\alpha"] \arrow[dd, "d"] & & C^k(\mathcal{V},\mathcal{F}) \arrow[rr, "\beta"] \arrow[dd, "d"] & & \bar{C}^k(\mathcal{V},\mathcal{G}) \arrow[dd, "d"] & \\
 & C^{k+1}(\mathcal{U},\mathcal{E}) \arrow[rr] \arrow[dl, "t"] & & C^{k+1}(\mathcal{U},\mathcal{F}) \arrow[rr] \arrow[dl, "t"] & & \bar{C}^{k+1}(\mathcal{U},\mathcal{G}) \arrow[dl, "t"] \\
C^{k+1}(\mathcal{V},\mathcal{E}) \arrow[rr, "\alpha"] & & C^{k+1}(\mathcal{V},\mathcal{F}) \arrow[rr, "\beta"] & & \bar{C}^{k+1}(\mathcal{V},\mathcal{G}) &
\end{tikzcd}$$

Let $g \in \bar{C}^k(\mathcal{U},\mathcal{G})$ and $e \in C^{k+1}(\mathcal{U},\mathcal{E})$ given by $\alpha(e) = df$ where $\beta(f) = g$. Then $\delta(g) = e$.

Define $\tilde{e} = t(e) \in C^{k+1}(\mathcal{V},\mathcal{E})$. Again, $\tilde{e}$ is a cocycle since t and d commute.

Define $\tilde{f} = t(f) \in C^k(\mathcal{V},\mathcal{F})$ and $\tilde{g} = t(g) \in \bar{C}^k(\mathcal{V},\mathcal{G})$. We like to show: $\delta(\tilde{g}) = \tilde{e}$.

Since $\beta(\tilde{f}) = \beta(t(f)) = t(\beta(f)) = t(g) = \tilde{g}$ it suffices to show $\alpha(\tilde{e}) = d\tilde{f}$. But

$$\alpha(\tilde{e}) = t(\alpha e) = t(df) = d\tilde{f}.$$

Notice that this proof actually works for each homomorphism t between short exact sequences.

4. Remains to show: If M is paracompact then $\bar{H}^k(\mathcal{G}) \cong H^k(\mathcal{G})$.

Define $\tilde{C}^k(\mathcal{U}) := C^k(\mathcal{U},\mathcal{G})/\bar{C}^k(\mathcal{U},\mathcal{G})$ with induced coboundary operator d. Then the following sequence is exact

$$0 \longrightarrow \bar{C}^k(\mathcal{U},\mathcal{G}) \hookrightarrow C^k(\mathcal{U},\mathcal{G}) \longrightarrow \tilde{C}^k(\mathcal{U}).$$

Hence we can apply (2) and have a connecting homomorphism and a long exact sequence on cohomology. We want to show that $\tilde{H}^k(\mathcal{G}) = 0$ for all k. For this purpose it suffices to show that for all $g \in C^k(\mathcal{U},\mathcal{G})$ there exists a refinement $\mathcal{V} \prec_t \mathcal{U}$ with $t(g) \in \bar{C}^k(\mathcal{V},\mathcal{G})$.

Choose first a refinement $\mathcal{O} \prec \mathcal{U}$ such that $O_i \subset \bar{O}_i \subset U_i$ for all i. Since M is paracompact it is also possible to choose for each $p \in M$ a neighborhood V_p of p such that

(a) $V_p \subset O_i$ for some $i \in I$

(b) If $V_p \cap O_i \neq$ then $V_p \subset U_i$

(c) $V_p \subset \cap_{p\in U_i} U_i$ and

(d) if $p \in U := \cap_{l=0}^k U_{i_l}$ then $g\big|_{V_p}^{U} = \beta(f)$ for some $f \in \mathcal{F}(V_p)$. ★ ja?

Thus $\mathcal{V} = \{V_p\}_{p\in M}$ is an open cover of M. In fact $\mathcal{V} \prec_t \mathcal{U}$ with refinement map t which maps V_p to U_p where $U_p \in \mathcal{U}$ and $O_p \in \mathcal{O}$ are chosen with $V_p \subset O_p \subset U_p$.

Let now $V_{p_0}, \ldots, V_{p_k} \in \mathcal{V}$ with $V = \cap V_{p_i} \neq \emptyset$. Then $V_{p_0} \cap O_{p_i} \neq \emptyset$ and hence $V_{p_0} \subset U_{p_i}$. In particular, $V_{p_0} \subset \cap U_{p_i} = U$. This gives

$$t(g)_{p_0,\ldots,p_k} = g_{p_0,\ldots,p_k}\big|_V = \rho_V^{V_{p_0}} \rho_{V_{p_0}}^U(g).$$

But $g\big|_{V_{p_0}} \in \operatorname{Im} \beta$.

$\square$

Remark B.17. We emphasize the following facts which we already mentioned in the preceding proof:

1. Given short exact sequences

$$0 \longrightarrow C^k(\mathcal{U}, \mathcal{E}) \longrightarrow C^k(\mathcal{U}, \mathcal{F}) \longrightarrow C^k(\mathcal{U}, \mathcal{G})$$

   for each k there exists a connecting homomorphism and a long exact sequence on cohomology.

2. Let t be a homomorphism between short exact sequences. Then δ commutes with t.

Example B.18.

★ *Skyscraper sheaf* $H^0(\mathcal{G}) = V$, $H^k(\mathcal{G}) = 0$, $k \geq 1$.

? Let $E \to M$ be a holomorphic vector bundle over a Riemann surface M. We claim that $H^k(\mathcal{O}_E) = 0, k \geq 2$. Consider the exact sequence

$$O \longrightarrow \mathcal{O}_E \longrightarrow C^\infty_E \longrightarrow \Omega^{(0,1)}_E \longrightarrow 0$$

★ then there is the long exact sequence

noch nicht fertig

$$O \longrightarrow H^0(\mathcal{O}_E) = H^0(E) \longrightarrow H^0(C^\infty_E) = C^\infty(E) \longrightarrow \Omega^{(0,1)}(E) \longrightarrow 0$$

★

$H^1(\mathcal{O}^*)$=hol. line bundles

B.4 Fine Sheaves

Definition B.19. A sheaf $\mathcal{G}$ over M is said to be *fine* if and only if for each locally finite cover $\mathcal{U} := \{ U_i \}_{i \in I}$ of M there exists a family $\{ \alpha_i \colon \mathcal{G} \to \mathcal{G} \mid i \in I, \alpha_i \text{ sheaf endomorphism}\}$ such that

1. $\operatorname{supp}(\alpha_i) := \overline{\{p \in M \mid \alpha_i|_{\mathcal{G}_p} \neq 0\}} \subset U_i$

2. $\sum_{i \in I} \alpha_i = id$

Remark B.20. This family is a *partition of unity* for $\mathcal{G}$ subordinate to the cover $\mathcal{U}$ of M.

Lemma B.21. *Let $E \to M$ be an arbitrary C^k vector bundle over a C^k manifold M. The sheaf $\mathcal{C}^k_E$) of germs of C^k sections of E is a fine sheaf.*

Proof. Let $\{ U_i \}_{i \in I}$ be an open cover of M and $\{\varphi_i | i \in I\}$ be a partition of unity for M subordinate to the cover $\{ U_i \}_{i \in I}$.[1]
Define $\alpha_i \colon \mathcal{C}^k_E \to \mathcal{C}^k_E$ by $\alpha_i([f]) := [\varphi_i|_U]f$ where $f \in C^k(U, E)$ □

[1]The existence for such an partition of unity is a basic result from Topology.

Lemma B.22. *Let $\mathcal{G} \to M$ be a fine sheaf where M is paracompact* [2]*.*
Then $H^q(M, \mathcal{G}) = 0$ for all $q \geq 1$.

Proof. Since $H^q(M, \mathcal{G}) := \bigcup^{\bullet}_{\mathcal{U}\,\text{open cover of}\,M} H^q(\mathcal{U}, \mathcal{G})/_{\sim}$, it is enough to show that $H^q(\mathcal{U}, \mathcal{G})$ vanishes for all open covers $\mathcal{U}$. But since M is paracompact, it is sufficient to prove it for all locally finite open covers.

From the Homotopy Lemma B.12 we know that for two cochain maps

$$\psi^q, \phi^q \colon C^q(\mathcal{U}, \mathcal{G}) \to C^q(\mathcal{U}, \mathcal{G}),$$

and a cochain homomorphism

$$h^q \colon C^q(\mathcal{U}, \mathcal{G}) \to C^{q-1}(\mathcal{U}, \mathcal{G})$$

satisfying

$$h^{q+1} \circ d + d \circ h^q = \psi^q - \phi^q.$$

The induced maps $\psi^q, \phi^q \colon H^*(\mathcal{U}, \mathcal{G}) \to H^*(\mathcal{U}, \mathcal{G})$ are equal[3]. If we choose $\psi^q := id^q$ and $\phi^q := 0$, then the induced id on cohomolgy vanishes. Hence $H^q(\mathcal{U}, \mathcal{G}) = 0$ and therefore $H^q(M, \mathcal{G}) = 0$.

It remains to define such a cochain homomorphism h and verify that the condition $h \circ d + d \circ h = id$ is satisfied.

We define $h^{q+1} \colon C^{q+1}(\mathcal{U}, \mathcal{G}) \to C^q(\mathcal{U}, \mathcal{G})$ by

$$(h^{q+1} g)_{i_0, \cdots, i_q} := \sum_{l \in I} \alpha_l \; g_{l, i_0, \cdots, i_q} \; , \tag{B.9}$$

where $g := \{g_{i_0, \cdots, i_{q+1}} \colon U_{i_0} \cap \cdots U_{i_{q+1}} \to \mathcal{G} \mid (i_0, \cdots, i_{q+1}) \in I^{q+1}\}$ and α_l forms the partition of unity subordinate $\mathcal{U}$. Remark h^q is well defined because $\mathcal{U}$ is locally finite.
Recall that $d^q \colon C^q \to C^{q+1}$ was defined by

$$(d^q f)_{i_0, \cdots, i_{q+1}} := \sum_{k=0}^{q+1} (-1)^k \; f_{i_0, \cdots, \hat{i_k}, \cdots, i_{q+1}}. \tag{B.10}$$

Now we will show the Homotopy Lemma condition $h \circ d + d \circ h = id \colon C^q \to C^q$,

[2] A topological space M is called *paracompact* if and only if every open cover of M has a locally finite refinement.

[3] For a cochain map $\psi \colon C^*(\mathcal{G}) \to C^*(\mathcal{G}')$, $f \mapsto \psi(f)$ the induced map $\psi \colon H^*(\mathcal{G}) \to H^*(\mathcal{G}')$, $[f] \mapsto \psi([f]) = [\psi(f)]$ is a homomorphism too.

for all $q \geq 1$. To do so, we calculate first $h \circ d$ and $d \circ h$.

$$\begin{aligned}
(h^{q+1}(d^q f))_{i_0,\cdots,i_q} &\overset{(B.9)}{=} \sum_{l \in I} \alpha_l \; (d^q f)_{l,i_0,\cdots,i_q} \\
&\overset{i \rightsquigarrow j}{=} \sum_{j_0 \in I} \alpha_{j_0} \; (d^q f)_{j_0,\cdots,j_{q+1}} \\
&\overset{(B.10)}{=} \sum_{j_0 \in I} \alpha_{j_0} \sum_{k=0}^{q+1} (-1)^k \; f_{j_0,\cdots,\hat{j_k},\cdots,j_{q+1}}
\end{aligned} \tag{B.11}$$

$$\begin{aligned}
(d^{q-1}(h^q f))_{i_0,\cdots,i_q} &\overset{(B.10)}{=} \sum_{k=0}^{q} (-1)^k \; (h^q f)_{i_0,\cdots,\hat{i_k},\cdots,i_q} \\
&\overset{(B.9)}{=} \sum_{k=0}^{q} (-1)^k \sum_{j_0 \in I} \alpha_{j_0} \; f_{j_0,i_0,\cdots,\hat{i_k},\cdots,i_q} \\
&\overset{i \rightsquigarrow j}{=} \sum_{k=0}^{q} (-1)^k \sum_{j_0 \in I} \alpha_{j_0} \; f_{j_0,j_1,\cdots,\widehat{j_{k+1}},\cdots,j_{q+1}} \\
&= \sum_{k=1}^{q+1} (-1)^{k+1} \sum_{j_0 \in I} \alpha_{j_0} \; f_{j_0,\cdots,\hat{j_k},\cdots,j_{q+1}} \\
&= \sum_{j_0 \in I} \alpha_{j_0} \sum_{k=1}^{q+1} (-1)^{k+1} \; f_{j_0,\cdots,\hat{j_k},\cdots,j_{q+1}}
\end{aligned} \tag{B.12}$$

Now we add the equation (B.11) and (B.12), substitute the indices $j_1, \cdots, j_{q+1}$ back to $i_0, \cdots, \hat{i_k}, \cdots, i_q$ and use the fact that α_l is a partition of unity:

$$\begin{aligned}
((hd + dh)f)_{i_0,\cdots,i_q} &= \sum_{j_0 \in I} \alpha_{j_0} \Big(\sum_{k=0}^{q+1} (-1)^k \; (d^q f)_{j_0,\cdots,\hat{j_k},\cdots,j_{q+1}} + \sum_{k=1}^{q+1} (-1)^{k+1} \; f_{j_0,\cdots,\hat{j_k},\cdots,j_{q+1}} \Big) \\
&= \sum_{j_0 \in I} \alpha_{j_0} \; f_{j_1,\cdots,j_{q+1}} \overset{j \rightsquigarrow i}{=} \sum_{l \in I} \alpha_l \; f_{i_0,\cdots,i_q} = f_{i_0,\cdots,i_q}
\end{aligned} \tag{B.13}$$

i.e the Homotopy Lemma condition is satisfied. This completes the proof. □

Example B.23. *(*Dolbeault *Sequence)*
Let $E \to M$ be a holomorphic vector bundle on a Riemann surface M. Using the Snake Lemma B.16, we get from the exact Dolbeault sequence

$$0 \to \mathcal{O}_E \hookrightarrow \mathcal{C}^\infty_E \xrightarrow{\overline{\partial}} \mathcal{C}^\infty_{\overline{K}E} = \Omega^{(0,1)}_E \to 0$$

the associated long exact sequence. Hence from this long exact sequence and the previous Lemma B.22 we get the following exact (sub)sequences.

$$0 \to H^0(\mathcal{O}_E) \xrightarrow{incl} H^0(\mathcal{C}_E^\infty) \xrightarrow{\overline{\partial}} H^0(\Omega_E^{(0,1)}) \xrightarrow{\delta} H^1(\mathcal{O}_E) \to H^1(\mathcal{C}_E^\infty) = 0 \tag{B.14}$$

$$0 = H^q(\Omega_E^{(0,1)}) \xrightarrow{\delta} H^{q+1}(\mathcal{O}_E) \to H^{q+1}(\mathcal{C}_E^\infty) = 0; \;\; q \geq 1 \tag{B.15}$$

From the exact sequence (B.15) we can deduce

$$H^q(\mathcal{O}_E) = 0 \text{ for all } q \geq 2.$$

And we can use these sequences (B.14), to conclude that the first Dolbeault cohomology group coincides with the first sheaf cohomomolgy group[4] $H^1(\mathcal{O}_E)$.

In detail: We know that for any sheaf $\mathcal{G}$, $H^0(\mathcal{G})$ is isomorphic to the global sections $\Gamma(\mathcal{G})$. We also know that the connecting map δ from the exact sequence (B.14) is onto. Hence the claim follows from:

$$\begin{aligned} H^1_{\overline{\partial}}(M,E) &:= \frac{\Omega^{(0,1)}(E)}{\overline{\partial}(\Gamma(E))} \\ &= \frac{\Gamma(M,\Omega_E^{(0,1)})}{\overline{\partial}\Gamma(M,\mathcal{C}_E^\infty)} \\ &\cong \frac{H^0(\Omega_E^{(0,1)})}{\overline{\partial}(H^0(\mathcal{C}_E^\infty))} \\ &= \frac{H^0(\Omega_E^{(0,1)})}{\ker\delta} \\ &\cong H^1(\mathcal{O}_E) \end{aligned} \tag{B.16}$$

B.5 Resolution

Definition B.24. A *resolution* of a sheaf $\mathcal{G}$ is an exact sequence of sheaves of the form $0 \to \mathcal{G} \to \mathcal{G}_0 \to \mathcal{G}_1 \to \cdots$, where $H^q(\mathcal{G}_i)$ vanishes for all $q \geq 1$ and $i \geq 0$. correct ?

Definition B.25. A resolution of a sheaf $\mathcal{G}$ is called a *fine resolution* if and only if each $\mathcal{G}_i$ is fine.

Theorem B.26. *For every sheaf there is a fine resolution*[5].

[4] The generalization of this fact is known as the Dolbeault Theorem: $H^{p,q}_{\overline{\partial}}(M,E) = \frac{\Omega^{(0,q)}(E)}{\overline{\partial}(\Omega^{(0,q-1)}(E))} \cong H^q(\mathcal{O}_E)$, for more details see Gunning and Rossi, [GR, page 183f] or Griffiths and Harris,[GH, page 45].

[5] We can find a proof in [GR, page 178].

Proposition B.27. *A resolution of a sheaf $\mathcal{G}$ on a paracompact space M is fine if and only if $H^q(\mathcal{G}_i)$ vanishes for all $q \geq 1$ and $i \geq 0$.*

Proof. One direction follows immediately from Lemma B.22. The other direction from the definition. ☐

Example B.28. *(The* DERHAM *Sequence)*
Let M be a smooth manifold and V be a vector space over $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Then the DERHAM sequence

$$0 \to \underline{V} \xrightarrow{incl} C^\infty_V \xrightarrow{d} \Omega^1_V \xrightarrow{d} \Omega^2_V \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\dim M}_V \to 0,$$

is a fine resolution of the constant sheaf $\underline{V} := M \times V$ with the discrete topology on the fibres[6]. The constant sheaf was declared on page 126. If M is a paracompact manifold, we deduce from Theorem B.30

$$H^q(M, \mathcal{O}_{\underline{\mathbb{C}}}) \cong H^q_{dR}(M, \underline{\mathbb{C}}).$$

Example B.29. *(The* DOLBEAULT *Sequence)*
Let $E \to M$ be a holomorhic vector bundle over a Riemann surface. The DOLBEAULT sequence

$$0 \to \mathcal{O}_E \overset{\text{incl}}{\hookrightarrow} \mathcal{C}^\infty_E \xrightarrow{\overline{\partial}} \mathcal{C}^\infty_{\overline{K}E} = \Omega^{(0,1)}_M \to 0\ ,$$

is a fine resolution of the holomorphic sheaf $\mathcal{O}_E$, i.e a fine resolution of the sheaf of germs of holomorphic sections in E.

Theorem B.30. *For a fine resolution $0 \to \mathcal{G} \xrightarrow{h} \mathcal{G}_0 \xrightarrow{h_0} \mathcal{G}_1 \xrightarrow{h_1} \mathcal{G}_2 \to \cdots$ of a sheaf $\mathcal{G}$ over a paracompact space M, the cohomology group $H^q(\mathcal{G})$ is isomorphic to $\ker h_q / \mathrm{Im}\ h_{q-1}$, for all $q \geq 0$. Where $h_q \colon H^0(\mathcal{G}_q) \to H^0(\mathcal{G}_{q+1})$ are the induced maps.*

Proof. First we are collecting some equations. For $\mathcal{K}_i := \ker h_i \subset \mathcal{G}_i$, we have following exact sequences

$$0 \longrightarrow \mathcal{G} \xrightarrow{h} \mathcal{G}_0 \xrightarrow{h_0} \mathcal{K}_1 \longrightarrow 0 \tag{B.17}$$

$$0 \longrightarrow \mathcal{K}_i \overset{\text{incl}}{\hookrightarrow} \mathcal{G}_i \xrightarrow{h_i} \mathcal{K}_{i+1} \longrightarrow 0 \ \text{ for all } i > 0 \tag{B.18}$$

While short exact sequences are inducing long exact sequences we get from the equation (B.17) the long exact sequence:

$$\begin{gathered} 0 \hookrightarrow H^0(\mathcal{G}) \xrightarrow{h} H^0(\mathcal{G}_0) \xrightarrow{h_0} H^0(\mathcal{K}_1) \xrightarrow{\delta^0} H^1(\mathcal{G}) \to \cdots \\ \cdots \to H^q(\mathcal{G}_0) \xrightarrow{h_0} H^q(\mathcal{K}_1) \xrightarrow{\delta^q} H^{q+1}(\mathcal{G}) \xrightarrow{h} H^{q+1}(\mathcal{G}_0) \end{gathered} \tag{B.19}$$

[6] We must pay attention, because both the constant sheaf and trivial bundle are denote by the same symbol $\underline{V}$. Even the underlining set is the same, they still differ by topology, which could not enough emphasised.

where δ_i are the connecting homomorphisms.
And, similarly from (B.18), for each $i > 0$ the long exact sequence

$$\begin{aligned} 0 \hookrightarrow H^0(\mathcal{K}_i) \xrightarrow{incl} H^0(\mathcal{G}_i) \xrightarrow{h_i} H^0(\mathcal{K}_{i+1}) \xrightarrow{\delta^0} H^1(\mathcal{K}_i) \to \cdots \\ \cdots \to H^q(\mathcal{G}_i) \xrightarrow{h_i} H^q(\mathcal{K}_{i+1}) \xrightarrow{\delta^q} H^{q+1}(\mathcal{K}_i) \xrightarrow{incl} H^{q+1}(G_i) \end{aligned} \tag{B.20}$$

Therefore our first equation to remember comes from (B.20):

$$H^0(\mathcal{K}_i) \cong \ker h_i \colon H^0(\mathcal{G}_i) \to H^0(\mathcal{G}_{i+1}). \tag{B.21}$$

Since G_0 is fine, we know from Lemma B.22 that $H^q(\mathcal{G}_0)$ vanishes for all $q \geq 1$. Using this fact we get a bunch of exact (sub)sequences from the long exact sequence (B.19).

$$0 \hookrightarrow H^0(\mathcal{G}) \xrightarrow{h} H^0(\mathcal{G}_0) \xrightarrow{h_0} H^0(\mathcal{K}_1) \xrightarrow{\delta^0} H^1(\mathcal{G}) \xrightarrow{h} H^1(\mathcal{G}_0) = 0 \tag{B.22}$$

$$0 = H^q(\mathcal{G}_0) \longrightarrow H^q(\mathcal{K}_1) \xrightarrow{\delta^q} H^{q+1}(\mathcal{G}) \longrightarrow H^{q+1}(\mathcal{G}_0) = 0 \tag{B.23}$$

As a first result we get the claim for the case $q = 0$:

$$H^0(\mathcal{G}) \underset{\text{(B.19)}}{\cong} \ker h_0 \colon H^0(\mathcal{G}_0) \to H^0(\mathcal{G}_1) \tag{B.24}$$

and for the case $q = 1$:

$$H^1(\mathcal{G}) \underset{\text{(B.22)}}{\cong} \frac{H^0(\mathcal{K}_1)}{h_0(H^0(\mathcal{G}_0))} \underset{\text{(B.21)}}{\cong} \frac{\ker h_1 \colon H^0(,\mathcal{G}_1) \to H^0(\mathcal{G}_1)}{\text{Im } h_0 \colon H^0(,\mathcal{G}_0) \to H^0(\mathcal{G}_1)}. \tag{B.25}$$

Further from sequence (B.23) we get our second equation:

$$H^q(\mathcal{K}_1) \cong H^{q+1}(\mathcal{G}), \quad q \geq 1. \tag{B.26}$$

Observe again all G_i are fine, i.e. $H^q(\mathcal{G}_i)$ vanishes for all $q \geq 1$ and $i \geq 0$. Hence the (sub)sequences coming from (B.20) are

$$0 \hookrightarrow H^0(\mathcal{K}_i) \xrightarrow{h} H^0(\mathcal{G}_i) \xrightarrow{h_i} H^0(\mathcal{K}_{i+1}) \xrightarrow{\delta^0} H^1(\mathcal{K}_i) \xrightarrow{h} H^1(\mathcal{G}_i) = 0 \tag{B.27}$$

$$0 = H^q(\mathcal{G}_i) \xrightarrow{h_i} H^q(\mathcal{K}_{i+1}) \xrightarrow{\delta^q} H^{q+1}(\mathcal{K}_i) \longrightarrow H^{q+1}(\mathcal{G}_i) = 0 \tag{B.28}$$

Therefore

$$\begin{aligned} H^q(\mathcal{K}_{i+1}) &\cong H^{q+1}(\mathcal{K}_i), \quad q \geq 1; \\ H^q(\mathcal{K}_0) &\cong H^1(\mathcal{K}_{q-1}), \quad q \geq 2. \end{aligned} \tag{B.29}$$

Now, since we got all these equations, we are in the position to join everything together:

$$
\begin{aligned}
H^q(\mathcal{G}) &\underset{(B.26)}{\cong} H^{q-1}(\mathcal{K}_1) \\
&\underset{(B.29)}{\cong} H^q(\mathcal{K}_0) \\
&\underset{(B.29)}{\cong} H^1(\mathcal{K}_{q-1}) \\
&\underset{(B.27)}{\cong} H^0(\mathcal{K}_q)/h_{q-1}(H^0(\mathcal{G}_i)) \\
&\underset{(B.21)}{\cong} \ker h_q/\mathrm{Im}\ h_{q-1}
\end{aligned}
\tag{B.30}
$$

□

B.6 Leray's Theorem on Cohomology

Definition B.31. Let $\mathcal{U} := \{ U_i \}_{i\in I}$ be an open cover for M. $\mathcal{U}$ will be called a *Leray cover* if and only if $H^q(U_{i_0}\cap\cdots\cap U_{i_l}, \mathcal{G}) = 0$ for all $q \geq 1$ and any $i_0, \cdots, i_l$.

Theorem B.32 (Leray)**.** *Let M be paracompact, $\mathcal{G}$ be a sheaf and $\mathcal{U}$ be a* Leray *cover. Then $H^q(\mathcal{U}, \mathcal{G}) \cong H^q(\mathcal{G})$.*

Proof. First select a fine resolution $0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}_0 \xrightarrow{h_o} \mathcal{G}_1 \xrightarrow{h_1} \mathcal{G}_2 \cdots$ of the sheaf $\mathcal{G}$ over M. The existence is guaranteed by Theorem B.26. From Theorem B.30 we know for the induced homomorphism $h_i\colon \Gamma(M, \mathcal{G}_i) \to \Gamma(M, \mathcal{G}_{i+1})$ that $H^q(M, \mathcal{G}) \cong \ker h_q/\mathrm{Im}\ h_{q-1}$ for all $q \geq 1$.
For any $\sigma := U_{i_0} \cap \cdots U_{i_l}$, we get $H^q(\sigma, \mathcal{G}) \cong \ker h_q \mid_\sigma /\mathrm{Im}\ h_{q-1} \mid_\sigma$ from Theorem B.30. And by hypothesis $H^q(\sigma, \mathcal{G})$ vanishes for all q, i.e.

$$0 \longrightarrow \Gamma(\sigma, \mathcal{G}) \longrightarrow \Gamma(\sigma, \mathcal{G}_0) \xrightarrow{h_0} \Gamma(\sigma, \mathcal{G}_1) \xrightarrow{h_1} \cdots$$

is exact for all σ. Therefore

$$0 \longrightarrow C^q(\mathcal{U}, \mathcal{G}) \longrightarrow C^q(\mathcal{U}, \mathcal{G}_0) \xrightarrow{h_0} C^q(\mathcal{U}, \mathcal{G}_1) \xrightarrow{h_1} \cdots$$

is exact for all $q \in \mathbb{Z}$. Grouping all these exact sequences together the following commutative diagram appears:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(M,\mathcal{G}) & \xrightarrow{h} & \Gamma(M,\mathcal{G}_0) & \xrightarrow{h_0} & \Gamma(M,\mathcal{G}_1) & \xrightarrow{h_1} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^0(\mathcal{U},\mathcal{G}) & \xrightarrow{h} & C^0(\mathcal{U},\mathcal{G}_0) & \xrightarrow{h_0} & C^0(\mathcal{U},\mathcal{G}_1) & \xrightarrow{h_1} & \cdots \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & C^1(\mathcal{U},\mathcal{G}) & \xrightarrow{h} & C^1(\mathcal{U},\mathcal{G}_0) & \xrightarrow{h_0} & C^1(\mathcal{U},\mathcal{G}_1) & \xrightarrow{h_1} & \cdots \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

All columns and all row are exact, except the first row. Since the entire diagram is commutative the desired isomorphism between $H^q(\mathcal{G})$ and $H^q(\mathcal{U},\mathcal{G})$ follows from a straightforward[7] diagram chase.

□

B.7 Axiomatic sheaf cohomology theory

Last but not least we present an axiomatic definition of a sheaf cohomology theory for sheaves of R-modules. Of course, Čech cohomology satisfies all these axioms. It can be shown that on a paracompact Hausdorff space (e.g. a second countable differentiable manifold), any two sheaf cohomology theories are canonically isomorphic (cf. [Wa]). Therefore, this can be seen as a summary of the really important properties of Čech cohomology.

Definition B.33. A *sheaf cohomology theory* for a topological space M with coefficients in the sheaves of R-modules consists of the following objects.

a) For each sheaf $\mathcal{G}$ of R-modules over M and each $k \in \mathbb{Z}$ there is a R-modules $H^k(\mathcal{G})$ called the k^{th} cohomology group.

b) For each homomorphism $\mathcal{G} \to \mathcal{G}'$ of sheaves and every $k \in \mathbb{Z}$ there is a homomorphism $H^k(\mathcal{G}) \to H^k(\mathcal{G}')$ of R-modules.

c) For each short exact sequence $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ of sheaves and every $k \in \mathbb{Z}$ there is a homomorphism $H^k(\mathcal{G}'') \to H^{k+1}(\mathcal{G}')$.

The following axioms must be satisfies by a sheaf cohomology.

[7] Even Gunning and Rossi, see [GR, page 189-191], only did it for $q = 1$

1. (Functorality) For all sheaves $\mathcal{G}$ and every $k \in \mathbb{Z}$, the identity homomorphism $\mathrm{id}\colon \mathcal{G} \to \mathcal{G}$ induces the identity $id\colon H^k(\mathcal{G}) \to H^k(\mathcal{G})$. Furthermore, if the diagram
$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathcal{G}' \\ & \searrow & \downarrow \\ & & \mathcal{G}'' \end{array}$$
commutes, then for each $k \in \mathbb{Z}$ so does the diagram
$$\begin{array}{ccc} H^k(M,\mathcal{G}) & \longrightarrow & H^k(M,\mathcal{G}') \\ & \searrow & \downarrow \\ & & H^k(M,\mathcal{G}''). \end{array}$$
2. (Long Exact Sequence) If $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ is a short exact sequence of sheaves, the sequence
$$\ldots \to H^k(\mathcal{G}') \to H^k(\mathcal{G}) \to H^k(\mathcal{G}'') \to H^{k+1}(\mathcal{G}') \to \ldots$$
is exact.
3. (Naturality) For each homomorphism of short exact sequences of sheaves
$$\begin{array}{ccccccccc} 0 & \to & \mathcal{G}'' & \to & \mathcal{G} & \to & \mathcal{G}' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{S}'' & \to & \mathcal{S} & \to & \mathcal{S}' & \to & 0 \end{array}$$
the following diagram commutes for all $k \in \mathbb{Z}$
$$\begin{array}{ccc} H^k(M,\mathcal{G}'') & \to & H^{k+1}(M,\mathcal{G}') \\ \downarrow & & \downarrow \\ H^k(M,\mathcal{S}'') & \to & H^{k+1}(M,\mathcal{S}'). \end{array}$$
4. (Normalization) $H^k(M,\mathcal{G}) = 0$ for all $k < 0$ and $H^0(M,\mathcal{G}) = \Gamma(\mathcal{G})$. Furthermore, if $\mathcal{G}$ is a fine sheaf, then $H^k(M,\mathcal{G}) = 0$ for all $k > 0$.

The uniqueness theorem for sheaf cohomology theories over paracompact spaces provides a simple method to prove that many of the classical cohomology theories are isomorphic, even if at the first sight no sheaves are involved in their definitions. For more details on this subject, see [Wa]. We will only explain one important application, a proof of the deRham theorem.

Remark B.34. The Theorem B.30 showing that the cohomology of a sheaf on a paracompact space can be calculated using a fine resolution holds for arbitrary sheaf cohomologies. This follows from the fact that the proof only uses the axioms given in this section.

Fine resolutions are in fact a very important tool to define new sheaf cohomology theories! [8] One very important example is the *singular cohomology for sheaves*. Using this cohomology theory, one can easily see that for an arbitrary R module G, the "classical" singular cohomology coincides with the cohomology of the corresponding constant sheaf $\mathcal{G}$, i.e.

$$H^*_{sing}(M, G) \cong H^*(\mathcal{G}).$$

In particular, we have $H^*_{sing}(M, \mathbb{Z}) \cong H^*(\underline{\mathbb{Z}})$ where $H^*(M, \mathbb{Z})$ is the integer singular cohomology of M and $H^*(\underline{\mathbb{Z}})$ is the cohomology of the corresponding constant sheaf. Similarly, $H^*_{sing}(M, \mathbb{R}) \cong H^*(\underline{\mathbb{R}})$ and $H^*_{sing}(M, \mathbb{C}) \cong H^*(\underline{\mathbb{C}})$.

Another example is the DERHAM *cohomology of sheaves* over differentiable manifolds. This theory uses the DERHAM sequence (see Example A.19 and B.28), which is a fine resolution of the constant sheaf $\underline{\mathbb{R}}$ (or $\underline{\mathbb{C}}$). Taking a tensor product of sheaves, this yields a fine resolution of all sheaves of $\mathbb{R}$ and $\mathbb{C}$ modules. The DERHAM Theorem follows immediately[9] since

$$H^*_{sing}(M, \mathbb{R}) \cong H^*(\underline{\mathbb{R}}) \cong H^*_{dR}(M, \mathbb{R}).$$

Here, the first isomorphism follows using singular cohomology for sheaves as sketched above. The second isomorphism has already been proved in Example B.28. (The same holds for coefficients in $\mathbb{C}$ or in arbitrary vector spaces instead of $\mathbb{R}$.)

[8] It is remarkable, that ČECH-cohomology is quite exceptional in the sense that no fine resolution of the sheaf is needed for the definition of its cohomology, i.e. the cohomology is defined using the sheaf alone! All other sheaf cohomology theories that can be found in [Wa] for example are defined using some special choices of fine resolution.

[9] To be honest there is still some work to be done in order to show that the isomorphisms preserve the algebra structures.

Appendix C

The Riemann–Roch theorem

This chapter start with the formulation of the SERRE duality theorem in the language of sheaves.

We use the first statement of the SERRE duality theorem, the fact that $H^0(\mathcal{O}_E)$ is finite dimensional for any holomorphic vector bundle E, to prove the existence of non trivial meromorphic sections of a holomorphic vector bundle. This in turn is used to show that holomorphic line bundles can be described by divisors. The second statement of the SERRE duality theorem, that $H^1(\mathcal{O}_E) \cong H^0(\mathcal{O}_{KE^*})$ is used to formulate the RIEMANN–ROCH theorem using $h^0(KE^*)$ instead of $h^1(E)$. Another way to see this, is that the formulation of the RIEMANN–ROCH theorem with $h^1(E)$ is just a theorem about linear algebra of sheaves, and that the RIEMANN–ROCH theorem follows as a corollary of the SERRE duality theorem.

C.1 Serre Duality

This chapter start with the formulation of the SERRE duality theorem in the language of sheaves. ★ neu

Theorem C.1 (SERRE Duality Theorem (sheaf formulation))**.** *Let M be a compact Riemann surface, $E \to M$ a holomorphic vector bundle. Then*

i) $H^0(\mathcal{O}_E)$ is finite dimensional and

ii)
$$(H^1(\mathcal{O}_E))^* \cong H^0(\mathcal{O}_{KE^*}).$$

Proof. In Example B.23 we have seen that the long exact sequence of the short exact DOLBEAULT sequence

$$0 \to \mathcal{O}_E \hookrightarrow \mathcal{C}^\infty_E \xrightarrow{\bar\partial} \mathcal{C}^\infty_{\bar K E} \to 0$$

and the fact that $\mathcal{C}_E^\infty$ and $\mathcal{C}_{\bar{K}E}^\infty$ are fine sheaves imply

$$H^1(\mathcal{O}_E) \cong \frac{\Gamma(\bar{K}E)}{\bar{\partial}\Gamma(E)} = \operatorname{coker}\bar{\partial}.$$

Hence the result follows with Theorem 6.24. □

C.2 The Riemann–Roch theorem for line bundles

Definition C.2. For a sheaf $\mathcal{G}$ of vector spaces we denote

$$h^q(\mathcal{G}) := \dim H^q(\mathcal{G}), \quad q \in \mathbb{N},$$

and use the abbriviation $h^q(E) = h^q(\mathcal{O}_E)$.

Example C.3.

1. If E is a holomorphic vector bundle over a compact Riemann surface then the Dolbeault sequence yields $h^k(E) = 0$ for $k \geq 2$.
2. Holomorphic functions on a compact Riemann surface are constant. Thus $h^0(M \times \mathbb{C}) = 1$.
3. By Hodge decomposition we know that the real dimension of the space of real valued harmonic 1–forms on a compact Riemann surface of genus g is equal to $2g$. Let $\omega = a + ib$ be a complex valued 1–form, and a and b its decomposition into its real and imaginary part. ω lies in K if and only if $*a = -b$. Thus the elemts of K are the 1–forms $\omega = a - i * a$, where a is a real valued 1–Form. ω is holomorphic if and only if $d\omega = 0$, that means if and only if $da = 0$ and $d * a = 0$. Thus the holomorphic sections of K are of the form $\omega = a - i * a$ with a real valued harmonic 1–form a. Thus $h^0(K) = g$.

Theorem C.4 (Riemann–Roch for line bundles)**.** *Let $L \to M$ be a holomorphic line bundle on a compact Riemann surface M. Then*

$$h^0(L) - h^1(L) = \deg(L) - g + 1,$$

or equivalently

$$h^0(L) - h^0(KL^{-1}) = \deg(L) - g + 1.$$

Remark C.5. If one defines the Euler *characteristic* of a sheaf $\mathcal{F} \to M$ to be $\chi(\mathcal{F}) = \sum_k (-1)^k h^k(\mathcal{F})$, then the proposition reads

$$\chi(\mathcal{O}_L) - \chi(\mathcal{O}_{M\times\mathbb{C}}) = \deg L,$$

since $H^k(\mathcal{O}_L) = 0$ for $k \geq 2$ and $\chi(\mathcal{O}_{M\times\mathbb{C}}) = h^0(M \times \mathbb{C}) - h^1(M \times \mathbb{C}) = h^0(M \times \mathbb{C}) - h^0(K) = 1 - g$, by Serre duality.

Proof. The equivalence of the two formulations follows from Serre duality. The validity of the for $L = M \times \mathbb{C}$ follows then from the facts collected in Example C.3.

For arbitrary holomorphic line bundles L we will proof the theorem by induction over the factors of the decomposition $L = L(p_1) \dots L(p_r) L(-p_{r+1}) \dots L(-p_k)$, Proposition 8.19.

First step. We assume that the theorem holds for a holomorphic vector bundle L. Then the skyscraper sequence

$$0 \to \mathcal{O}_L \to \mathcal{O}_{LL(p)} \to LL(p)\big|_p \to 0$$

is exact, example 4.4 and $E = LL(p)$. This implies the long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(\mathcal{O}_L) \longrightarrow H^0(\mathcal{O}_{LL(p)}) \longrightarrow \underbrace{H^0(LL(p)\big|_p)}_{=LL(p)\big|_p} \\ \longrightarrow H^1(\mathcal{O}_L) \longrightarrow H^1(\mathcal{O}_{LL(p)}) \longrightarrow \underbrace{H^1(LL(p)\big|_p)}_{=0} \longrightarrow \dots \quad (*) \end{aligned}$$

$H^1(LL(p)\big|_p) = 0$ follows from the fact that the skyscraper sheaf of a vector space is fine. Indeed let $\mathcal{U}$ be a local finite cover, and r the number of open sets in $\mathcal{U}$ that contain p, and define Das k"onnte woanders hin.

$$\alpha_i(f) := \begin{cases} \dfrac{1}{r} f, & \text{if } \pi(f) = p \text{ and } p \in U_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\operatorname{supp}(\alpha_i) = \{p\}$ if $p \in U_i$, $\operatorname{supp}(\alpha_i) = \emptyset$ if $p \notin U_i$, $\sum_{i\in I} \alpha_i(f) = 0 = \operatorname{id}(f)$ if $\pi(f) \in M \setminus \{p\}$, and $\sum_{i\in I} \alpha_i(f) = \sum_{i=1}^{r} \frac{1}{r} f = f = \operatorname{id}(f)$ if $\pi(f) = p$.

Thus $(*)$ is an exact sequence that starts and ends with zero. Consequently the alternating sum of the dimensions of the vector spaces involved is zero. Hence

$$0 = h^0(L) - h^0(LL(p)) + 1 - h^1(L) + h^1(LL(p)),$$

which implies

$$\begin{aligned} h^0(LL(p)) - h^1(LL(p)) &= h^0(L) - h^1(L) + 1 \\ &= \deg(L) - g + 1 + 1 = \deg(LL(p))) - g + 1. \end{aligned}$$

Second step. Now we want to prove the theorem for $LL(-p)$. Therefore we apply the skyscraper sequence

$$0 \to \mathcal{O}_{LL(-p)} \to \mathcal{O}_L \to L\big|_p \to 0.$$

which yields

$$0 = h^0(LL(-p)) - h^0(L) + 1 - h^1(LL(p)) + h^1(L),$$

thus

$$\begin{aligned} h^0(LL(-p)) - h^1(LL(-p)) &= h^0(L) - h^1(L) - 1 \\ &= \deg(L) - g + 1 - 1 = \deg(LL(-p))) - g + 1. \end{aligned}$$

□

C.3 The Riemann–Roch theorem for vector bundles

Theorem C.6 (Riemann–Roch for vector bundles)**.** *Let E be a holomorphic vector bundle over a compact Riemann surface M. Then*

$$h^0(E) - h^0(KE^*) = \deg E - (g-1)\operatorname{rank} E.$$

Proof by induction over $\operatorname{rank} E$. There exists a holomorphic subbundle $L \subset E$, by Corollary 8.2. Then E/L is a holomorphic vector bundle and $\operatorname{rank} E/L = \operatorname{rank} E - 1$. By assumption the theorem is true for E/L. Now

$$0 \longrightarrow L \longrightarrow E \longrightarrow E/L \longrightarrow 0$$

is exact. This induces the long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(L) \longrightarrow H^0(E) \longrightarrow H^0(E/L) & \\ \longrightarrow H^1(L) \longrightarrow H^1(E) \longrightarrow H^1(E/L) \longrightarrow \underbrace{H^2(L)}_{=0} \end{aligned}$$

Hence the alternating sum of the dimensions of the vector spaces involved is 0, i.e.

$$\begin{aligned} & h^0(L) - h^0(E) + h^0(E/L) - h^1(L) + h^1(E) - h^1(E/L) = 0 \\ \Longrightarrow \quad & h^0(E) - h^1(E) = \deg L - g + 1 + \deg(E/L) - (g-1)(\operatorname{rank} E - 1) \\ \Longrightarrow \quad & h^0(E) - h^1(E) = \deg E - (g-1)\operatorname{rank} E. \end{aligned}$$

□

Appendix D

Existence of isothermic coordinates

Appendix E

Chern classes

We already have seen how to define the degree of a vector bundle over a Riemann surface by using the trace of a Riemann curvature tensor. The degree defines a (topological) invariant of the Riemann surface.

In a more general context one should consider the so–called CHERN classes of a complex manifold.

In this section, let M be a complex manifold of dimension m, let $E \to M$ be a complex vector bundle of rank r and denote by ∇ a complex connection, i.e. $\nabla J = 0$ for the complex structure on E. The *absolute exterior differential* $d^\nabla \colon \Omega^k(E) \longrightarrow \Omega^{k+1}(E)$ with respect to ∇ is defined by

$$(d^\nabla \omega)(X_0, \ldots, X_k) := \sum_i (-1)^i \nabla_{X_i}(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k)) + \\ + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k).$$

For $\psi \in \Omega^0(E) = \Gamma(E)$ we have $d^\nabla_X \psi = \nabla_X \psi$, and for $\omega \in \Omega^1(E) = \Gamma(T^*M \otimes E)$ the usual formula,

$$d^\nabla \omega(X, Y) = \nabla_X(\omega(Y)) - \nabla_Y(\omega(X)) - \omega([X, Y]) \tag{E.1}$$

holds. The *curvature* $\mathcal{R}^\nabla \in \Omega^2(\mathrm{End}(E))$ is defined by $\mathcal{R}^\nabla := d^\nabla \circ d^\nabla |_{\Gamma(E)}$. Thus

$$\mathcal{R}^\nabla(X, Y)\psi = \nabla_X \nabla_Y \psi - \nabla_Y \nabla_X \psi - \nabla_{[X,Y]} \psi.$$

It is well known that $\mathcal{R}^\nabla$ satisfies the BIANCHI *identity*

$$d^\nabla \mathcal{R}^\nabla = 0$$

where $d^\nabla \mathcal{R}^\nabla := [d^\nabla, \mathcal{R}^\nabla]$, i.e. $(d^\nabla \mathcal{R})\psi = d^\nabla(\mathcal{R}\,\psi) - \mathcal{R}^\nabla(d^\nabla \psi)$.

Exactly the same way as for holomorphic structures we can construct, given a complex connection on the vector bundle E, a canonical complex connection on E^* via

$$(\nabla\alpha)\psi := d(\alpha\psi) - \alpha(\nabla\psi)$$

where $\alpha \in \Gamma(E^*)$ and $\psi \in \Gamma(E)$. Similarly, given two complex connections, ∇_1 and ∇_2 on E_1 and E_2, we define complex connections on $E_1 \otimes E_2$ and $\mathrm{Hom}(E_1, E_2) = E_1^* \otimes E_2$ by

$$\begin{aligned} \nabla(\psi_1 \otimes \psi_2) &:= (\nabla^1\psi_1) \otimes \psi_2 + \psi_1 \otimes \nabla^2\psi_2, \\ (\nabla T)\psi_1 &:= \nabla^2(T\psi_1) - T(\nabla^1\psi_1), \end{aligned}$$

where $\psi_1 \in \Gamma(E_1)$, $\psi_2 \in \Gamma(E_2)$ and $T \in \Gamma(\mathrm{Hom}(E_1, E_2))$.

Let V be a complex vector space of dimension r. We define the *elementary invariant polynomial maps* $P_k\colon \mathrm{End}(V) \to \mathbb{C}$ by

$$\begin{aligned} \det(X \cdot \mathrm{Id} - A) &= \sum_k (-1)^k P_k(A) X^{n-k} \\ &= \underbrace{P_0(A)}_{1} X^n - \underbrace{P_1(A)}_{\mathrm{tr}(A)} X^{n-1} + P_2(A)X^{n-2} - \ldots \pm \underbrace{P_r(A)}_{\det(A)} \end{aligned}$$

where $A \in \mathrm{End}(V)$. In the case of a 2-dimensional vector space V, this is just the well known formula

$$\det(X \cdot \mathrm{Id} - A) = X^2 - \mathrm{tr}(A)X + \det(A)$$

for the characteristic polynomial of A. The polynomials defined this way are Ad-invariant, i.e.

$$P_k(Ad(T)A) = P_k(TAT^{-1}) = P_k(A).$$

It can be shown that $P_1, \ldots, P_r$ are algebraically independent and they generate the algebra of invariant polynomial maps $\mathrm{End}(V) \to \mathbb{C}$ (cf. [KN]). The homogeneous invariant polynomial maps $P\colon \mathrm{End}(V) \to \mathbb{C}$ of degree k form a vector space isomorphic to the space of invariant symmetric k-linear maps $\tilde{P}\colon \mathrm{End}(V) \times \ldots \times \mathrm{End}(V) \to \mathbb{C}$. To a symmetric k-linear map $\tilde{P}$ we assign $P(X) := \tilde{P}(X, \ldots, X)$. With respect to a basis $X_1, \ldots, X_r$ of $\mathrm{End}(V)$ and a dual basis $\sigma_1, \ldots, \sigma_r$ we have

$$\tilde{P} = \sum_{I=(i_1,\ldots,i_k)} a_I\, \sigma_{i_1} \otimes \ldots \otimes \sigma_{i_k}$$

with $a_I = \tilde{P}(X_{i_1}, \ldots, X_{i_k})$. Thus, the corresponding $P(X) = \tilde{P}(X, \ldots, X)$ has the form

$$P = \sum_I a_I \sigma_I.$$

Conversely, given a homogeneous invariant polynomial map $P\colon \operatorname{End}(V) \to \mathbb{C}$ of degree k. Then P can be written in the form $P = \sum_I a_I\sigma_I$ with $a_{i_1,\dots,i_k}$ symmetric in the $i_1,...,i_k$. Now we define by $\tilde{P} = \sum_{I=(i_1,\dots,i_k)} a_I\sigma_{i_1}\otimes\dots\otimes\sigma_{i_k}$ the corresponding invariant symmetric k-linear map. In the following we denote by P_k not only the homogeneous polynomial, but also the invariant symmetric k-linear map $\tilde{P}_k$ obtained by the polarization just described.

The k^{th} Chern *form* of the complex vector bundle E with complex connection ∇ is the $2k$-form defined by

$$c_k := P_k(\frac{i}{2\pi}\mathcal{R}^\nabla \wedge\dots\wedge \frac{i}{2\pi}\mathcal{R}^\nabla) \in \Omega^{2k}(M).$$

The following formulas hold for the first and for the second Chern form

$$\begin{aligned} c_1 &= \frac{i}{2\pi}\operatorname{tr}\mathcal{R}^\nabla \\ c_2 &= P_2(\frac{i}{2\pi}\mathcal{R}^\nabla \wedge \frac{i}{2\pi}\mathcal{R}^\nabla) \end{aligned}$$

where $P_2(\omega\wedge\eta)(X_1,\dots,X_4) = \sum_\sigma(-1)^\sigma P_2(\omega(X_{\sigma(1)},X_{\sigma(2)}),\eta(X_{\sigma(3)},X_{\sigma(4)}))$.

Theorem E.1. *Let E be a complex vector bundle.*

1. *The* Chern *form c_k is closed for any complex connection ∇ on E, i.e.*

$$d(P_k(\frac{i}{2\pi}\mathcal{R}^\nabla \wedge\dots\wedge \frac{i}{2\pi}\mathcal{R}^\nabla)) = 0.$$

2. *The* Chern *forms of two complex connections ∇ and $\tilde{\nabla}$ are cohomologous, i.e.*

$$P_k(\frac{i}{2\pi}\mathcal{R}^\nabla \wedge\dots\wedge \frac{i}{2\pi}\mathcal{R}^\nabla) - P_k(\frac{i}{2\pi}\mathcal{R}^{\tilde{\nabla}} \wedge\dots\wedge \frac{i}{2\pi}\mathcal{R}^{\tilde{\nabla}}) \in d\Omega^{2k-1}(M,\mathbb{C}).$$

Proof. 1. Follows from the Bianchi identity $d^\nabla\mathcal{R}^\nabla = 0$ by

$$\begin{aligned} dP_k(\frac{i}{2\pi}\mathcal{R}^\nabla \wedge\dots\wedge \frac{i}{2\pi}\mathcal{R}^\nabla) = P_k(\frac{i}{2\pi}d^\nabla\mathcal{R}^\nabla \wedge \frac{i}{2\pi}\mathcal{R}^\nabla \wedge\dots\wedge \frac{i}{2\pi}\mathcal{R}^\nabla) + \\ +\dots+ P_k(\frac{i}{2\pi}\mathcal{R}^\nabla \wedge\dots\wedge \frac{i}{2\pi}\mathcal{R}^\nabla \wedge \frac{i}{2\pi}d^\nabla\mathcal{R}^\nabla) = 0. \end{aligned}$$

2. Let $B := \tilde{\nabla} - \nabla$. Then we have a curve connecting ∇ and $\tilde{\nabla}$ in the space of complex connections given by $\nabla^t = \nabla + tB$. For the curvature we have

$$\mathcal{R}^{\nabla^t} = \mathcal{R}^{\nabla^{t_0}} + (t-t_0)d^{\nabla^{t_0}}B + (t-t_0)^2 B\wedge B.$$

In 1. we have seen that the CHERN forms $c_k(\mathcal{R}^{\nabla^t})$ are closed. Therefore we can consider the curve (the factors $i/2\pi$ are left out in this calculation)

$$[c_k(\mathcal{R}^{\nabla^t})] = [P_k(\mathcal{R}^{\nabla^{t_0}} \wedge ... \wedge \mathcal{R}^{\nabla^{t_0}})] + (t-t_0)[P_k(d^{\nabla^{t_0}}B \wedge \mathcal{R}^{\nabla^{t_0}} \wedge ... \wedge \mathcal{R}^{\nabla^{t_0}})+ \\ ... + P_k(\mathcal{R}^{\nabla^{t_0}} \wedge ... \wedge \mathcal{R}^{\nabla^{t_0}} \wedge d^{\nabla^{t_0}}B)] + \text{ h.o.t. in } (t-t_0)$$

in the finite dimensional vector space $H^{2k}_{dR}(M,\mathbb{C})$. The derivative of this curve at the point t_0 is

$$\begin{aligned}&\frac{\partial}{\partial t}[c_k(\mathcal{R}^{\nabla^t})]|_{t_0} = \\ &\quad [P_k(d^{\nabla^{t_0}}B \wedge \mathcal{R}^{\nabla^{t_0}} \wedge ... \wedge \mathcal{R}^{\nabla^{t_0}}) + ... + P_k(\mathcal{R}^{\nabla^{t_0}} \wedge ... \wedge \mathcal{R}^{\nabla^{t_0}} \wedge d^{\nabla^{t_0}}B)] \\ &\quad = [d(P_k(B \wedge \mathcal{R}^{\nabla^{t_0}} \wedge ... \wedge \mathcal{R}^{\nabla^{t_0}} + ... + P_k(\mathcal{R}^{\nabla^{t_0}} \wedge ... \wedge \mathcal{R}^{\nabla^{t_0}} \wedge B))].\end{aligned}$$

Since the cohomology class of an exact form is zero, the curve $[c_k(\mathcal{R}^{\nabla^t})]$ is constant in $H^{2k}_{dR}(M,\mathbb{C})$.

□

Definition E.2. Let $E \to M$ be a complex vector bundle over M. The k^{th} CHERN *class* of E is defined by

$$c_k(E) := [P_k(\frac{i}{2\pi}\mathcal{R}^{\nabla} \wedge \ldots \wedge \frac{i}{2\pi}\mathcal{R}^{\nabla})] \in H^{2k}_{dR}(M,\mathbb{C})$$

where ∇ is an arbitrary complex connection on E. The total CHERN class of E is defined by

$$c(E) := \sum_{j\geq 0} c_j(E) \in H^{2*}_{dR}(M,\mathbb{C}).$$

The CHERN class associates to each complex vector bundle over M an element of the cohomology ring $H^{*}_{dR}(M,\mathbb{C})$. The product in this ring is given by

$$[\omega][\eta] := [\omega \wedge \eta].$$

In fact, CHERN classes live in the even part $H^{2*}_{dR}(M,\mathbb{C})$ of the cohomology ring which is commutative.

Proposition E.3. *The* CHERN *class* $c(E) = 1 + c_1(E) + \ldots + c_r(E) \in H^{2*}_{dR}(M,\mathbb{C})$ *of a rank r complex vector bundle E has the following properties:*

1. $f^*c(E) = c(f^*E)$ *for a complex vector bundle* $E \to N$ *and* $f\colon M \to N$ (naturality).

2. *Isomorphic vector bundles have the same* CHERN *classes.*

3. $c(E_1 \oplus E_2) = c(E_1)c(E_2)$ *for two complex vector bundles* E_1 *and* E_2 (WHITNEY Sum Formula).

4. $c_1(\Sigma) = -1$ *for the tautological bundle* $\Sigma \to \mathbb{C}P^1$ (normalization).

5. *The* k^{th} Chern *class of the dual bundle* E^* *satisfies* $c_k(E^*) = (-1)^k c_k(E)$.

6. *The* Chern *class of the trivial bundle* $\underline{\mathbb{C}}^r = M \times \mathbb{C}^r$ *satisfies* $c(\underline{\mathbb{C}}^r) = 1$.

Furthermore, the first Chern *class* c_1 *satisfies*

7. $c_1(E/E_1) = c_1(E) - c_1(E_1)$ *for a complex subbundle* E_1 *of* E.

8. $c_1(E_1 \otimes E_2) = \operatorname{rank}(E_1)c_1(E_2) + \operatorname{rank}(E_2)c_1(E_1)$ *for two bundles* E_1 *and* E_2.

Proof. 1. We take the induced connection $f^*\nabla$ on f^*E characterized by

$$(f^*\nabla)_X(\psi \circ f) = \nabla_{df(X)}\psi$$

for all $\psi \in \Gamma(E)$ and $X \in TM$. Then $\mathcal{R}^{f^*\nabla} = f^*\mathcal{R}^{\nabla}$ and the statement follows.

2. Suppose we have an isomorphism $T\colon E_1 \to E_2$ of complex vector bundles. Then a complex connection ∇^1 on E_1 induces a connection $\nabla^2 = T\nabla^1 T^{-1}$ on E_2. For this connection we have $\mathcal{R}^{\nabla^2} = T\mathcal{R}^{\nabla^1}T^{-1}$ and the statement follows by Ad-invariance of P_k.

3. If we take as the connection on $E_1 \oplus E_2$ the direct sum $\nabla^1 \oplus \nabla^2$ of the two complex connections on E_1 and E_2 , then

$$\mathcal{R}^{\nabla^1 \oplus \nabla^2} = \mathcal{R}^{\nabla^1} \oplus \mathcal{R}^{\nabla^2} \in \Omega^2(\operatorname{End}(E_1 \oplus E_2)).$$

From linear algebra we know

$$\det(X\,\mathrm{Id} - \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}) = \det(X\,\mathrm{Id} - A_1)\det(X\,\mathrm{Id} - A_2).$$

Hence, for our choice of connection on $E_1 \oplus E_2$, the Whitney sum formula even holds on the level of Chern forms. ★ details?

4. See next section, Remark 5.19.

5. Follows from the fact that the connection ∇^* on the dual bundle E^* defined above satisfies $\mathcal{R}^{\nabla^*} = -(\mathcal{R}^{\nabla})^*$.

6. There is a flat connection ∇ on the trivial rank r bundle $\underline{\mathbb{C}}^r$. But $\mathcal{R}^{\nabla} = 0$ implies $c(E) = 1$.

7. E/E_1 is isomorphic to the orthogonal complement of E_1 for any hermitian product on E. Therefore, E is isomorphic to $E_1 \oplus E/E_1$ as complex vector bundles and 3. yields the formula.

8. Let ∇_1 and ∇_2 be complex connections on E_1 and E_2. The tensor product connection $\nabla_1 \otimes \nabla_2$ defined above satisfies

$$\mathcal{R}^{\nabla_1 \otimes \nabla_2} = \mathcal{R}^{\nabla_1} \otimes \mathrm{Id}_{E_2} + \mathrm{Id}_{E_1} \otimes \mathcal{R}^{\nabla_2} .$$

Therefore

$$\mathrm{tr}(\mathcal{R}^{\nabla_1 \otimes \nabla_2}) = \mathrm{tr}(\mathcal{R}^{\nabla_1})\, \mathrm{rank}_{E_2} + \mathrm{rank}_{E_1}\, \mathrm{tr}(\mathcal{R}^{\nabla_2}).$$

□

Remark E.4.

1. Chern classes are real cohomology classes, i.e. $c(E) \in H^{2*}_{dR}(M, \mathbb{R})$. This can be seen by taking an hermitian product $< . >$ on E and a compatible complex connection ∇, i.e. $d < \psi, \varphi > = < \nabla\psi, \varphi > + < \psi, \nabla\varphi >$. Then $\mathcal{R}^\nabla$ is skew-symmetric (i.e. $< \mathcal{R}^\nabla \psi, \varphi > = - < \psi, \mathcal{R}^\nabla \varphi >$) and we have

$$c_k(E) = [P_k(\frac{i}{2\pi}\, \mathcal{R}^\nabla)] = \overline{[P_k(\frac{i}{2\pi}\, \mathcal{R}^\nabla)]} \in H^{2k}_{dR}(M, \mathbb{R}).$$

2. Topologists define Chern classes as integer cohomology classes. In fact, one can give the following axiomatic definition: the Chern class is the (unique) map assigning to a complex vector bundle E of rank r the class $c(E) = 1 + c_1(E) + \ldots + c_r(E) \in H^{2*}(M, \mathbb{Z})$ such that 1. through 4. of the preceding proposition are satisfied (see [MS] for details).

3. The so called Pontrjagin *class* of a real vector bundle can be defined to be the Chern class of its complexification. Since every second Chern class of a complexified bundle is zero, the Pontrjagin class lives in $H^{4*}(M, \mathbb{Z})$.

Appendix F

The $\bar\partial$–Lemma for $g \in C^1(\mathbb{C})$

This appendix is devoted to the proof of the $\bar\partial$–Lemma.

Theorem F.1 ($\bar\partial$–Lemma [GH, p. 5]). *Let $D \subset U \subset \mathbb{C}$ be two open sets, $\bar D \subset U$, $\bar U$ compact and ∂D a 1–dimensional C^∞–submanifold, $g \in C^1(U, \mathbb{C})$. Then*

$$f(z) := \frac{1}{2\pi i}\int_D g(\xi)\frac{d\xi \wedge d\bar\xi}{\xi - z}$$

is well defined, $f \in C^1(D, \mathbb{C})$ and

$$f_{\bar z} = g.$$

Lemma F.2 (Change of variables for polar coordinates).
A function $f\colon \mathbb{C} \to \mathbb{R}$ is integrable over $\mathbb{C}$ if and only if $(r, \varphi) \mapsto rf(re^{i\varphi})$ is integrable over $\mathbb{R}_{\geq 0} \times [0, 2\pi]$. Then

$$\int_{\mathbb{C}} f dx \wedge dy = -\frac{1}{2i}\int_{\mathbb{C}} f dz \wedge d\bar z = \int_{\mathbb{R}_{\geq 0}\times[0,2\pi]} rf(re^{i\varphi})dr \wedge d\varphi.$$

Proof. The map $\Phi\colon \mathbb{R}_{>0}\times]0, 2\pi[\to \mathbb{C}\setminus \mathbb{R}_{>0}$, $(r, \varphi) \mapsto re^{i\varphi}$ is a diffeomorphism. From $z = x + iy = r\cos(\varphi) + ir\sin(\varphi)$ it follows that $dx = \cos(\varphi)dr - r\sin(\varphi)d\varphi$ and $dy = \sin(\varphi)dr + r\cos(\varphi)d\varphi$, thus

$$-\frac{1}{2i}dz \wedge d\bar z = dx \wedge dy = rdr \wedge d\varphi.$$

Since the sets omitted by Φ are of measure zero, the claim follows from the change of variables formula. ☐

Corollary F.3. *Let $f \in C(\mathbb{C})$ with compact support. Then $\frac{f(z)}{z-z_0}$ is integrable for any constant $z_0 \in \mathbb{C}$ and*

$$\int_{\mathbb{C}} f(z)\frac{dz \wedge d\bar z}{z - z_0} = -2i\int_{\mathbb{R}_{\geq 0}\times[0,2\pi]} f(re^{i\varphi} + z_0)e^{-i\varphi}dr \wedge d\varphi.$$

Proof. The formula follows as in Lemma F.2 using the transformation $(r, \varphi) \mapsto re^{i\varphi} + z_0$. The latter integral exists because a continuous function with compact support is integrable. $\square$

Lemma F.4 (Differentiation under the integral)**.** *Let $V \subset \mathbb{C}$ be compact and $U \subset \mathbb{C}$ open and let $f\colon U \times V \to \mathbb{C}$, $(z, \xi) \mapsto f(z, \xi)$ be C^1. Then*

$$g(z) = \int_V f(z, \xi) d\xi$$

is C^1 and

$$g_x(z) = \int_V f_x(z, \xi) d\xi, \qquad g_y(z) = \int_V f_y(z, \xi) d\xi$$

for all $z = x + iy \in U$.

Proof. Fix $z \in U$,let $\epsilon > 0$ be small enough such that for $h \in (-\epsilon, \epsilon)$, $z + h \in U$. We have

$$g_x(z) = \lim_{\substack{h \to 0 \\ |h| \leq \epsilon}} \int_V \frac{f(z+h, \xi) - f(z, \xi)}{h} d\xi$$

if the limes exists. Since f is C^1 in the first variable we get

$$\left| \frac{f(z+h, \xi) - f(z, \xi)}{h} \right| \leq |f_x(z + \tilde{h}, \xi)|$$

for some $\tilde{h}$ between 0 and h, and

$$|f_x(z + \tilde{h}, \xi)| \leq \max_{|h| \leq \epsilon} |f_x(z + h, \xi)|.$$

The function $\xi \mapsto \max_{|h| \leq \epsilon} |f_x(z + h, \xi)|$ is a continuous function for any $z \in \mathbb{C}$, hence integrable over the compact set V. Applying the dominated convergence theorem we get (since f is C^1)

$$g_x(z) = \int_V f_x(z, \xi) d\xi.$$

The continuity of g_x follows from the dominated convergence theorem as well. $\square$

Lemma F.5 (Cauchy integral theorem for C^1 functions)**.** *Let $U \subset \mathbb{C}$ be an open set, $f \in C^1(U, \mathbb{C})$, $D \subset U$ open, $\bar{D}$ compact and ∂D a 1–dimensional C^∞–submanifold of $\mathbb{C}$. Then*

$$\int_{\partial D} f \, dz = -\int_D f_{\bar{z}} dz \wedge d\bar{z}.$$

Proof. By Stokes theorem

$$\int_{\partial D} f\,dz = \int_D d(f\,dz) = \int_D df \wedge dz = \int_D f_{\bar z} d\bar z \wedge dz.$$

$\square$

Lemma F.6 (Cauchy integral formula for C^1 functions)**.** *Let $D \subset U \subset \mathbb{C}$ be two open sets, $\bar D \subset U$, $f \in C^1(U, \mathbb{C})$, $\bar U$ compact and ∂D a 1–dimensional C^∞–submanifold of $\mathbb{C}$. Then*

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\xi)}{\xi - z} d\xi + \frac{1}{2\pi i}\int_D \frac{f_{\bar \xi}(\xi)}{\xi - z} d\xi \wedge d\bar\xi,$$

for all $z \in D$.

Proof. Let $B_\epsilon(z) = \{\, w \in \mathbb{C} \mid |z-w| < \epsilon \,\}$. Apply Lemma F.5 to $D_\epsilon := D \setminus \overline{B_\epsilon(z)}$ and $\xi \mapsto \frac{f(\xi)}{\xi - z}$:

$$\int_{\partial D} \frac{f(\xi)}{\xi - z} d\xi - \int_{\partial B_\epsilon(z)} \frac{f(\xi)}{\xi - z} d\xi = \int_{\partial D_\epsilon} \frac{f(\xi)}{\xi - z} d\xi = -\int_{D_\epsilon} f_{\bar\xi}(\xi) \frac{d\xi \wedge \bar d\xi}{\xi - z}. \qquad (*)$$

Since $f_{\bar\xi}(\xi)$ can be continued to a continuous function with compact support on $\mathbb{C}$, the map $\xi \mapsto \frac{f_{\bar\xi}(\xi)}{\xi - z}$ is integrable by Corollary F.3. Thus

$$\lim_{\epsilon\to 0} \int_{D_\epsilon} f_{\bar\xi}(\xi) \frac{d\xi \wedge \bar d\xi}{\xi - z} = \int_D f_{\bar\xi}(\xi) \frac{d\xi \wedge \bar d\xi}{\xi - z}.$$

On the other hand

$$\int_{\partial B_\epsilon(z)} \frac{f(\xi)}{\xi - z} d\xi = 2\pi i f(z).$$

Now

$$\begin{aligned}
\left| \int_{\partial B_\epsilon(z)} \frac{f(\xi)}{\xi - z} d\xi - 2\pi i f(z) \right| & \\
&= \left| \int_0^{2\pi} i(f(\epsilon e^{it} + z) - f(z)) dt \right| \\
&\le \int_0^{2\pi} |f(\epsilon e^{it} + z) - f(z)| dt \\
&\le \int_0^{2\pi} \max_{h \in [0,2\pi]} |f(\epsilon e^{ih} + z) - f(z)| dt \\
&= 2\pi \max_{h \in [0,2\pi]} |f(\epsilon e^{ih} + z) - f(z)|
\end{aligned}$$

and $\lim_{\epsilon\to 0} \max_{h\in[0,2\pi]} |f(\epsilon e^{ih} + z) - f(z)| = 0$. This and $(*)$ proves the lemma. $\square$

Proof of Theorem F.1. We fix a point $z \in D$. Since g can be continued to a continuous function with compact support on $\mathbb{C}$ the function $\frac{g(\xi)}{\xi - z}$ is integrable by Corollary F.3. If $\mathrm{supp}(g) \subset D$ then

$$f(z) = \frac{1}{2\pi i}\int_D g(\xi)\frac{d\xi \wedge d\bar{\xi}}{\xi - z} = -\frac{1}{\pi}\int_{\mathbb{R}_{\geq 0}\times[0,2\pi]} g(re^{i\varphi} + z)e^{-i\varphi} dr \wedge d\varphi.$$

Now we can differentiate under the integral (Lemma F.4):

$$\begin{aligned}
f_{\bar{z}}(z) &= -\frac{1}{\pi}\int_{\mathbb{R}_{\geq 0}\times[0,2\pi]} g_{\bar{z}}(re^{i\varphi} + z)e^{-i\varphi} dr \wedge d\varphi \\
&= \frac{1}{2\pi i}\int_D g_{\bar{z}}(\xi)\frac{d\xi \wedge d\bar{\xi}}{\xi - z}, && \text{by Corollary F.3} \\
&= g(z) - \frac{1}{2\pi i}\int_{\partial D} \frac{g(\xi)}{\xi - z} d\xi, && \text{by the Cauchy integral formula (Lemma F.6)} \\
&= g(z), && \mathrm{supp}(g) \subset D. \qquad (*)
\end{aligned}$$

We fix a point $z \in D$.

Additionally we know that $f \in C^1$.

Now we chose $\epsilon > 0$ such that $B_{2\epsilon}(z) \subset D$, decompose $g = g_1 + g_2$, such that $g_1\big|_{D \setminus B_{2\epsilon}(z)} \equiv 0$ and $g_2\big|_{B_\epsilon(z)} \equiv 0$. Then $f = f_1 + f_2$,

$$f_i = \frac{1}{2\pi i}\int_D g_i(\xi)\frac{d\xi \wedge d\bar{\xi}}{\xi - z}, \qquad i = 1, 2.$$

The map $z \mapsto \frac{g_2(\xi)}{\xi - z}$ is C^1 on $B_\epsilon(z)$ for any $\xi \in D$. Differentiating under the integral (Lemma F.4) implies f_2 is C^1 and:

$$f_{2\bar{z}}(z) = \frac{1}{2\pi i}\int_D \left(\frac{(g_2(\xi)}{\xi - z}\right)_{\bar{z}} d\xi \wedge d\bar{\xi} = 0.$$

Since $\mathrm{supp}(g_1) \subset D$ we can use $(*)$ to complete the proof:

$$f_{\bar{z}}(z) = f_{1\bar{z}}(z) + f_{2\bar{z}}(z) = g_1(z) = g(z).$$

$\square$

Appendix G

Some technical details about the Abel map and Theta function

Let M be a RIEMANN surface of genus g, Δ its fundamental domain, a_j, b_j, its basis curves and ω_j the dual holomorpic one forms to a_j. The reader not familiar with these picture should read the section differential topological background, 10.1.
Further let $\zeta \in \mathrm{Jac}(M)$ and Ψ the abbreviation for $\Psi_\zeta = \theta(A_{P_0}(.) - \zeta)$ and A the ABEL map without emphasizing the base point P_0.

Lemma G.1. *Let $P_j \in a_j$ and P'_j the corrosponding point in a_j^{-1} and Q_j, Q'_j analog in b_j respective b_j^{-1}. Further R_1, R_2, R_3 the vertexes of the fundamental domain Δ where a_j lies between R_1, R_2 and b_j lies between R_2, R_3. Note that everything which works for a Q – couples works for R_1, R_2 and analogous with the P – couples and R_2, R_3. See picture* **??**.

1. $A_k(P'_j) = A_k(P_j) + B_{j,k}$

2. $A_k(Q'_j) = A_k(Q_j) + \delta_{j,k}$

3. $d \log \Psi(Q'_j) = d \log \Psi(Q_j)$

4. $d \log \Psi(P'_j) = d \log \Psi(P_j) - 2\pi i \, \omega_j(P_j)$

5. $\Psi(P'_j) = \Psi(P_j) \exp(-2\pi i \, (A_j(P_j) - \zeta_j + \frac{1}{2} B_{j,j}))$

6. $\int_{a_j} d \log \Psi(P) = 0$

7. $\int_{b_j} d\log\Psi(Q) = 2\pi i\ (\zeta_j - A_j(R_2) - \frac{1}{2}B_{j,j}))$

8. $\int_{a_j+a_j^{-1}} (d\log\Psi(P)A_k(P)) = 2\pi i\ (\int_{a_j}(\omega_j(P)A_k(P)) + B_{j,k})$

9. $\int_{b_j+b_j^{-1}} (d\log\Psi(Q)A_k(Q)) = \delta_{j,k}\ 2\pi i\ (\zeta_j - A_j(R_2) - \frac{1}{2}B_{j,j}))$

10. $\int_{a_j}(\omega_k(P)A_k(P)) = \delta_{j,k}(A_k(R_2) - \frac{1}{2})$

Proof. Take $D(Z) = d\log\Psi(Z)$ as abbreviation.

1. $A_k(P_j) - A_k(P'_j) = \int_{P_0}^{P_j}\omega_k - \int_{P_0}^{P'_j}\omega_k = \int_{P'_j}^{P_j}\omega_k = -\int_{b_j}\omega_k = -B_{j,k}$

2. $A_k(Q_j) - A_k(Q'_j) = \int_{Q_0}^{Q_j}\omega_k - \int_{Q_0}^{Q'_j}\omega_k = \int_{Q'_j}^{Q_j}\omega_k = -\int_{a_j}\omega_k = -\delta_{j,k}$

3. $\Psi(Q'_j) = \theta(A(Q'_j) - \zeta)$. Using 2. this is $\theta(A(Q_j) - e_j - \zeta)$ and by the periodicity of θ this is $\theta(A(Q_j) - \zeta) = \Psi(Q_j)$.

4. Since $\Psi(P'_j) = \theta(A(P'_j) - \zeta)$. Using 1. this is $\theta(A(P_j) - B_j - \zeta)$ and by the periodicity of θ this is $\exp(-2\pi i\ (A_j(P_j) - \zeta_j) - \pi i\ B_{i,i})\ \theta(A(P_j) - \zeta)$. Since we are intressed in $D(P'_j) = d\log\Psi(P'_j)$ we calculate $d\log$ from the exp term above:

$$2\pi i\ d(-A_j(P_j)+\zeta_j-\frac{1}{2}B_{i,i}) = -2\pi i\ dA_j(P_j) = -2\pi i\ d\int_{P_0}^{P_j}\omega_j = -2\pi i\ \omega_j(P_j)$$

5. $\Psi(P'_j) = \theta(A(P'_j) - \zeta) = \theta(A(P_j) + B_{j,j} - \zeta) = \exp(-2\pi i\ (A_j(P_j) - \zeta_j + \frac{1}{2}B_{j,j}))\theta(A(R_2) - \zeta) = \exp(-2\pi i\ (A_j(P_j) - \zeta_j + \frac{1}{2}B_{j,j}))\Psi(P_j)$

6. $\int_{a_j} d\log\Psi(P)) = \log\Psi(R_2) - \log\Psi(R_1) = \log\Psi(R_2) - \log\theta(A(R_1) - \zeta) \overset{2.}{=} \log\Psi(R_2) - \log\theta(A(R_2) + e_j - \zeta) = \log\Psi(R_2) - \log\theta(A(R_2) - \zeta) = 0$

7. $\int_{b_j} d\log\Psi(Q) = \log\Psi(R_3) - \log\Psi(R_2) = \log\frac{\Psi(R_3)}{\Psi(R_2)} \overset{1.}{=} -2\pi i\ (A_j(R_2) - \zeta_j + \frac{1}{2}B_{j,j}))$

8.

$$\begin{aligned}
\int_{a_j+a_j^{-1}}(D(P)\int_{P_0}^{P}\omega_k) &= \int_{a_j}(D(P)\int_{P_0}^{P}\omega_k)+\int_{a_j^{-1}}(D(P')\int_{P_0}^{P'}\omega_k)\\
&\overset{4.}{=} \int_{a_j}(D(P)\int_{P_0}^{P}\omega_k)-\int_{a_j}((D(P)-2\pi i\;\omega_j(P))(\int_{P_0}^{P}\omega_k+\int_{b_j}\omega_k\\
&= \int_{a_j}(D(P)\int_{P_0}^{P}\omega_k)-\int_{a_j}((D(P)-2\pi i\;\omega_j(P))(\int_{P_0}^{P}\omega_k+B_{j,k})\\
&= 2\pi i\;\int_{a_j}(\omega_j(P)\int_{P_0}^{P}\omega_k)+B_{j,k}(2\pi i-\int_{a_j}D(P))\\
&\overset{6.}{=} 2\pi i\;\int_{a_j}(\omega_j(P)\int_{P_0}^{P}\omega_k)+2\pi i\;B_{j,k}
\end{aligned}$$

9.

$$\begin{aligned}
\int_{b_j+b_j^{-1}}(D(Q)\int_{P_0}^{Q}\omega_k) &= \int_{b_j}(D(Q)\int_{P_0}^{Q}\omega_k)+\int_{b_j^{-1}}(D(Q')\int_{P_0}^{Q'}\omega_k)\\
&\overset{3.}{=} \int_{b_j}(D(Q)\int_{P_0}^{Q}\omega_k)-\int_{b_j}(D(Q)\delta_{i,k}\int_{b_j}D(Q)\int_{P_0}^{Q'}\omega_k-\delta_{j,k})\\
&= \delta_{j,k}\int_{b_j}D(Q)\\
&\overset{7.}{=} \delta_{j,k}\;2\pi i\;(\zeta_j-A_j(R_2)-\frac{1}{2}B_{j,j})
\end{aligned}$$

10. $2\int_{a_j}(\omega_k(P)A_k(P)) = 2\int_{a_j}(\omega_k(P)\int_{P_0}^{P}\omega_k) = \int_{a_j}d(A_k(P))^2 = (A_k(R_2))^2 - (A_k(R_1))^2 \overset{2.}{=} (A_k(R_2))^2-(A_k(R_2)-\int_{a_j}\omega_k)^2 = 2A_k(R_2)\int_{a_j}\omega_k-(\int_{a_j}\omega_k)^2 = 2\delta_{j,k}(A_k(R_2)-\frac{1}{2})$

$\square$

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