Appendix A. Calculation Details for Different Statistical Structures

Normal Distribution

In order to reveal what is the population density tail at large distances when the population statistical structure is described by a normal distribution, we need to estimate how the following integral,

$$n(\mathbf{R},t) \;=\; \frac{N_0 A_0}{4\pi t}\,\exp(-\eta^2/4)\int_0^\infty \frac{1}{z}\,\exp\left(-\frac{p}{z}-z^2+\eta z\right)dz\;, \tag{A.1}$$

depends on the parameter $p = r^2/(4\mu t$.

Having introduced a new variable $\xi = zp^{-1/3}$, the integral is transformed as follows:

$$\int_0^\infty \frac{1}{z}\,\exp\left(-\frac{p}{z}-z^2+\eta z\right)dz \;=\; \int_0^\infty \frac{1}{\xi}\exp\left(\eta p^{1/3}\xi\right)\exp\left[p^{2/3}\left(-\frac{1}{\xi}-\xi^2\right)\right]d\xi\;. \tag{A.2}$$

By denoting

$$\lambda = p^{2/3}\;,\; f(\xi,\lambda) = \frac{1}{\xi}\exp\left(\eta\xi\sqrt{\lambda}\right)\;\text{ and }\; S(\xi) = -\frac{1}{\xi}-\xi^2\;, \tag{A.3}$$

the integral on the right-hand side of (A.2) takes the following form:

$$\int_0^\infty f(\xi,\lambda)\exp\left[\lambda S(\xi)\right]d\xi\;, \tag{A.4}$$

where function $f\exp(\lambda S)$ obviously tends to zero for $\xi \to 0$ and $\xi \to +\infty$, λ is a large positive parameter, and function $S(\xi)$ has a unique maximum (reached for $\xi = \xi_0 = 2^{-1/3}$). These are the conditions when the Laplace method can be applied to estimate the integral's leading asymptotics when the parameter $\lambda = p^{2/3}$ tends to infinity:

$$\int_0^\infty f(\xi,\lambda)\exp\left[\lambda S(\xi)\right]d\xi \;\;\simeq\;\; \left[-\frac{2\pi}{\lambda S''(\xi_0)}\right]^{1/2} f(\xi_0,\lambda)\exp[\lambda S(\xi_0)] \tag{A.5}$$

(e.g. see Fedoryuk, 1987) where the omitted terms are of higher order with respect to $1/\lambda$.

Correspondingly, with (A.3) and (A.5), from (A.1) we arrive at the following expression for the population density:

$$n(\mathbf{R},t) \;\;\simeq\;\; C\cdot\left(\frac{p}{2}\right)^{-1/3}\exp\left[-3\left(\frac{p}{2}\right)^{2/3}+\eta\left(\frac{p}{2}\right)^{1/3}\right], \tag{A.6}$$

where $C = (N_0A_0/(4t\sqrt{3\pi}))\exp(-\eta^2/4)$, which is valid when $p = r^2/(4\mu t) \gg 1$, the larger p the higher is its accuracy.

Exponential Distribution

Now we are going to addresses the situation when the rate of decay of $\phi(D)$ at large D is lower than Gaussian. Specifically, here we consider the case when $\phi(D)$ decreases exponentially, so that $\phi(D) \sim \exp(-D/\nu)$ where ν is a characteristic diffusivity for the given species.

A lower asymptotical rate of decay of ϕ is the main difference from the previous case; however, in order to calculate the integral in (2.3), we need to make some additional assumptions about the properties of $\phi(D)$ also at small and intermediate values of D. Specifically, we assume the following functional form:

$$\phi(D) \;\;=\;\; A_2 D^\beta \exp\left(-\frac{D}{\nu}\right), \tag{A.7}$$

where $A_2 = [\nu^{\beta+1}\Gamma(\beta+1)]^{-1}$ is a coefficient, Γ is the standard Gamma-function, and β is a parameter describing the rate of decay in ϕ at small D.

Having substituted (A.7) into (2.3), we obtain

$$n(\mathbf{R},t) \;=\; \frac{N_0 A_2}{4\pi t}\,\nu^{\beta}\int_0^{\infty}\exp\left(-\frac{q}{z}-z\right)z^{\beta-1}dz\ , \tag{A.8}$$

where $z = D/\nu$ and $q = r^2/(4\nu t)$.

The integral in (A.8) can now be calculated exactly resulting in $2q^{\beta/2}K_{\beta}(2\sqrt{q})$ where $K_{\beta}(z)$ is the modified Bessel function (cf. Abramowitz and Stegun, 1972); for the calculation details see online Appendix B. Correspondingly, for any r and t, the population density is given by the following expression:

$$n(\mathbf{R},t) \;=\; \frac{N_0}{2\pi\nu\Gamma(\beta+1)t}\cdot q^{\beta/2}K_{\beta}(2\sqrt{q})\ . \tag{A.9}$$

Since

$$K_{\beta}(z) \;\simeq\; \sqrt{\frac{\pi}{2z}}\left[1+O\left(\frac{1}{z}\right)\right]\exp(-z)\ , \tag{A.10}$$

for large z, the tail of the distribution (A.9) is given by[4]

$$n(\mathbf{R},t) \;\simeq\; \frac{1}{4\sqrt{\pi}}\cdot\frac{N_0}{\nu\Gamma(\beta+1)t}\;q^{(2\beta-1)/4}\exp(-2\sqrt{q})\ , \tag{A.11}$$

that is, for any given t,

$$n(\mathbf{R},t) \;\sim\; r^{\beta-\frac{1}{2}}\exp\left(-\frac{r}{\sqrt{\nu t}}\right)\ . \tag{A.12}$$

Therefore, for the diffusivity distribution function $\phi(D)$ given by (A.7), the population density at large distances exhibits an exponential rate of decay. Note that the additional parameter β affects only the pre-exponential factor but not the exponent itself. That agrees very well with the intuitive expectation that the asymptotical behavior of $n(\mathbf{R},t)$ should depend more on the rate of decay of $\phi(D)$ at large D rather than on its other properties.

Power-law Distribution

Finally, we consider the third generic case when the asymptotical rate of decay in $\phi(D)$ is lower than exponential, e.g. when it is described by a power law, $\phi(D)\sim D^{-\gamma}$ for $D\to\infty$ where γ is a positive parameter. In order to ensure that $\int_0^{\infty}\phi(D)dD$ exists (which is necessary because $\phi(D)$ has the meaning of a probability density), we have to assume that $\gamma>1$.

As well as in the previous case, in order to make the integration in (2.3) possible, we need to make an additional assumption about the behavior of $\phi(D)$ also at small and intermediate D. Specifically, for convenience of mathematical treatment, here we consider the following parametrization:

$$\phi(D) \;=\; A_3 D^{-\gamma}\exp\left(-\frac{\alpha}{D}\right),\qquad D>0\ , \tag{A.13}$$

where $A_3 = \alpha^{\gamma-1}/\Gamma(\gamma-1)$ is a coefficient and α is an auxiliary parameter. In order to avoid the singularity at $D=0$, we additionally define $\phi(0)=0$.

Correspondingly, Eq. (2.3) turns into

$$n(\mathbf{R},t) \;=\; \frac{N_0 A_3}{4\pi t}\int_0^{\infty}\frac{1}{D^{1+\gamma}}\;\exp\left[-\left(\alpha+\frac{r^2}{4t}\right)\frac{1}{D}\right]dD\ . \tag{A.14}$$

The integral in (A.14) can be readily calculated resulting in

$$n(\mathbf{R},t) \;=\; \frac{N_0}{4\pi t}\cdot\frac{\alpha^{\gamma-1}\Gamma(\gamma)}{\Gamma(\gamma-1)}\left(\alpha+\frac{r^2}{4t}\right)^{-\gamma}\ . \tag{A.15}$$

[4]Note that asymptotic expression (A.11) can also be obtained by applying the Laplace method to the integral in (A.8).

Obviously, at any given moment of time and for sufficiently large r (so that $r^2 \gg \alpha t$), from (A.15) we arrive at

$$n(\mathbf{R}, t) \quad \sim \quad r^{-2\gamma}. \tag{A.16}$$

Thus, function $\phi(D)$ given by (A.13) results in a spatial distribution of the population density with the large-distance asymptotical behavior described by a power law.

Appendix B. Exact Solution for Exponential Decay

Consider the following integral as a function of parameter p:

$$F(p) \quad = \quad \int_0^\infty \exp\left(-\frac{p}{z} - z\right) z^{\gamma-1} dz \ , \tag{B.1}$$

where $p > 0$.

By differentiating $F(p)$ twice, we observe that F is a solution of the following differential equation:

$$pF''(p) - (\gamma - 1)F'(p) - F(p) \quad = \quad 0 \ . \tag{B.2}$$

Obviously, the above equation is of the type

$$(a_2 p + b_2)F''(p) + (a_1 p + b_1)F'(p) + (a_0 p + b_0)F(p) \quad = \quad 0 \ , \tag{B.3}$$

which is very well known and has been studied thoroughly. In our case, i.e. for $a_0 = a_1 = b_2 = 0$, $a_2 = 1$, $b_0 = -1$ and $b_1 = 1 - \gamma$, its general solution can be expressed through modified Bessel functions (e.g. see Polyanin and Zaitsev, 1995, p.155):

$$F(p) \quad = \quad p^{\gamma/2} \left[C_1 I_\gamma(2\sqrt{p}) + C_2 K_\gamma(2\sqrt{p})\right] \ , \tag{B.4}$$

where $I_\gamma(z)$ and $K_\gamma(z)$ are the modified Bessel function of the γth order (cf. Abramowitz and Stegun, 1972) and $C_1, \ C_2$ are arbitrary constants.

Now, in order to find the value of C_1 and C_2, we need to make use of the boundary conditions, i.e., what is $F(p = 0)$ and $F(p \to +\infty)$. It is straightforward to see that

$$\text{(a) } F(0) \ = \ \Gamma(\gamma) \quad \text{and} \quad \text{(b) } F(+\infty) \ = \ 0 \ , \tag{B.5}$$

where $\Gamma(z)$ is the standard gamma-function.

Since $I_\gamma(z)$ grows unboundedly for $z \to +\infty$, condition (B.5b) results in $C_1 = 0$. In order to make use of condition (B.5a), we take into account the asymptotic behavior of $K_\gamma(z)$ at small $z > 0$:

$$K_\gamma(z) \quad \simeq \quad \frac{1}{2}\Gamma(\gamma)\left(\frac{1}{2}z\right)^{-\gamma} \tag{B.6}$$

(Abramowitz and Stegun, 1972). From (B.4) and (B.6), we then obtain that, at small p,

$$F(p) \ = \ C_2 p^{\gamma/2} K_\gamma(2\sqrt{p}) \ \simeq \ C_2 p^{\gamma/2} \cdot \frac{1}{2}\Gamma(\gamma)\left(\sqrt{p}\right)^{-\gamma} \ = \ \frac{C_2}{2}\,\Gamma(\gamma) \tag{B.7}$$

and, having compared it to (B.5a), we arrive at $C_2 = 2$. Therefore, from (B.1) and (B.4) we obtain $F(p) = 2p^{\gamma/2} K_\gamma(2\sqrt{p})$.

Appendix C. Effect of Diffusivity Boundedness

Diffusivity finiteness means that the integration in (2.3) (as well as in all its particular cases, see the first column of Table 1) should be done over the domain $(0, D_*)$ where $D_* < \infty$ is the maximum theoretically possible diffusivity for a given population. Then, for sufficiently large r, it may happen that the maximum of the function under the integral is not contained inside the integration domain any more. The dependence of the integral on the parameter then becomes essentially different; the Laplace method in the form of equation (A.5) does not apply and should be modified.

Detailed analysis of the general case (2.3) is rather laborious; for the sake of simplicity, here we focus on the special cases. Consider, for instance, Eq. (2.5). The integral still has the structure as given by Eq. (A.4) in online Appendix A, where the integration is now taken from 0 to $\xi_* = D_*/(p^{1/3}\mu) < \infty$. Since $p \sim r^2$, for sufficiently large distances the integration domain shrinks to the left; as a result the global maximum of $S(\xi)$ (reached at $\xi = \xi_0$ which does not depend on p) slips out of the integration domain. It means that, over the integration domain $0 \le \xi \le \xi_*$, $S(\xi)$ reaches its maximum at $\xi = \xi_*$ and $S'(\xi_*) > 0$. The asymptotical analysis (Fedoryuk, 1987) then leads to the following estimate:

$$\int_0^{\xi_*} f(\xi,\lambda)\exp[\lambda S(\xi)]d\xi \quad \simeq \quad \frac{f(\xi_*,\lambda)}{\lambda S'(\xi_*)}\ \exp[\lambda S(\xi_*)]\ . \tag{C.1}$$

It is straightforward to see that application of the standard Laplace method to the case of normal distribution (2.5) ceases working for $p > 2z_*^3$ where $z_* = D_*/\mu$, i.e. for $r^2 > 8z_*^3\mu t$. For these values of r, applying (C.1) instead of (A.4), it is readily seen that the integral in the right-hand side of (A.2) is estimated differently, so that we finally arrive at

$$n(\mathbf{R},t) \quad \sim \quad \exp\left(-\frac{r^2}{4D_* t}\right) . \tag{C.2}$$

A similar result can be obtained for other special cases from Table 1; we omit the technical details for the sake of brevity.

Remarkably, this analysis predicts not only that at sufficiently large distances the tail of the population distribution always remains normal, but also gives the "critical distance" r_* where the fat tail induced by the statistical variations gives way to the thin Gaussian one. Recall that the integral properties change at $p_* = 2z_*^3$, i.e. for

$$\frac{r_*^2}{4\mu t} \quad = \quad 2\left(\frac{D_*}{\mu}\right)^3 . \tag{C.3}$$

From (C.3), we immediately obtain the value of the critical distance:

$$r_* \quad = \quad \left(\frac{8D_*^3 t}{\mu^2}\right)^{1/2} . \tag{C.4}$$