Galois Theory problems, #5.

1. Let $L \supset K$ be a finite Galois extension with Galois group G. For any $x \in L$ we define $N(x)$ by

$$N(x) = \prod_{\sigma \in G} \sigma(x).$$

   Show that $N(x) \in K$.

2. Let $L \supset K$ be a finite Galois extension with Galois group G. For any $x \in L$ we define a polynomial

$$f_x(t) = \prod_{\sigma \in G} (t - \sigma(x)).$$

   Show that $f_x(t)$ has coefficients in K. (Notice that this generalises the previous problem because the constant term of $f_x(t)$ is $\pm N(x)$.)

   Prove that if $\sigma(x) \neq \tau(x)$ whenever $\sigma \neq \tau$ then $f_x(t)$ is the minimal polynomial of x over K. What can we say if this condition doesn't hold?

3. Use the previous problem to calculate the minimal polynomial of $2\sqrt{2} - \sqrt{3}$ over $\mathbf{Q}$.

4. Let $L \supset K$ be a finite Galois extension of degree 90. Show that there exists an intermediate field M of degree 10 over K.

5. Let $L \supset K$ be a Galois extension of even degree. Show that there exists a subfield M of L containing K such that $[L : M] = 2$. What if we do not assume the extension is Galois?