Homotopy invariant notions of complete intersection in algebra and topology

Growth and periodicity

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Outline

Commutative algebra

- Philosophy
- Hierarchies.
- Three styles

Convenient categories.

- Convenient categories of spaces
- Convenient categories of modules

Regular spaces.

Finite generation

Complete intersections

- Three styles
- zci
- Spaces

A proof.

Philosophy

View spaces and groups through their ‘rings of functions’ using the eyes of commutative algebra.

Consequences

Seek to find homotopy invariant versions of standard notions.

The Hierarchy 1.

Commutative algebra

- Regular local rings.
- Complete intersections.
- Gorenstein rings

Rational homotopy theory

- Products *KV* of even Eilenberg-MacLane spaces
- X in a fibration $F \to X \to KV$ and $\pi_*(F)$ finite and odd.
- Manifolds, finite Postnikov systems (Félix-Halperin-Thomas)

The Hierarchy 2.

Commutative algebra

- Regular local rings.
- Complete intersections.
- Gorenstein rings

Group theory

- p-nilpotent groups.
- Many groups but not all ($(C_p \times C_p) \rtimes C_3$ Levi)!
- All finite groups (Dwyer-G-Iyengar).

Styles of regularity.

First style: regular sequences (s)

Maximal ideal generated by a regular sequence:

$$R \xrightarrow{x_1} R \to R/(x_1)$$

$$R/(x_1) \xrightarrow{x_2} R/(x_1) \to R/(x_1, x_2)$$

$$R/(x_1, \ldots, x_{n-1}) \xrightarrow{x_n} R/(x_1, \ldots, x_{n-1}) \to k$$

Styles of regularity.

Second style: modules (h)

Any finitely generated module M has a finite resolution by finitely generated projectives (i.e., it is *small* in the sense that

$$\bigoplus_i [M, T_i] \xrightarrow{\cong} [M, \bigoplus_i T_i].$$

is an isomorphism)

Third style: growth (g)

The Ext algebra $\mathrm{Ext}_R^*(k, k)$ is finite dimensional

Styles of regularity.

Equivalence

Auslander-Buchsbaum-Serre: The three styles of definition give equivalent notions

Commutative cochains.

Rings of functions

$R = C^*(X; k)$: we need a *commutative* model, with an internal tensor product of R-modules

Good models

Rational	PL polynomial differential forms $C^*(X; \mathbb{Q}) := \mathcal{A}_{PL}(X)$
Generally	Commutative ring spectra $C^*(X; k) := \mathrm{map}(X, Hk)$

Morita equivalents.

Suppose either X is 1-connected or X is 0-connected and p-complete, $\pi_1(X)$ is finite and $k = \mathbb{F}_p$

Rothenberg-Steenrod

$$\mathrm{Hom}_{C_*(\Omega X)}(k, k) \simeq C^*(X)$$

Eilenberg-Moore

$$\mathrm{Hom}_{C^*(X)}(k, k) \simeq C_*(\Omega X)$$

Counterparts of module concepts.

Regular sequences The (additive) exact sequence $Q \xrightarrow{x} Q \to Q/(x) = R$ gives (multiplicative) exact sequence

$$Q \to R \to R \otimes_Q k$$

corresponds to a *spherical fibration*

$$Y \leftarrow X \leftarrow S^n$$

Modules Clear!

Growth $H_*(\Omega X)$

Convenient categories of modules

From topology to algebra.

Want algebraic counterpart of $R = C^*(BG; k)$ and its category of modules.

Models with internal tensor products

$$\mathrm{K}(\mathrm{Inj}\, kG)$$

Models for loop spaces.

Benson's squeezed resolutions: in short, for $n \geq 1$

$$H_{n+1}(\Omega(BG_p^\wedge)) = \mathrm{Tor}_n^{e\cdot kG\cdot e}(kG\cdot e, e\cdot kG)$$

where e is the idempotent complementary to the one corresponding to the projective cover of the trivial module.

Styles of regular spaces.

First style: spherical fibrations (s)

$S^{n_1} \to X_1 \to X, S^{n_2} \to X_2 \to X_1, \ldots, S^{n_d} \to * \to X_{d-1}$

Example: $X = BU(n)$ is s-regular

$$S^{2n-1} \to BU(n-1) \to BU(n)$$

$$S^{2n-3} \to BU(n-2) \to BU(n-1)$$

$$S^1 \to * \to BU(1)$$

Styles of regularity.

Second style: modules (h)

Any *finitely generated* module M is *small*

Styles of regularity.

Third style: growth (g)

$H_*(\Omega X)$ is finite dimensional

Example

$X = BG$ for a compact connected Lie group G.

Relationship: growth and modules

- Suppose k is small over $C^*(X)$.

$$C^*(X) \models k$$

- Apply $\mathrm{Hom}_{C^*(X)}(\cdot, k)$
-

$$k = \mathrm{Hom}_{C^*(X)}(C^*(X), k) \models \mathrm{Hom}_{C^*(X)}(k, k) \simeq C_*(\Omega X)$$

- Conversely, suppose $H_*(\Omega X)$ is finite dimensional

$$k \models C_*(\Omega X)$$

- Apply $\mathrm{Hom}_{C_*(\Omega X)}(\cdot, k)$
-

$$C^*(X) = \mathrm{Hom}_{C_*(\Omega X)}(k, k) \models \mathrm{Hom}_{C_*(\Omega X)}(C^*(\Omega X), k) \simeq k$$

Relationship

- s-regularity implies g-regularity.
- Equivalent rationally

Examples of regularity

Rational homotopy theory.

$C^*(X;\mathbb{Q})$ is g-regular if and only if X is a finite product of even Eilenberg-MacLane spaces: $X = KV$

Mod *p* cochains.

For simply connected p-complete X, $C^*(X;\mathbb{F}_p)$ is g-regular if and only if X is the classifying space of a p-compact group in the sense of Dwyer-Wilkerson

Representation theory.

For a finite group G, $C^*(BG;\mathbb{F}_p)$ is g-regular if and only if G is p-nilpotent.

Noether normalizations.

The definition

If R is a commutative ring spectrum, a *Noether normalization* is a ring map $Q \to R$ with Q g-regular and R small over Q. In this case $R \otimes_Q k$ is the associated *Noether fibre*.

An example

If $R = C^*(BG)$ and $G \to U(n)$ is faithful representation then $Q = C^*(BU(n)) \to C^*(BG) = R$ is a Noether normalization with Noether fibre $C^*(U(n)/G)$.

Noether normalizations.

Finitely generated

If R is a commutative ring spectrum, and M is an R-module, we say M is *finitely generated* if there is a Noether normalization $Q \to R$ so that M is small over Q

An example

In rational homotopy theory (i.e., if $R = C^*(X; \mathbb{Q})$) then M is finitely generated if and only if $H^*(M)$ is finitely generated over $H^*(R) = H^*(X)$.

Styles of ci.

First style: regular sequences (s)

There is a regular ring Q and elements $x_1, \ldots, x_c$

$$Q \xrightarrow{x_1} Q \to Q/(x_1)$$

$$Q/(x_1) \xrightarrow{x_2} Q/(x_1) \to Q/(x_1, x_2)$$

$$Q/(x_1, \ldots, x_{n-1}) \xrightarrow{x_n} Q/(x_1, \ldots, x_{n-1}) \to Q/(x_1, \ldots, x_c) = R$$

Styles of ci.

Second style: modules (z)

The module theory is 'eventually multiperiodic'.

Third style: growth (g)

The Ext algebra $\mathrm{Ext}_R^*(k,k)$ has polynomial growth

Hypersurfaces

All modules M are eventually periodic

- Suppose $0 \to Q \xrightarrow{f} Q \to R \to 0$.
- Resolve M over Q

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0.$$

- Apply $(\cdot) \otimes_Q R$ to obtain
  $0 \to \overline{F}_n \to \overline{F}_{n-1} \to \cdots \to \overline{F}_1 \to \overline{F}_0 \to M \to 0.$
- Splice in correction

$$\begin{tikzcd}
0 \arrow[r] & \overline{F}_n \arrow[r] & \overline{F}_{n-1} \arrow[r] & \cdots \arrow[r] & \overline{F}_2 \arrow[r] & \overline{F}_1 \arrow[r] & \overline{F}_0 \arrow[r] & M \arrow[r] & 0 \\
\oplus & \oplus & \oplus & & \oplus & & & & \\
\overline{F}_{n-1} \arrow[r] & \overline{F}_{n-2} \arrow[r] & \overline{F}_{n-3} \arrow[r] & \cdots \arrow[r] & \overline{F}_0 \arrow[uur] & & & &
\end{tikzcd}$$

Hypersurfaces

All modules *M* are eventually periodic

- Repeat, to obtain a resolution

$$\ldots \to G_3 \to G_2 \to G_1 \to G_0 \to M \to 0$$

  over R.
- Assuming n is even (wlg), for $2i \geq n$

$$G_{2i} = \overline{F}_n \oplus \overline{F}_{n-2} \oplus \cdots \oplus \overline{F}_2 \oplus \overline{F}_0$$

  in even degrees and

$$G_{2i+1} = \overline{F}_{n-1} \oplus \overline{F}_{n-3} \oplus \cdots \oplus \overline{F}_3 \oplus \overline{F}_1$$

zci

Smallness

If M is finitely generated $\mathrm{Cone}(\chi_f : M \to \Sigma^2 M)$ is small.

Proof

- $\chi_f : M \to \Sigma^2 M$ is is factoring out the first row
- The exact sequence $0 \to \overline{F}_\bullet \to G_\bullet \to \Sigma^2 G_\bullet \to 0$ realizes the triangles

$$R \otimes_Q M \to M \to \Sigma^2 M.$$

zci

Hochschild

$$\begin{tikzcd}
R \otimes_Q M \arrow[r] \arrow[d, "\simeq"'] & M \arrow[r] \arrow[d, "\simeq"'] & \Sigma^2 M \arrow[d, "\simeq"'] \\
R \otimes_Q R \otimes_R M \arrow[r] & R \otimes_R M \arrow[r] & \Sigma^2 R \otimes_R M
\end{tikzcd}$$

$$\begin{tikzcd}
R \otimes_Q R \arrow[r] & R \arrow[r] & \Sigma^2 R
\end{tikzcd}$$

The definition (B-G)

R is a *z-hypersurface* if there is a natural transformation $1 \rightarrow \Sigma^a 1$ of non-zero degree so that the mapping cone is small for any finitely generated module.

Styles of ci.

Equivalence

Avramov-Gulliksen (+Benson-G): The three styles of definition give equivalent notions

Definitions.

sci

There is a g-regular space $B\Gamma$ and fibrations $S^{n_1} \to X_1 \to B\Gamma, S^{n_2} \to X_2 \to X_1, \ldots, S^{n_c} \to X_c \to X_{c-1}$ with $X = X_c$

zci

There are natural transformations $z_1, z_2, \ldots, z_c$ of the identity functor of non-zero degree so that $M/z_1/z_2/\cdots/z_c$ is small for all finitely generated M.

gci

$H_*(\Omega X)$ has polynomial growth

Relationships

Regularity and ci

- h-regular $\Rightarrow$ zci
- g-regular $\Rightarrow$ gci

Types of ci

- sci $\Rightarrow$ zci $\Rightarrow$ gci
- Equivalent rationally

Examples of complete intersections

Rational homotopy theory.

$C^*(X;\mathbb{Q})$ is sci if and only if there is a fibration

$$F \to X \to KV$$

where $\pi_*(F)$ is finite dimensional and odd.

Examples of complete intersections

Representation theory.

The ring $C^*(BG(q);\mathbb{F}_p)$ is gci if $G(q)$ is a Chevalley group

Proof.

$$\begin{tikzcd}
BG(q) \arrow[r] \arrow[d] & BG \arrow[d, "\Delta"] \\
BG \arrow[r, "\{1,\Psi^q\}"] & BG \times BG
\end{tikzcd}$$

□

Example

BA_4 is zci at 2

Taking $p = 2$ and $X = BA_4$, the natural 3-dimensional representation $A_4 \to SO(3)$ gives a 2-adic fibration

$$S^3 \to BA_4 \to BSO(3),$$

and BA_4 is a hypersurface space at 2 with $B\Gamma = BSO(3)$, and $n = 3$.
The cofibre sequence of bimodules showing BA_4 is zci is

$$C^*(BA_4 \times_{BSO(3)} BA_4) \to C^*(BA_4) \to \Sigma^2 C^*(BA_4),$$

and the periodicity element will be

$$\chi \in THH^{-2}(C^*(BA_4)).$$

sci $\Rightarrow$ zci

Theorem

If X is an s-hypersurface space then X is a z-hypersurface space.

Proof

Given

$$S^n \to X \to B\Gamma$$

We want bimodules, i.e., modules over

$$C^*(X) \otimes_{C^*(B\Gamma)} C^*(X) = C^*(X \times_{B\Gamma} X)$$

Proof

Proof (cont)

Pull back to obtain

$$S^n \to X \times_{B\Gamma} X \to X,$$

split by the diagonal

$$\Delta : X \to X \times_{B\Gamma} X.$$

$C^*(X)$ becomes a bimodule by pulling back along Δ.

Theorem

Suppose given a split fibration $S^n \to E \to B$ with $n \geq 3$, and odd (Example: $B = X$, $E = X \times_{B\Gamma} X$, where a $C^*(E)$-module is a $C^*(X)$-bimodule). There is a cofibre sequence of $C^*(E)$-modules $\Sigma_{n-1} C^*(B) \leftarrow C^*(B) \leftarrow C^*(E)$.

Proof

Ingredients

- Form $\Omega S^n \to B \xrightarrow{s} E$.
- Cohomology level (Serre spectral sequence)
- There is a unique $C^*(E)$-module with homotopy $H^*(B)$.
- Lift cohomology level to cochains

Examples

$H^*(X)$ ci

If $H^*(X)$ is a complete intersection, then X is formal, and there is a fibration

$$S^{m_1} \times \cdots \times S^{m_c} \to X \to KV$$

with $m_1, m_2, \ldots, m_c$ odd. In particular, X is also sci.

Comment

Contrast with general sci space $F \to X \to KV$,

Examples

Gorenstein not ci

- Connected sum $M\#N = (M' \vee N') \cup e^n$ where M' is M with a small disc removed.
- $\pi_*(\Omega(M\#N)) = (\pi_*(\Omega M') * \pi_*(\Omega N'))/(\alpha + \beta)$, ($*$ is the coproduct of graded Lie algebras, α and β are the attaching maps for the top cells).
- $\pi_*(\Omega(\mathbb{C}P^2\#\mathbb{C}P^2\#\mathbb{C}P^2)) = \mathrm{Lie}(u_1, v_1, w_1)/([u_1, u_1] + [v_1, v_1] + [w_1, w_1])$,
- $\mathbb{C}P^2\#\mathbb{C}P^2\#\mathbb{C}P^2$ is not gci.

Examples

Noetherian essential

- X with model $(\Lambda(v_2, x_3, w_4), dw = vx)$.
- Not sci: else in a fibration $S^3 \to X \to KV$ where $V = \mathbb{Q}\{v, w\}$.
- In homotopy this gives a short exact sequence

  $$0 \to \pi_*(\Omega S^3) \to \pi_*(\Omega X) \to \pi_*(\Omega KV) \to 0$$

  of graded Lie algebras, so $\pi_*(\Omega S^3)$ is an ideal of $\pi_*(\Omega X)$.
- By contrast, since $dw = vx$, the corresponding elements $\overline{v}_1, \overline{x}_2$ and $\overline{w}_3$ in the Lie algebra $\pi_*(\Omega X)$ satisfy $\overline{w} = [\overline{v}, \overline{x}]$
- Contradiction since $\pi_*(\Omega S^3)$ is generated by $\overline{x}$.
- The cohomology ring is not Noetherian (all odd products are zero).

Examples

ci and Gorenstein 1

- X in a fibration

$$S^3 \times S^3 \to X \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty,$$

classified by

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \overset{\{u^2, uv\}}{\to} K(\mathbb{Q}, 4) \times K(\mathbb{Q}, 4)$$

- X is h-Gorenstein
- $H^*(X)$ is not Gorenstein.

Examples

ci and Gorenstein 2

- $H^*(X) = \mathbb{Q}[u, v, p]/(u^2, uv, up, p^2)$ where u, v and p have degrees $2, 2$ and 5.
- The dimensions of its graded components are $1, 0, 2, 0, 1, 1, 1, 1, 1, \ldots$ (i.e., its Hilbert series is $p_X(t) = (1 + t^5)/(1 - t^2) + t^2$, where t is of codegree 1).
- $\mathfrak{m} = \sqrt{(v)}$; $H^0_{\mathfrak{m}}(R) = \Sigma^2\mathbb{Q}$
- Cohen-Macaulay defect here is 1, we have a pair of functional equations $p_X(1/t) - (-t)t^{-4}p_X(t) = (1 + t)\delta(t)$ and $\delta(1/t) = t^4\delta(t)$.

Examples

nci not zci 1

- X with model

$$R = (\Lambda(x_3, y_3, z_3, a_8), dx = dy = dz = 0, da = xyz).$$

- Unravel an even cocycle that is not a generator to the cocycle xy to yield

$$R' = (\Lambda(x, y, z, w, a), da = xyz, dw = xy).$$

- $d(wz) = da$, so change of variables $a' = a - wz$ to see that

$$R' \cong (\Lambda(x, y, z, w, a'), da' = 0, dw = xy)$$

Examples

nci not zci 1b

- R' is zci; from its homotopy we see it is of codimension 4.
- Hence nci of length 4, and therefore
- R is nci of length ≤ 5.
- The cohomology ring is not Noetherian.
- Note also that the dual Hurewicz map is not surjective in codegree 8.

Examples

nci not zci 2

- $$R = (\Lambda(x_5, y_3, z_3, y'_3, z'_3, a_{10}), dx = yz + y'z', da = xyy').$$
- Unravel the cocycle yy', yielding:

  $$R' = (\Lambda(x_5, y_3, z_3, y'_3, z'_3, a_{10}, w_5), dx = yz + y'z', da = xyy', dw = yy'$$
- This yields $d(a + xw) = (dx)w$.

Examples

nci not zci 2b

- Similarly, for the cocycles wy and wy', yielding:

$$R'' = (\Lambda(x_5, y_3, z_3, y'_3, z'_3, a_{10}, w_5, t_7, t'_7), dx = yz + y'z', da = xyy', dw$$

- Finally we have $d(a + xw - zt - zt') = 0$.
- Change of variables $a' = a + xw - zt - zt'$ and see that R'' is zci of codimension 8, and hence nci of length 8.
- It follows that R is *nci* of length ≤ 11.
- The cohomology ring is not Noetherian.
- The dual Hurewicz map is not surjective in codegree 10.

Summary

- There are homotopy invariant definitions of ci.
- There is a derived level notion of multiperiodicity.
- All notions are equivalent in rational homotopy theory.
- There are some interesting examples, rationally, in mod *p* homotopy theory and in representation theory.

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